IE 102
Spring 2018

Operations Research

and

Mathematical Programming

Part 1

What is Operations Research (OR)?

OR is a **decision making tool** based on mathematical modeling and analysis to aid in determining **how best to design and operate a system** usually under the conditions requiring **allocation of scarce resources**.

Brief History of OR

- The main origin of Operations Research was during the **Second World War** in Britain.
- At the time of World War II, the military management in England invited a team of scientists to study the **strategic** and **tactical problems** related to air and land defense of the country.

Brief History of OR

- At that time the resources available with England was very limited and the objective was to win the war with available scarce resources.
- The resources such as food, medicines, ammunition, manpower etc., were required to manage war and for the use of the population of the country.

Brief History of OR

- It was necessary to decide upon the most effective utilization of the available resources to achieve the objective.
- It was also necessary to utilize the military resources cautiously.

Brief History of OR

- British military leaders asked an interdisciplinary group of scientists, engineers, psychologists, and sociologists to analyze some problems arising in military operations and find the optimum allocation of limited resources.
- These specialists had a brain storming session and came out with a method of solving the problem, which they coined the name **"Linear Programming"**.

Brief History of OR

- As the name indicates,

  Operations is used to refer to the problems of military,

  Research is use for inventing new method.

- As this method of solving the problem was invented during the war period, the subject is given the name **‘OPERATIONS RESEARCH’** and abbreviated as **‘O.R.’**

OR Delivers Significant Value

Today, organizations and the world in which they operate continue to become more complex. Organizations worldwide in business, the military, health care, and the public sector are realizing powerful benefits from O.R., including:

Business insight: Providing quantitative and business insight into complex problems.

OR Delivers Significant Value

Business performance: Improving business performance by embedding model-driven intelligence into an organization's information systems to improve decision making.

Cost reduction: Finding new opportunities to decrease cost or investment.

Forecasting: Providing a better basis for more accurate forecasting and planning

OR Delivers Significant Value

Improved scheduling: Efficiently scheduling staff, equipment, events, and more.

Pricing: Dynamically pricing products and services.

Productivity: Helping organizations find ways to make processes and people more productive.

Profits: Increasing revenue or return on investment; increasing market share.

OR Delivers Significant Value

Quality: Improving quality as well as quantifying and balancing qualitative considerations.

Resources: Gaining greater utilization from limited equipment, facilities, money, and personnel.

Risk: Measuring risk quantitatively and uncovering factors critical to managing and reducing risk.

Throughput: Increasing speed or throughput and decreasing delays.

OR Modeling

- In spite of the fundamental differences among several settings, many problems share similar characteristics.

  Ex:
  - A physical network vs. a computer network.
  - Parallel computing vs. queuing systems in a bank (scheduling, queuing).
  - Air traffic control vs. local area network (routing).

OR Modeling

- It is the OR modeler's responsibility to observe and understand the inner workings of a system (usually in conjunction with the experts in the setting in which the problem arises).
- **Model:** A structure which has been built purposely to exhibit features and characteristics of some other object.

  Ex: Road maps, model airplanes, simulation model, mathematical model, etc.

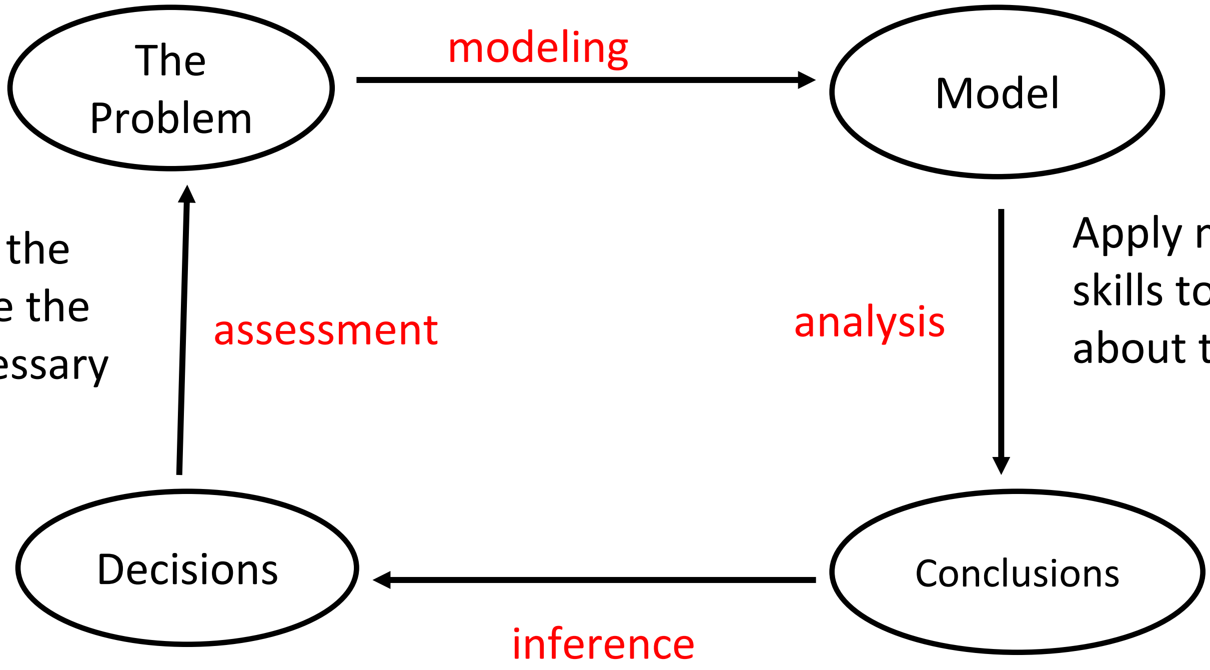
The **essential feature of a mathematical model** in OR is that it involves a set of mathematical relationships (such as equations, inequalities, logical dependencies, etc.) which correspond to the characterization of the relationships in the real world.

Why do we need mathematical modeling?

- To have a greater understanding of the object being modeled.
- To analyze it mathematically to help suggest courses that are not apparent otherwise.
- To simplify the inner workings of a complex system and to use the universally understandable language of algebra and math.

The Modeling Process

Define variables and mathematical relationships to represent the system behavior



Validation of the results, revise the model if necessary

Apply mathematical skills to make conclusions about the model

Translate the mathematical conclusions into the specific setting of the problem

The Diet Problem

The Diet Problem

	Food	Energy (kcal)	Protein (g)	Calcium (mg)	Price/serving (cents)
1	**Chicken**	205	32	12	24
2	**Egg**	160	13	54	13
3	**Whole milk**	160	8	285	9
4	**Cherry Pie**	420	4	22	20
5	**Stuffed Turkey**	260	14	80	19

The Diet Problem

Daily Nutrition Requirements:

- 2000 kcal energy
- 55 g protein
- 800 mg calcium

Balanced Diet Requirements:

\# of servings of:

- Chicken: ≥ 1 and ≤ 3
- Eggs: ≥ 1 and ≤ 2
- Milk: ≥ 2 and ≤ 4
- Cherry pie: ≥ 0 and ≤ 2
- Stuffed Turkey: ≥ 0 and ≤ 2

Design a diet with minimum total cost to ensure that daily nutrition and balance requirements are met.

The Diet Problem

First, we need to determine the # of servings of each type of food.

Let $\mathbf{X_i}$ denote the (unknown) # of servings of food type i, i=1,2,3,4,5. These are the “decision variables”

The Diet Problem

	Food	Energy (kcal)	Protein (g)	Calcium (mg)	Price/serving (cents)
1	**Chicken**	205	32	12	24
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Objective: minimize total cost

Total Cost= $24\,X_1 + 13\,X_2 + 9\,X_3 + 20\,X_4 + 19\,X_5$

The Diet Problem

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1	**Chicken**	205	32	12	24
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Daily Nutrition Requirements:

Energy: $205\ X_1 + 160\ X_2 + 160\ X_3 + 420\ X_4 + 260\ X_5$

needs to be at least 2000 kcal

The Diet Problem

	Food	Energy (kcal)	Protein (g)	Calcium (mg)	Price/serving (cents)
1	**Chicken**	205	32	12	24
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Daily Nutrition Requirements:

Protein: $32 X_1 + 13X_2 + 8 X_3 + 4 X_4 + 14 X_5$

needs to be at least 55 g

The Diet Problem

	Food	Energy (kcal)	Protein (g)	Calcium (mg)	Price/serving (cents)
1	**Chicken**	205	32	12	24
2	**Egg**	160	13	54	13
3	**Whole milk**	160	8	285	9
4	**Cherry Pie**	420	4	22	20
5	**Stuffed Turkey**	260	14	80	19

Daily Nutrition Requirements:

Calcium: $12\ X_1 + 54X_2 + 285\ X_3 + 22\ X_4 + 80\ X_5$

needs to be at least 800 mg

The Diet Problem

Balanced Diet Requirements:

\# of servings of:

- Chicken: ≥ 1 and ≤3
- Eggs: ≥ 1 and ≤2
- Milk: ≥ 2 and ≤4
- Cherry pie: ≥ 0 and ≤2
- Stuffed Turkey: ≥ 0 and ≤2

$X_1 \geq 1$, $X_1 \leq 3$,
$X_2 \geq 1$, $X_2 \leq 2$,
$X_3 \geq 2$, $X_3 \leq 4$,
$X_4 \geq 0$, $X_4 \leq 2$,
$X_5 \geq 0$, $X_5 \leq 2$.

The Diet Problem

minimize (min) $24\ X_1 + 13\ X_2 + 9\ X_3 + 20\ X_4 + 19\ X_5$
subject to (s.t.)

$$205\ X_1 + 160\ X_2 + 160\ X_3 + 420\ X_4 + 260\ X_5 \geq 2000$$

$$32\ X_1 + 13\ X_2 + 8\ X_3 + 4\ X_4 + 14\ X_5 \geq 55$$

$$12\ X_1 + 54\ X_2 + 285\ X_3 + 22\ X_4 + 80\ X_5 \geq 800$$

$X_1 \geq 1, \quad X_1 \leq 3,$
$X_2 \geq 1, \quad X_2 \leq 2,$
$X_3 \geq 2, \quad X_3 \leq 4,$
$X_4 \geq 0, \quad X_4 \leq 2,$
$X_5 \geq 0, \quad X_5 \leq 2.$

The Diet Problem

minimize (min) $24\ X_1 + 13\ X_2 + 9\ X_3 + 20\ X_4 + 19\ X_5$

subject to (s.t.)

constraints

$$205\ X_1 + 160\ X_2 + 160\ X_3 + 420\ X_4 + 260\ X_5 \geq 2000$$

$$32\ X_1 + 13\ X_2 + 8\ X_3 + 4\ X_4 + 14\ X_5 \geq 55$$

$$12\ X_1 + 54\ X_2 + 285\ X_3 + 22\ X_4 + 80\ X_5 \geq 800$$

$X_1 \geq 1,\quad X_1 \leq 3,$

$X_2 \geq 1,\quad X_2 \leq 2,$

$X_3 \geq 2,\quad X_3 \leq 4,$

$X_4 \geq 0,\quad X_4 \leq 2,$

$X_5 \geq 0,\quad X_5 \leq 2.$

High Step Sports Shoe

- Maximize profits
- Two styles of sports shoe: *Airheads* and *Groundeds*
- Production Plan – How much to produce?

High Step Sports Shoe

- Maximize profits
- Two styles of sports shoe: *Airheads* and *Groundeds*
- Production Plan – How much to produce?
- Facts:
  - $10 profit on each pair of *Airheads*
  - $8 profit on each pair of *Groundeds*
  - Limited source: # of workers
  - Limited source: Raw material availability
  - Maximum demand for a product ......

General Definitions:

- Decision variable: Represents a quantity that a decision-maker can change.
- Objective Function: The equation that represents a performance measure; is either maximizing or minimizing type
- Constraints: Are limitations created by scarce resources

High Step Sports Shoe

- Plan? **Decision variables**: Quantities that can change and decision maker(s) is able to control
  - In this case: # of pairs to produce

- **Performance measure**: Profit (revenue – cost)
  - **Objective function**: Maximize profit - optimization

- **Constraints**: Resources limit # of pairs to produce
  - In this case: Machine time, worker time

Also:

- The assembly plant works a 40 hour week.
- Each hour, each cutting machine can do 50 minutes of work.
- There are 6 machines that cut the materials.
- There are 850 workers who assemble the shoes.

- Each pair of
  - Airheads requires 3 minutes of cutting time
  - Groundeds require 2 minutes.
- It takes a single worker
  - 7 hours to assemble a pair of Airheads and
  - 8 hours to assemble a pair of Groundeds

High Step Sports Shoe

- 40-hour week – How many minutes of work can a machine do?

  50 mins/hr*40 hrs/week = 2 000 mins/week

  6 machines – **12 000 mins/week**

- 40-hour week – How many hours of work can one worker do?

  40 hrs/week

  850 workers – 850*40 = **34 000 hrs/week**

High Step Sports Shoe – Formulating, Modeling

Representing with symbols and expressions:

Decision variables:

A: # of pairs of *Airheads* to produce per week

G: # of pairs of *Groundeds* to produce per week

High Step Sports Shoe – Formulating, Modeling

Representing with symbols and expressions:

Decision variables:

A: # of pairs of *Airheads* to produce per week

G: # of pairs of *Groundeds* to produce per week

- Objective function: 10A + 8.5G

High Step Sports Shoe – Formulating, Modeling

Representing with symbols and expressions:

Decision variables:

A: # of pairs of *Airheads* to produce per week

G: # of pairs of *Groundeds* to produce per week

- Objective function: $10A + 8.5G$
- M/C Constraint: $3A + 2G \leq 12\,000$
- Worker constraint: $7A + 8G \leq 34\,000$
- Non-negativity constraints: $A,G \geq 0$
  - A, G need to be integer – never mind

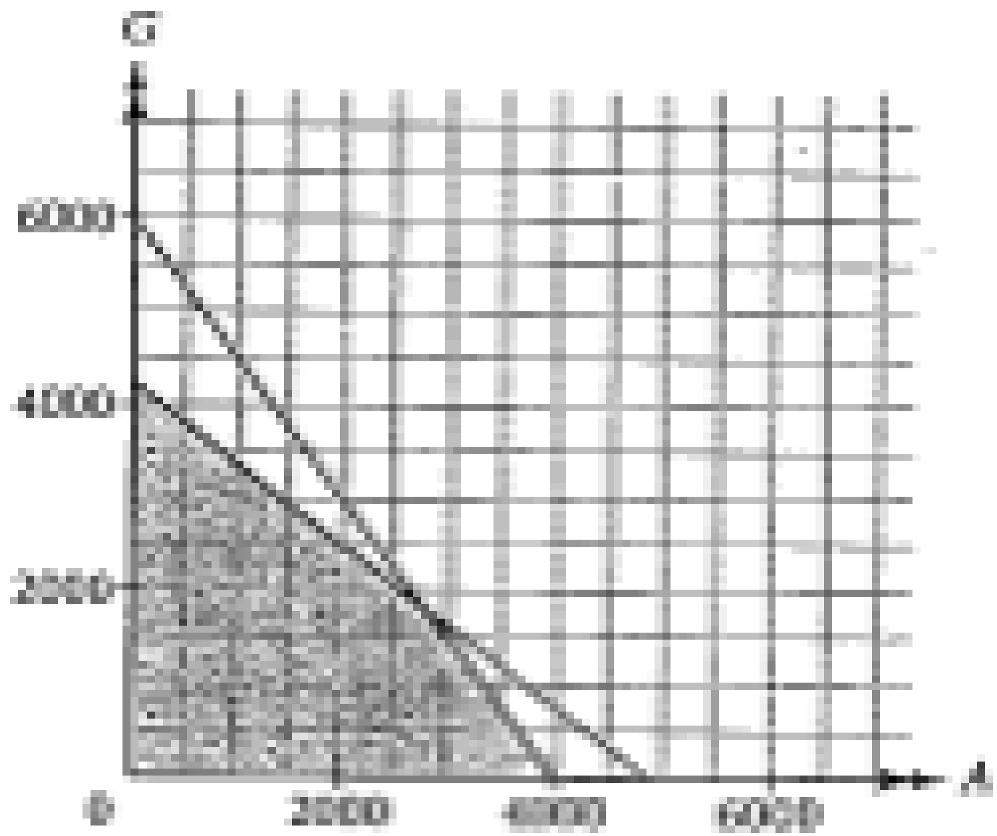
Linear Programming
Mathematical Programming

- Maximize (or minimize) a linear objective function, subject to linear constraints

- In general: Mathematical programming: Objective function can be a non-linear or linear function, with non-linear and/or linear constraints.

High Step Sports Shoe - Solution

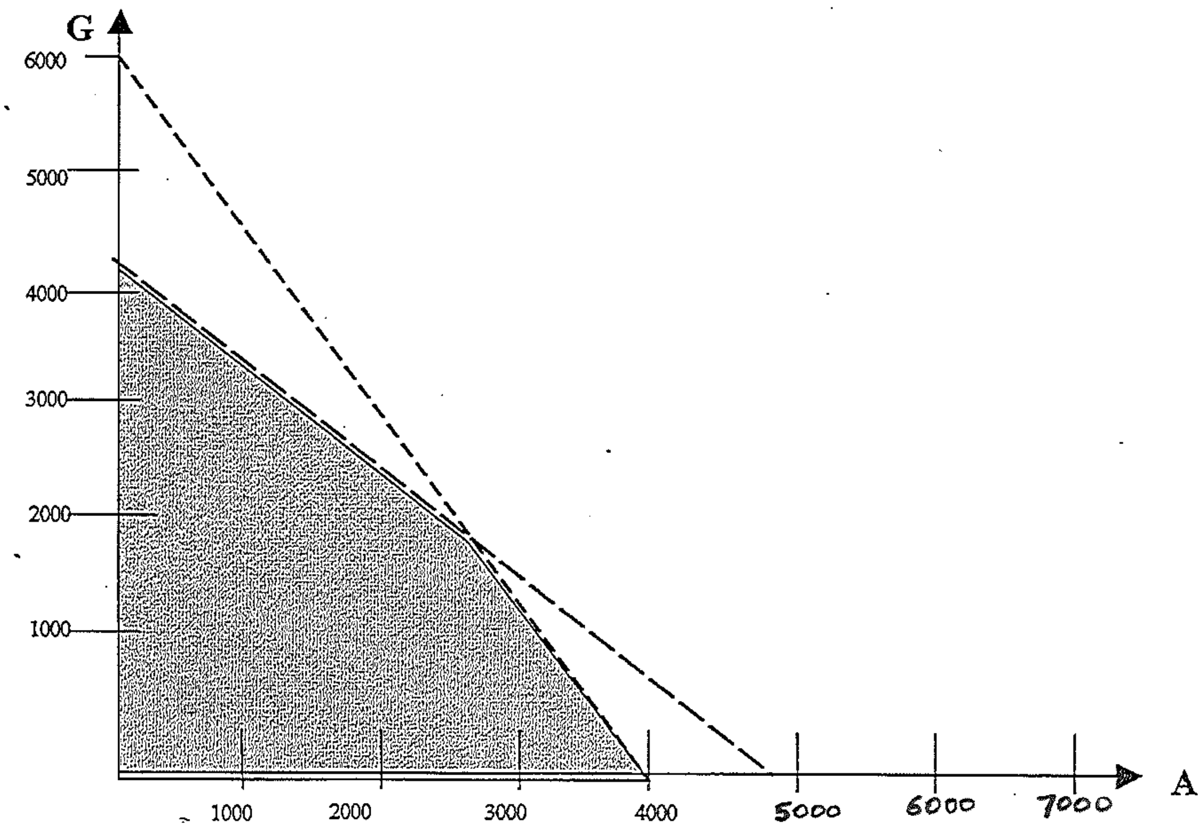
- Geometry:
- Draw the **feasible region** – the region identified by the inequalities



High Step Sports Shoe - Solution

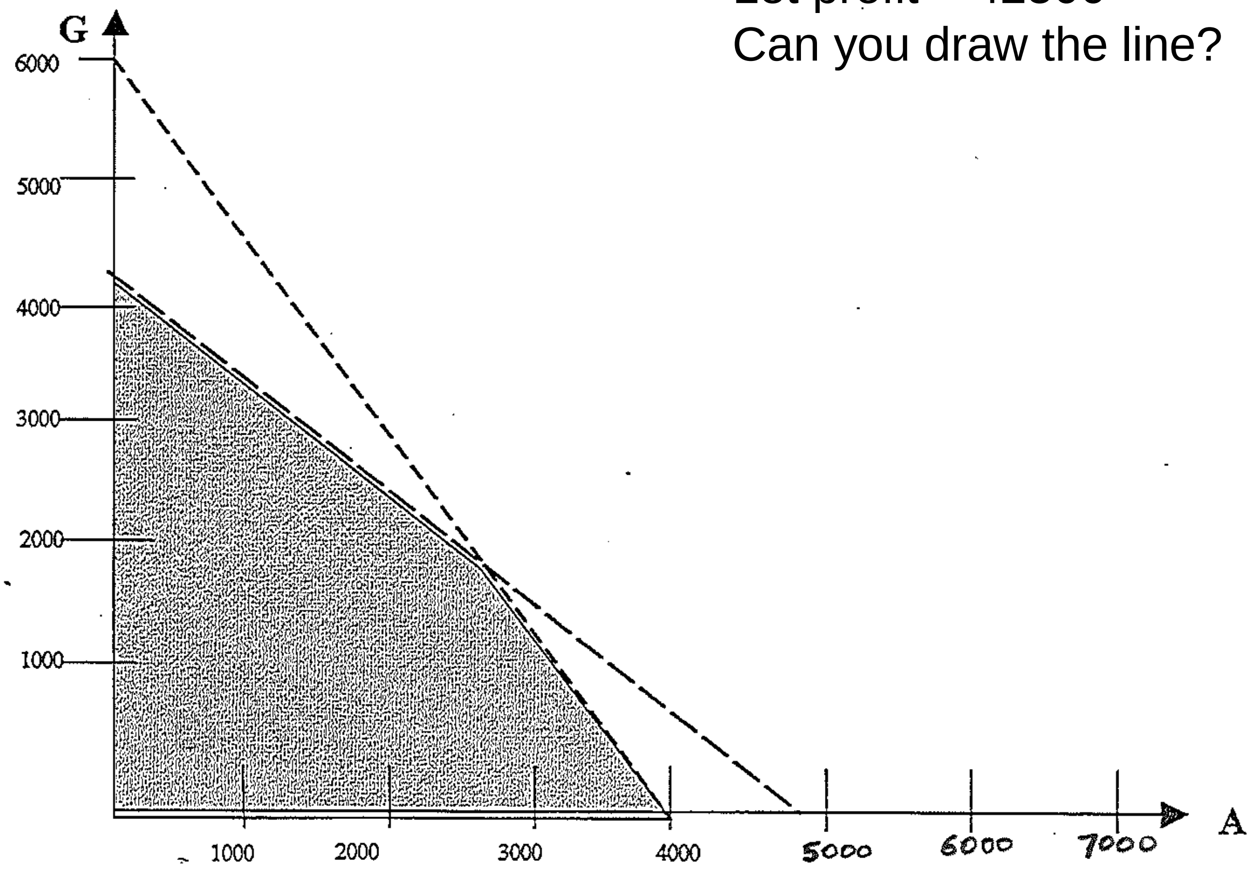
- Any point in the feasible region is a solution
- One can compute the profit of each solution
- Optimization: We want the solution that will yield the maximum profit

High Step Sports Shoe - Solution



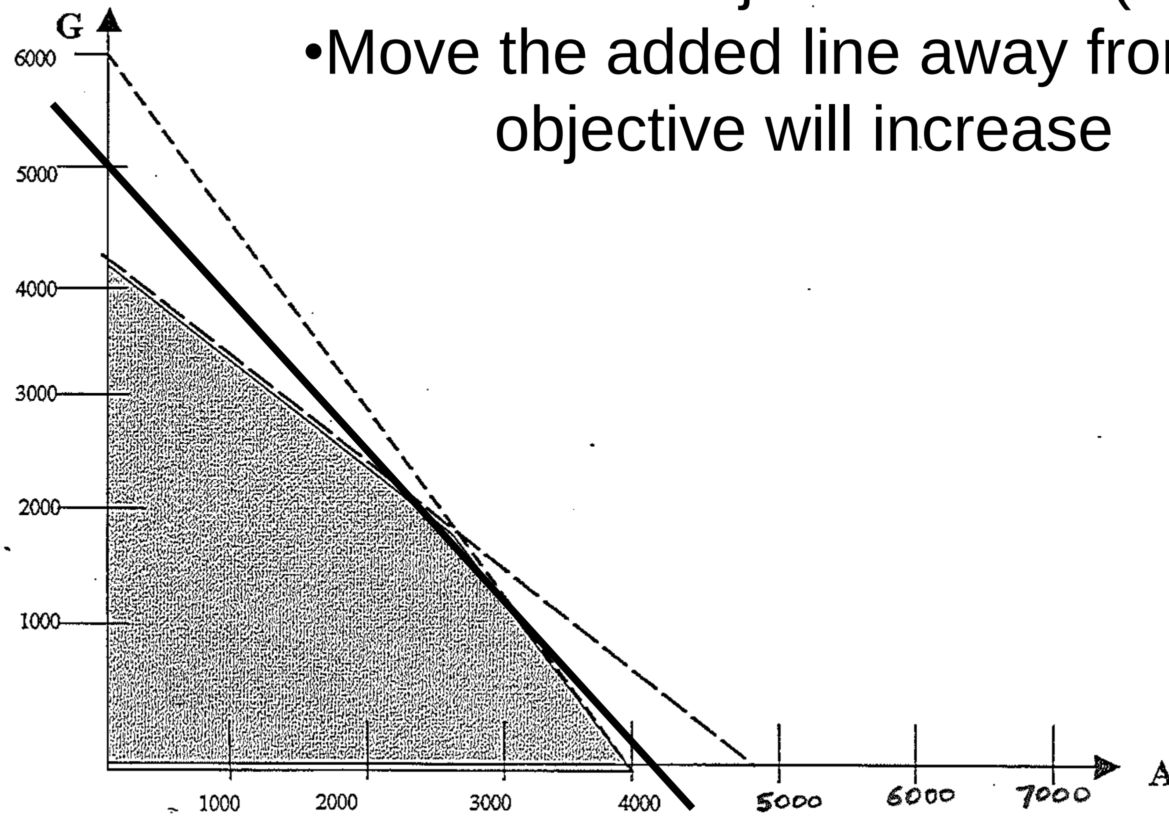
High Step Sports Shoe - Solution

profit = 10A + 8.5G
Let profit = 42500
Can you draw the line?



High Step Sports Shoe - Solution

- Line contains all solutions with the same objective value (iso-profit line)
- Move the added line away from (0.0) objective will increase



Two-dimensional Problem

- Geometric Intuition:
  - Observe the specific objective function
  - Place it over the feasible region, and move the function so to increase the profit (north east direction)
  - Eventually, will be out of the feasible region – just stop in the last possible point

High Step Sports Shoe - Solution

- One property – Corner principle – Extreme point: One optimal solution of an LP always occurs at a corner point (**extreme point**) of the feasible region

KAHOOT