Operations Research in Finance:
A Novice Introduction to Financial Engineering

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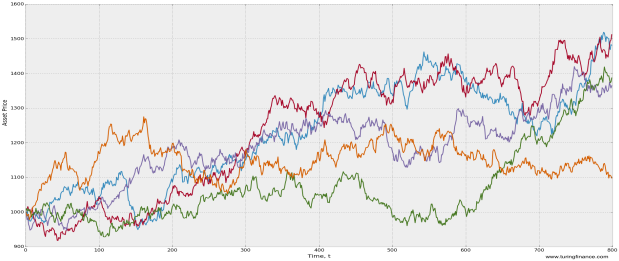
Finance

- "Science" of making money
  - making vs. losing money
  - time value of money
  - assets vs. liabilities
  - trading: buying (long position) vs. selling (short position)
  - trading with securities such as stocks, bonds, options, etc.
  - a lot of uncertainty (randomness)



Financial markets

- One of the challenging features of financial markets is the availability of a wide range of financial instruments.
- Any tradable financial instrument is called a security.
- Some of them are basic securities whose prices are observable in the market. These are called underlying securities.
  - Stocks of companies
  - Gold
  - Oil
  - Foreign currencies (USD, EUR, etc.)
- More complicated securities such as options, forward and futures contracts have payoffs that are based on the price of an underlying security. These are called derivative securities.
  - Options
  - Forward contracts
  - Futures contracts

Financial markets

- Where are these securities traded?
  - Exchanges: Regulated centralized markets where security buyers and sellers meet and negotiate on the prices. Participating companies must obey certain rules. All transactions are secured by the exchange. Many stocks and options are traded on exchanges.
  - Over-the-counter (OTC) markets: Less regulated decentralized markets with investment banks and individual investors. Stocks of smaller companies, gold, oil, foreign currencies are mostly traded on OTC markets.

- The ten largest stock exchanges in the world:
  - New York
  - NASDAQ
  - London
  - Tokyo
  - Shanghai
  - Hong Kong
  - Euronext (Belgium, France, Netherlands, Portugal, UK)
  - Toronto
  - Shenzen
  - Frankfurt

- Turkey
  - Exchange: Borsa İstanbul (stocks, gold, derivatives)
  - OTC market: Grand Bazaar Effective Market, Gold Market

- Consider the stock of Apple Inc. (AAPL on NASDAQ).
- Stock price data is publicly available. Just Google or use Google Finance, Yahoo Finance, etc.
- Stock prices fluctuate a lot!
- Prices from May 2, 2012 to May 2, 2017:

Financial engineering: modeling

- Prices are updated every milisecond at NASDAQ based on the investors willing to buy or sell their stocks. The exact mechanics of price formation are quite complicated.
- The numbers of buyers and sellers are affected by many economic, political, social, technological factors, which, in turn, affect the prices as well.
- Instead of explaining all these factors, as financial engineers, we take the data as is and try to find a mathematical model that is consistent with data.

Financial engineering: modeling

- Let us consider the prices during May 2, 2017 (yesterday):

- The data is taken at 3:11 pm at which the market was not closed yet.
- There is still uncertainty in the closing price of the day.
- Considering a full day, we may classify time as
  - Past: 9:30 am - 3:11 pm
  - Present: 3:11 pm
  - Future: 3:11 pm - 4:00 pm

Financial engineering: modeling

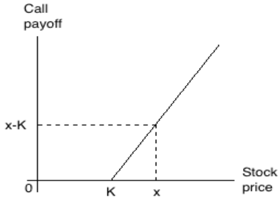
- Question: What is the probability that the closing price for the day is below 147 USD?
- To answer this question, we need a stochastic model for the prices of the stock. $\rightarrow$ Financial Engineering I
  - The model need not be consistent with economic theory but it has to be logically and mathematically sound.
  - Its parameters should be chosen so that the model is somehow consistent with data. $\rightarrow$ Calibration
  - You will learn more about general stochastic modeling in IE 325.
- Question: Can we use our model for other important purposes?
- Derivatives pricing...

Financial engineering: derivatives pricing

- An option is a derivative security written on an underlying security, typically a stock.
- A call option written on a stock is a contract which gives its holder the right but not the obligation to buy a stock at a fixed future date T for a predetermined price K.
- K is called the strike price of the call option.
- Let us denote by S_T the stock price at the future date T, which is uncertain at present.
- The holder of the call buys the stock at T if it is profitable, that is, if the strike price is cheaper than the market price: $K < S_T$. In this case, the payoff of the call holder is $S_T - K > 0$.
- If the strike price is more expensive, that is, if $S_T \leq K$, then the holder does nothing and gets zero payoff.
- Therefore, the payoff from the call option is given by

$$h(S_T) = \max\{S_T - K, 0\} = (S_T - K)^+.$$

- $x \mapsto h(x) = (x-K)^+$ is called the payoff function.



- h is a nonnegative and piecewise-linear function.
- Hence, the payoff of a call option is always nonnegative.
- If you were to write (issue) such a contract, would you sell it for free?
- How to determine the price of this call option?

Financial engineering: derivatives pricing

- Call payoff: $h(S_T) = (S_T - K)^+$
- Uncertainty in the call payoff is due to the uncertainty in the future stock price.
- Assuming a model for the stock prices and using a financial principle, we can determine the "reasonable" price of the call option. $\rightarrow$ Financial Engineering II
  - For simple models, you may be able to find an analytical formula for the option price.
  - In general, you need a more sophisticated procedure: run an algorithm, solve a system of equations, solve an optimization problem, etc. (More on optimization: IE 202, IE 303)
  - For more complex options, you can only calculate the price approximately via computational techniques.

A very simple market model

- Let us build the simplest possible financial model!
- One-period binomial model: observe the prices today, take an action, see what happens tomorrow.
- Two assets: a risky stock and a riskless bond
- Bond: Just like a bank account with known periodic return (interest rate) $r > -1$, e.g. $r = 0.01$
  - 1 lira today earns r liras as interest and becomes worth $1 + r$ liras tomorrow. (More in IE 342 Engineering Economic Analysis)
  - Suppose that the (unit) bond price is A_0 liras today so that the price will increase to $A_1 = A_0(1 + r)$ liras tomorrow.

bond price tree

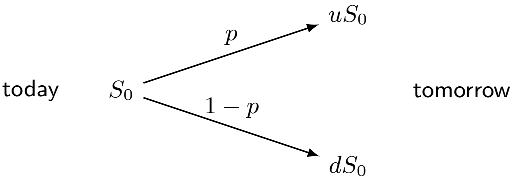
$$\text{today} \qquad A_0 \xrightarrow{\quad 1 \quad} (1+r)A_0 \qquad \text{tomorrow}$$

A very simple market model

- Stock: This is the asset that is subject to uncertainty.
  - 1 lira invested in stock today may be worth either u liras with probability p or else it will be worth d liras with probability $1-p$ tomorrow.
  - u is called the up factor and d is called the down factor.
  - We assume $0 < d < u$. (What's wrong with $d = u$?)
  - Just like flipping a coin in high school probability: up for heads, down for tails.
  - Suppose that the (unit) stock price is S_0 liras today. Let us denote by S_T the price tomorrow. So

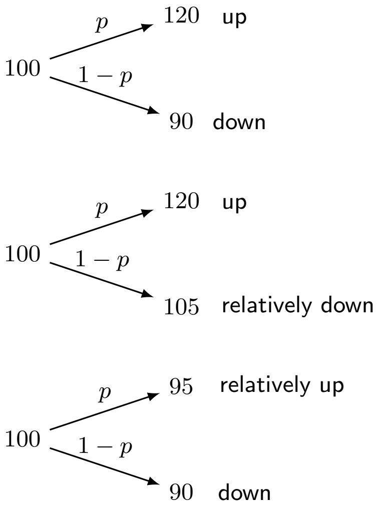
$$S_T = \begin{cases} uS_0 & \text{if the price goes up} \\ dS_0 & \text{if the price goes down} \end{cases}$$

stock price tree



A very simple market model

- The words *up* and *down* are used only in the relative sense.



A very simple market model

- Model parameters: u, d, r, p
- We assume $r > -1$, $0 < d < u$, $0 < p < 1$.
- The return on bond is r.
- The return on stock is either $u - 1$ or $d - 1$.
- Question: What should be the relationship among $u - 1, d - 1, r$ for the model to be *financially sound*?
- Possible cases:
  - Case 1: $d - 1 < u - 1 \leq r$. The bond always outperforms the stock.
  - Case 2: $r \leq d - 1 < u - 1$. The stock always outperforms the bond.
  - Case 3: $d - 1 < r < u - 1$. There is no clear winner. The performance of the bond is in between the good (up) and bad (down) performances of the stock.

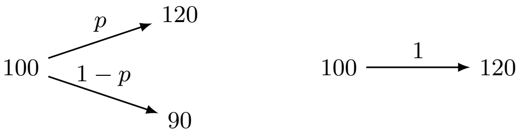
Kahoot time!

A very simple market model

- Let us focus on Case 1: $d - 1 < u - 1 \leq r$.
- The bond always outperforms the stock. As a rational investor, you would always invest money in the bond and never in the stock.
- An even better idea is to buy (long) bonds and sell (short) stocks. This way you can make free money.

A very simple market model

- Example: Consider the following trees for the stock and the bond.

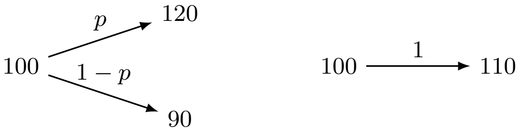


- Start with no money today. Borrow one stock to be given back tomorrow and immediately sell it stock for $S_0 = 100$. Deposit your 100 liras into the bank by buying $\frac{S_0}{A_0} = \frac{100}{100} = 1$ bond.
- Wait until tomorrow! The new value of your bond is $A_1 = 120$ liras. Withdraw this amount from the bank.
- If the stock price goes up to 120, then buy a stock with your 120 liras. Give the stock back to its original holder (lender). Your net gain is zero in this case.
- If the stock price goes down to 90, then buy a stock with 90 of your 120 liras. Give the stock back to its holder. Your net gain is $120 - 90 = 30$ liras in this case.

- Hence, when $d - 1 < u - 1 \leq r$, you never lose money but also make some money with probability $1 - p > 0$.
- Such an opportunity to make free money is called an arbitrage.
- A sound market should be free of arbitrage opportunities. This principle is called the principle of no-arbitrage (NA).
- Therefore, (NA) is violated in Case 1.
- Case 2: $r \leq d - 1 < u - 1$ is symmetric, you can make free money by longing the stock and shorting the bond. Hence, (NA) is violated in Case 2 as well.
- Case 3: $d - 1 < r < u - 1$ is the only case where (NA) is not violated.

A very simple market model

- Example: Consider the following trees for the stock and the bond.



- Start with no money today. Borrow one stock to be given back tomorrow and immediately sell it stock for $S_0 = 100$. Deposit your 100 liras into the bank by buying $\frac{S_0}{A_0} = \frac{100}{100} = 1$ bond.
- Wait until tomorrow! The new value of your bond is $A_1 = 110$ liras. Withdraw this amount from the bank.
- If the stock price goes up to 120, then borrow 10 liras from your friend and buy a stock all of your 120 liras. Give the stock back to the lender. You still owe 10 liras to your friend. Your net gain is -10 (loss) in this case.
- If the stock price goes down to 90, then buy a stock with 90 of your 110 liras. Give the stock back to its holder. Your net gain is $110 - 90 = 20$ liras in this case.
- Hence, you make money with probability $1 - p > 0$ and lose money with probability $p > 0$. This is not an arbitrage!

A very simple market model

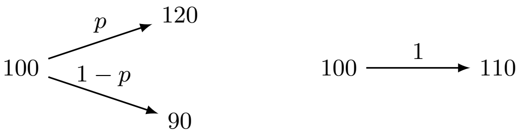
- Indeed, you can prove that an arbitrage cannot exist under Case 3.
- Hence,

$$\text{(NA) holds.} \quad \Leftrightarrow \quad d - 1 < r < u - 1.$$

- This result is called the fundamental theorem of asset pricing (FTAP). We will see why this is related to pricing assets (derivatives).

Pricing a call option

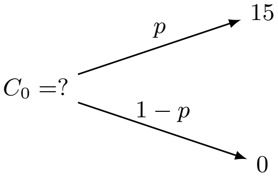
- Let us assume (NA) holds, that is, $d-1 < r < u-1$.
- So far, we have the stock and the bond in the market.
- Let us introduce a third asset to this market: a call option written on the stock with strike price K.
- Example: Consider the following trees for the stock and the bond.



- Suppose that the strike price of the call is $K = 105$.
- If the stock price goes up to 120 tomorrow, then the holder of the call option exercises the option and buys a stock for 105 liras and makes $120 - 105 = 15$ liras.
- If the stock price goes down to 90 tomorrow, then it is not profitable for the option holder to buy a stock for 105. Hence, the holder does not exercise the option and gets no payoff.

Pricing a call option

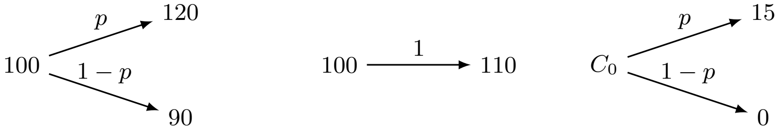
- The payoff structure of the call can be summarized by the following tree:



- Let C_0 be the price of the call today.
- What should C_0 be under (NA)?
- If C_0 is chosen to be too low or too high, then an arbitrage may arise.
- To determine C_0, we will first replicate the call using stocks and bonds. The price of the replicating strategy will determine C_0.

Pricing a call option

- Question of replication: Can we find a portfolio that consists of stocks and bonds and gives the same payoff as a call option?
- Let x be the number of stocks and y the number of bonds in such a portfolio. We need:



x stocks $+$ y bonds $=$ 1 call option

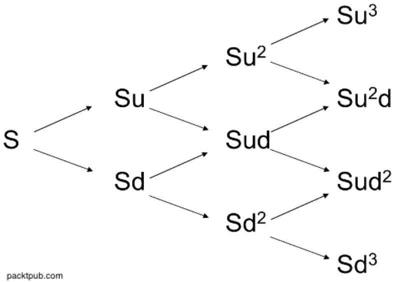
- Solve for (x, y) the system

$$
\begin{aligned}
120x + 110y &= 15 \\
90x + 110y &= 0.
\end{aligned}
$$

- There is a unique solution: $x = \frac{1}{2}, y = -\frac{9}{22}$. This is the replicating strategy.
- Hence, holding a call option is "equivalent" to holding $\frac{1}{2}$ stocks and owing $\frac{9}{22}$ bonds to somebody else.
- Hence, the price of the call should be given by $C_0 = 100x + 100y = \frac{100}{11}$. Done!

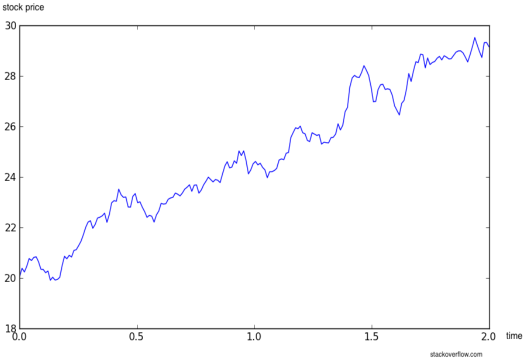
More general models

- Hence, assuming a one-period binomial model for the stock, we have priced a call option!
- The choice of the model is simplistic. As we have seen earlier, prices fluctuate more than once a day.
- More realistic models should be dynamic in the sense that multiple time steps are considered and the stock price fluctuates at every time step.
- Discrete time (multiple but finitely many periods): N-period binomial model



More general models

- Continuous time (real time): Black-Scholes model



- For details and more: IE 440 Introduction to Financial Engineering ☺

Thank you! Time for kahoot again!