Spring 2018
IE 102

Operations Research

and

Mathematical Programming

Part 2

Graphical Solution of 2-variable LP Problems

Consider an example

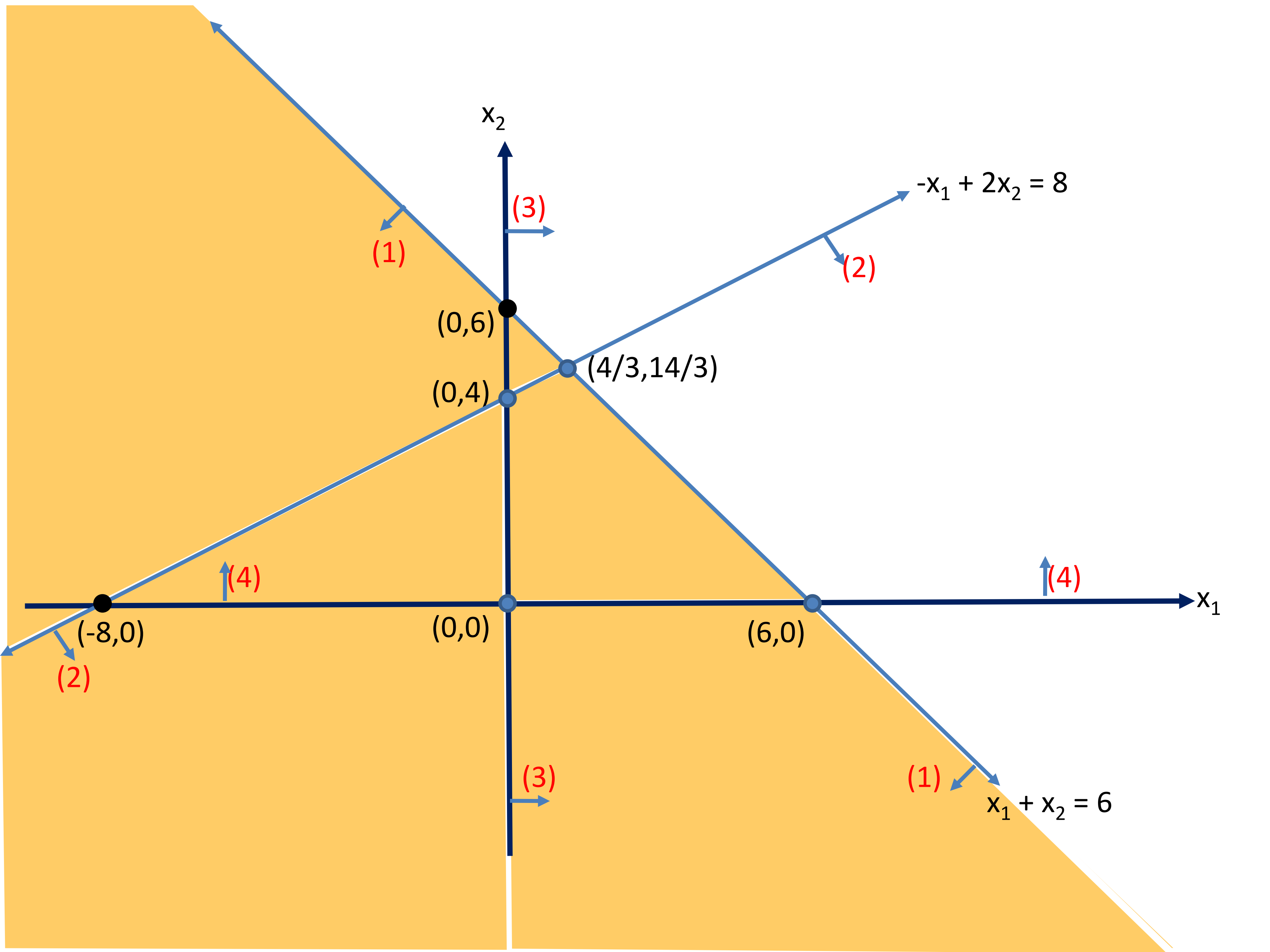
$$\max \quad x_1 + 3x_2$$

s.t.

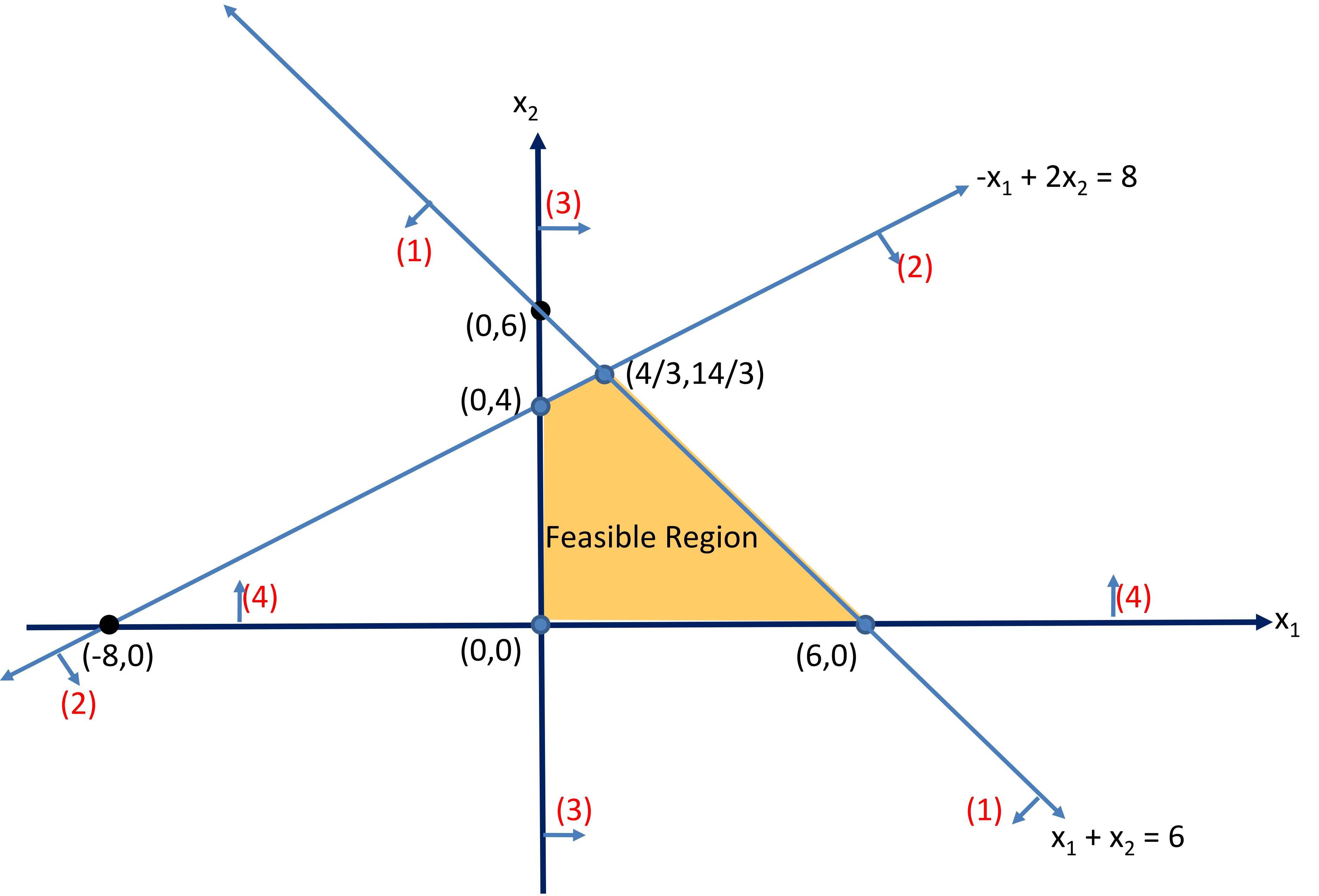
$$x_1 + x_2 \leq 6 \qquad (1)$$

$$-x_1 + 2x_2 \leq 8 \qquad (2)$$

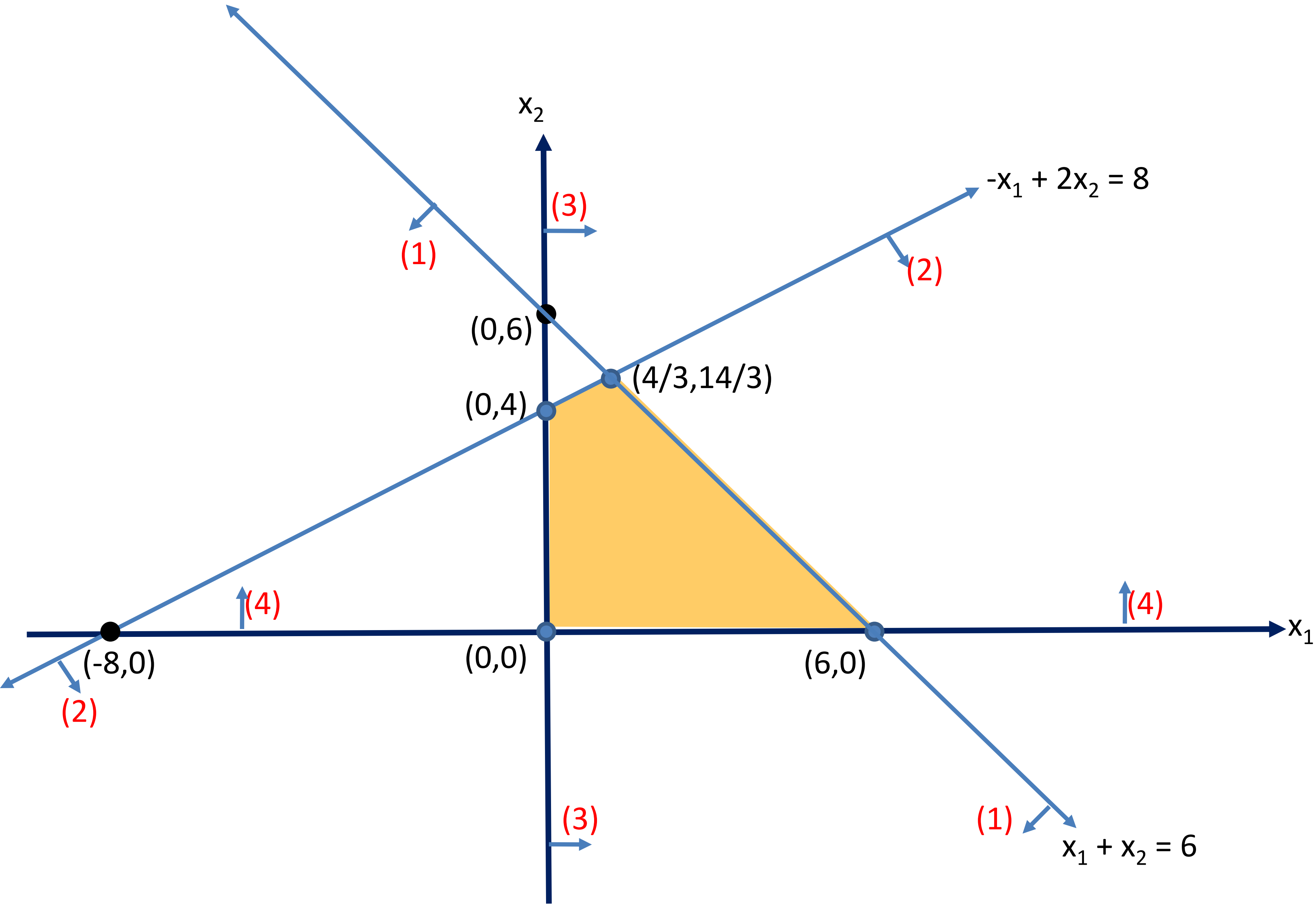
$$x_1, x_2 \geq 0 \qquad (3), (4)$$



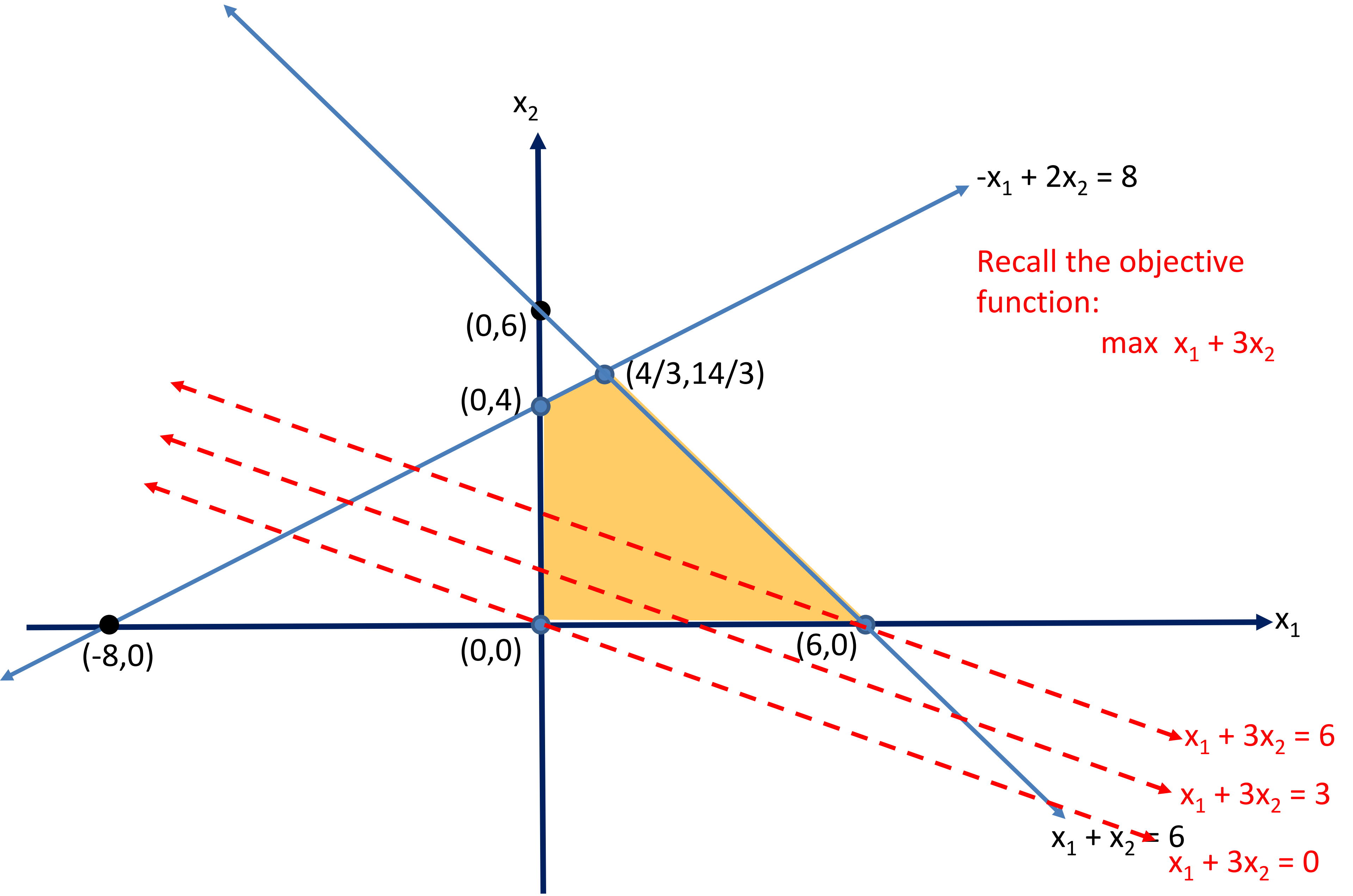
x_2
(3)
$-x_1 + 2x_2 = 8$
(1)
(2)
(0,6)
(4/3,14/3)
(0,4)
(4)
(4)
x_1
(-8,0)
(0,0)
(6,0)
(2)
(3)
(1)
$x_1 + x_2 = 6$



x_2
x_1
$-x_1 + 2x_2 = 8$
$x_1 + x_2 = 6$
(1)
(2)
(3)
(4)
(0,6)
(4/3,14/3)
(0,4)
Feasible Region
(-8,0)
(0,0)
(6,0)
(2)
(3)
(4)
(1)



x_2
$-x_1 + 2x_2 = 8$
(3)
(1)
(2)
(0,6)
(4/3,14/3)
(0,4)
(4)
(4)
x_1
(-8,0)
(0,0)
(6,0)
(2)
(3)
(1)
$x_1 + x_2 = 6$



x_2
x_1
$-x_1 + 2x_2 = 8$
Recall the objective function:
max $x_1 + 3x_2$
(0,6)
(4/3,14/3)
(0,4)
(-8,0)
(0,0)
(6,0)
$x_1 + 3x_2 = 6$
$x_1 + 3x_2 = 3$
$x_1 + x_2 = 6$
$x_1 + 3x_2 = 0$

Graphical Solution of 2-variable LP Problems

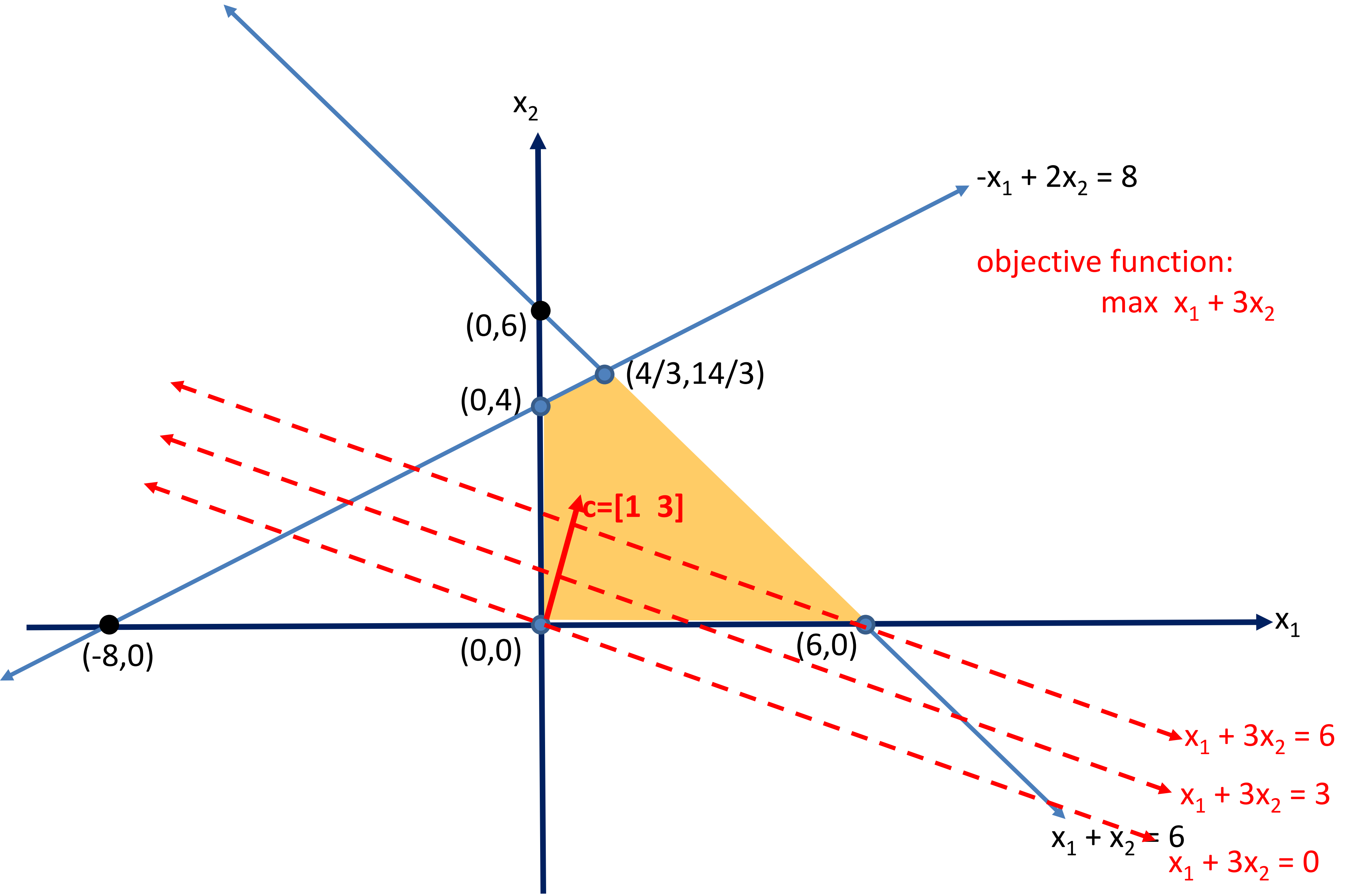
A line on which all points have the same z-value ($z = x_1 + 3x_2$) is called :

✦**Isoprofit line** for maximization problems,

✦**Isocost line** for minimization problems.

Graphical Solution of 2-variable LP Problems

Consider the coefficients of x_1 and x_2 in the objective function and let $\mathbf{c}=[1\ 3]$.



x_2
$-x_1 + 2x_2 = 8$
objective function:
max $x_1 + 3x_2$
(0,6)
(4/3,14/3)
(0,4)
c=[1 3]
x_1
(-8,0)
(0,0)
(6,0)
$x_1 + 3x_2 = 6$
$x_1 + 3x_2 = 3$
$x_1 + x_2 = 6$
$x_1 + 3x_2 = 0$

Graphical Solution of 2-variable LP Problems

Note that as move the isoprofit lines in the direction of **c**, the total profit increases for this max problem!

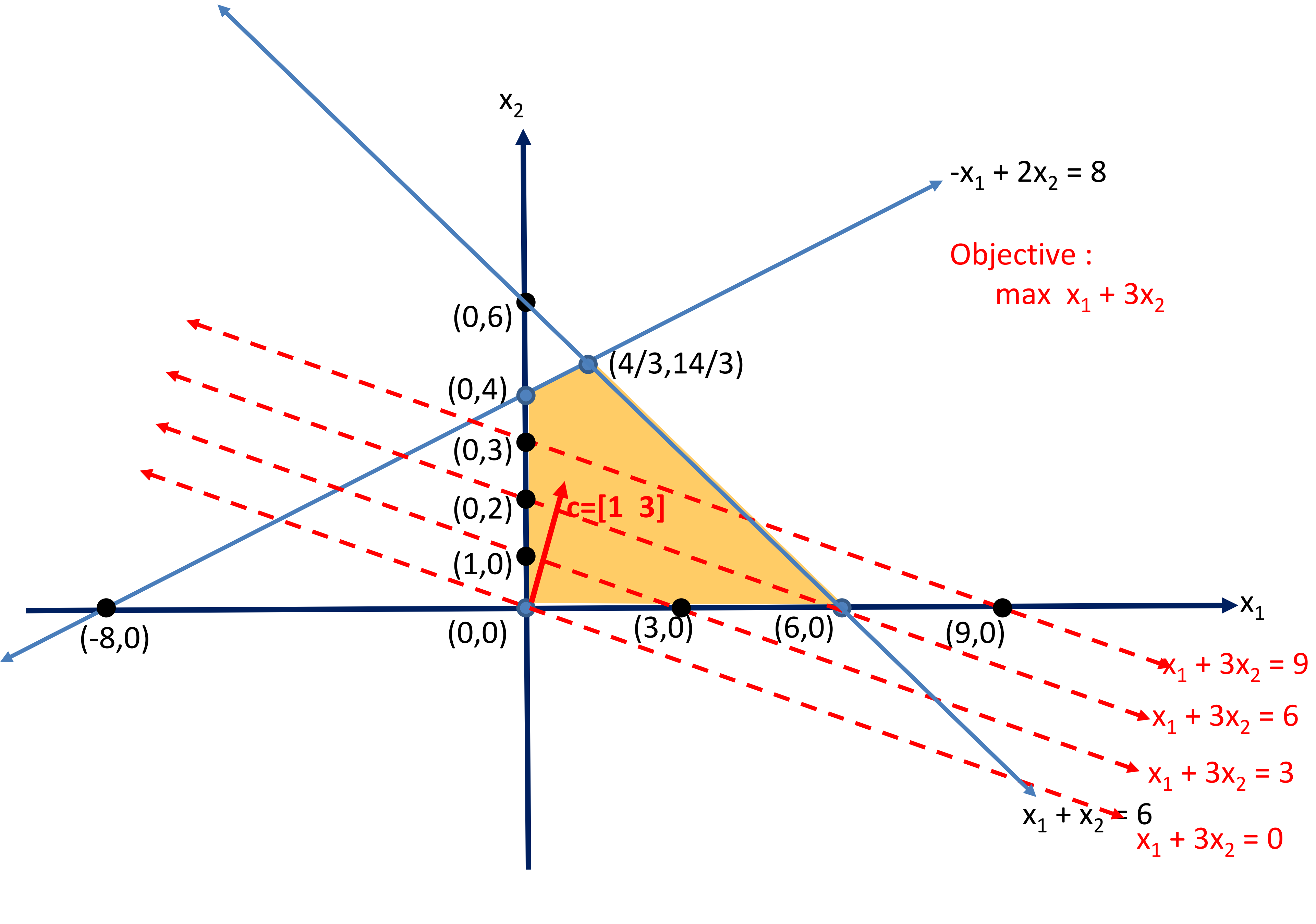
Graphical Solution of 2-variable LP Problems

For a **maximization problem**, the vector **c**, given by the coefficients of x_1 and x_2 in the objective function, is called the **improving direction**.

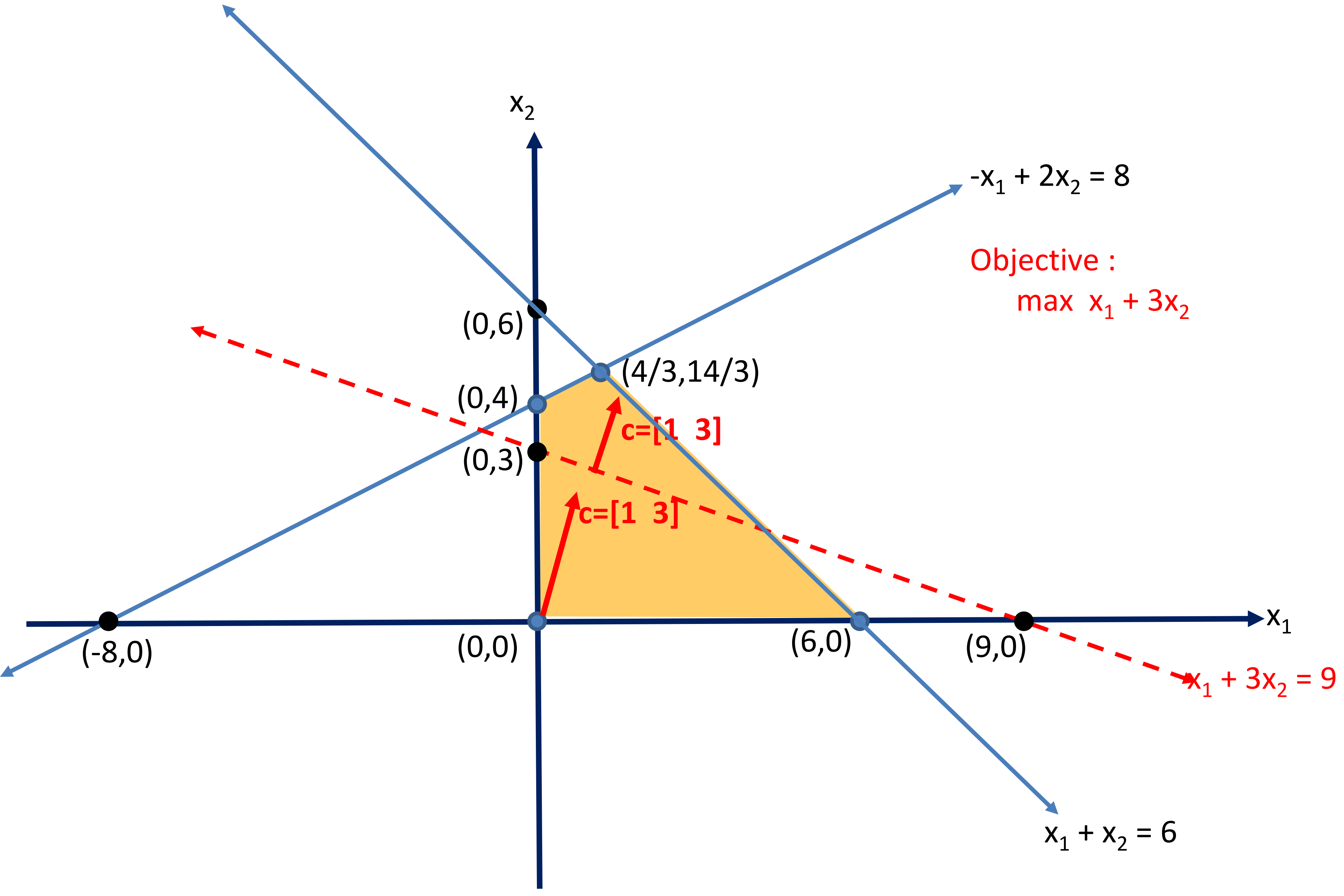
On the other hand, **-c is the improving direction for minimization problems**!

Graphical Solution of 2-variable LP Problems

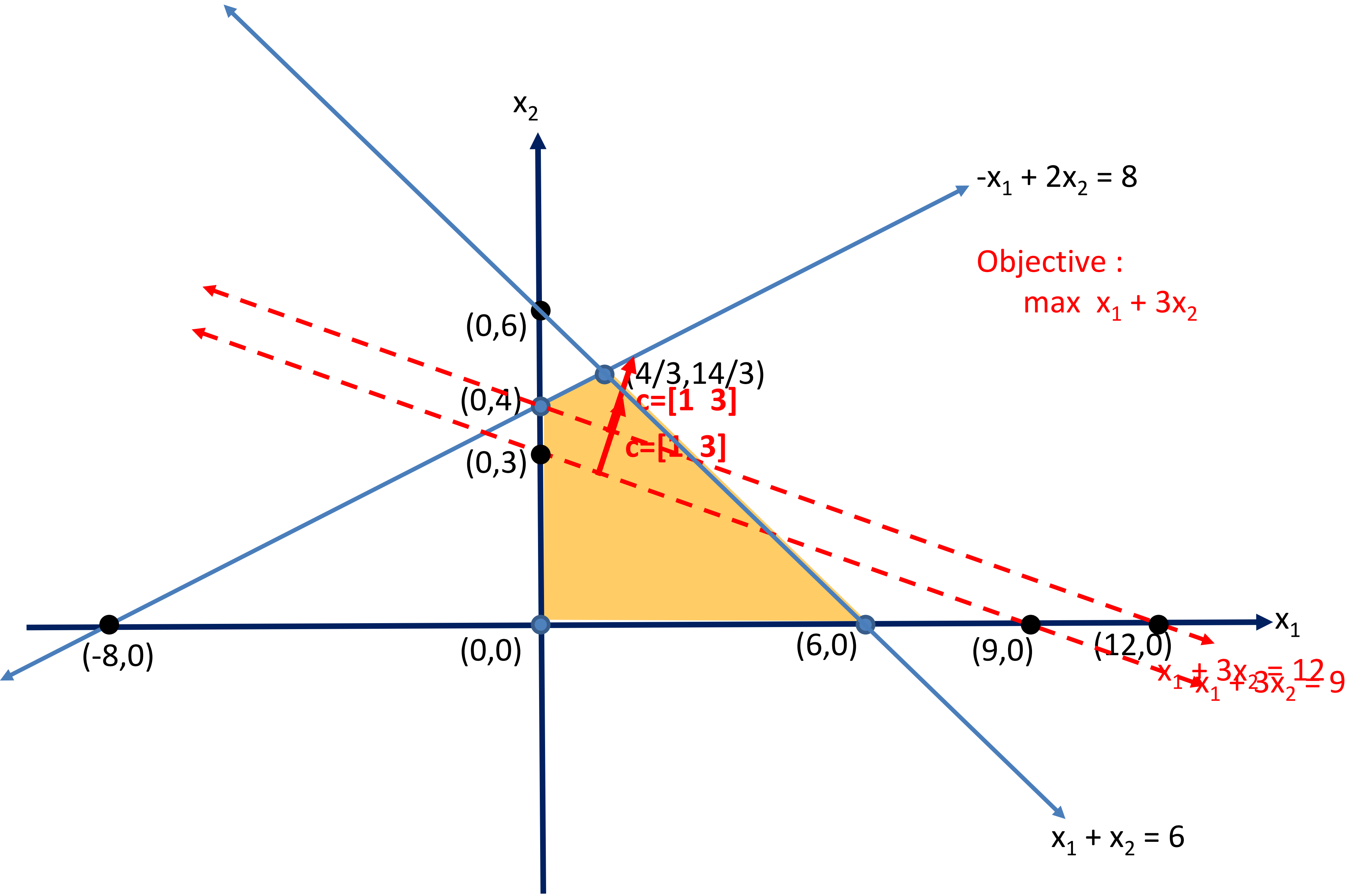
To solve an LP problem in the two variables, we should try to push isoprofit (isocost) lines in the improving direction as much as possible while still staying in the feasible region.



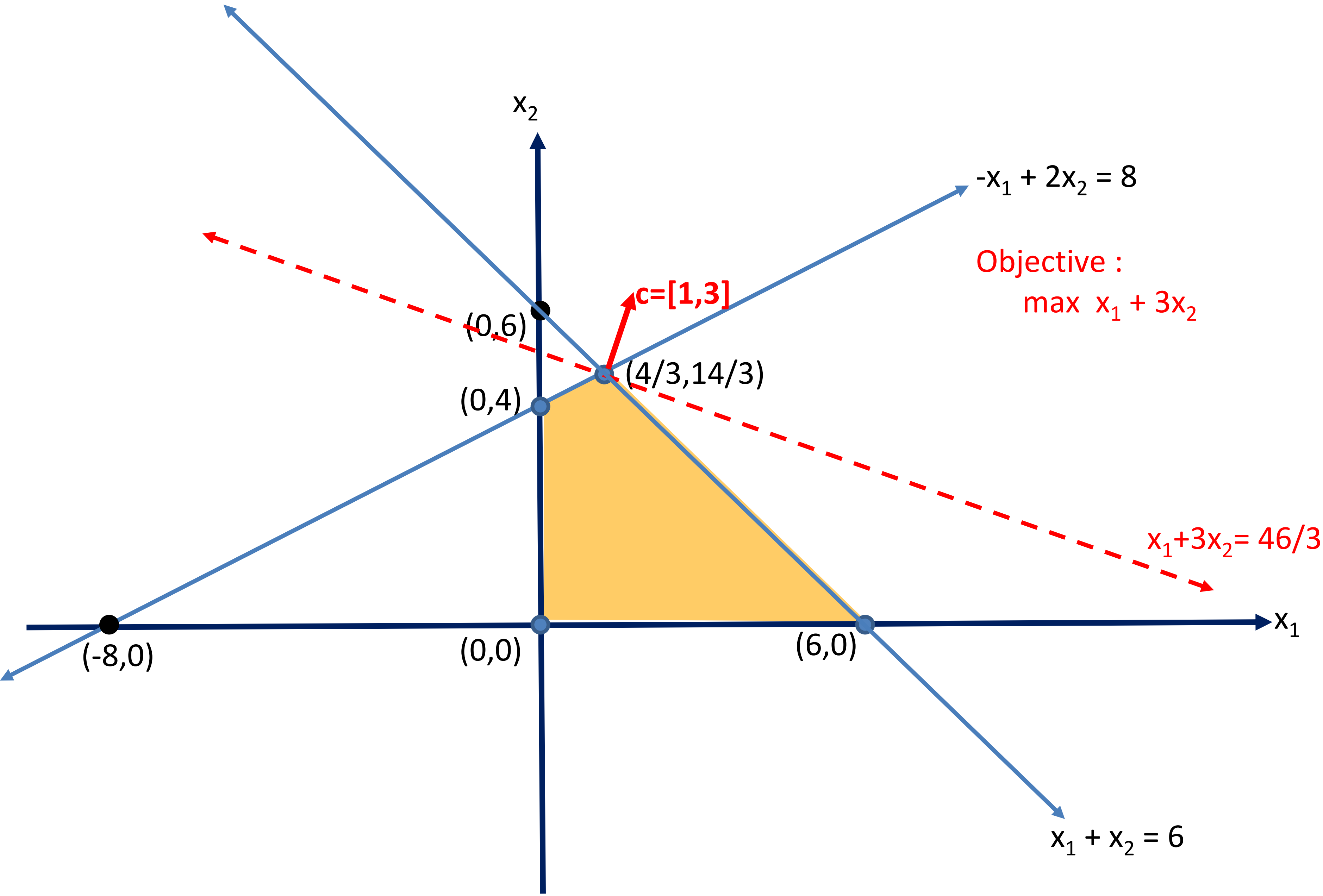
x_2
$-x_1 + 2x_2 = 8$
Objective :
max $x_1 + 3x_2$
(0,6)
(4/3,14/3)
(0,4)
(0,3)
(0,2)
c=[1 3]
(1,0)
x_1
(-8,0)
(0,0)
(3,0)
(6,0)
(9,0)
$x_1 + 3x_2 = 9$
$x_1 + 3x_2 = 6$
$x_1 + 3x_2 = 3$
$x_1 + x_2 = 6$
$x_1 + 3x_2 = 0$



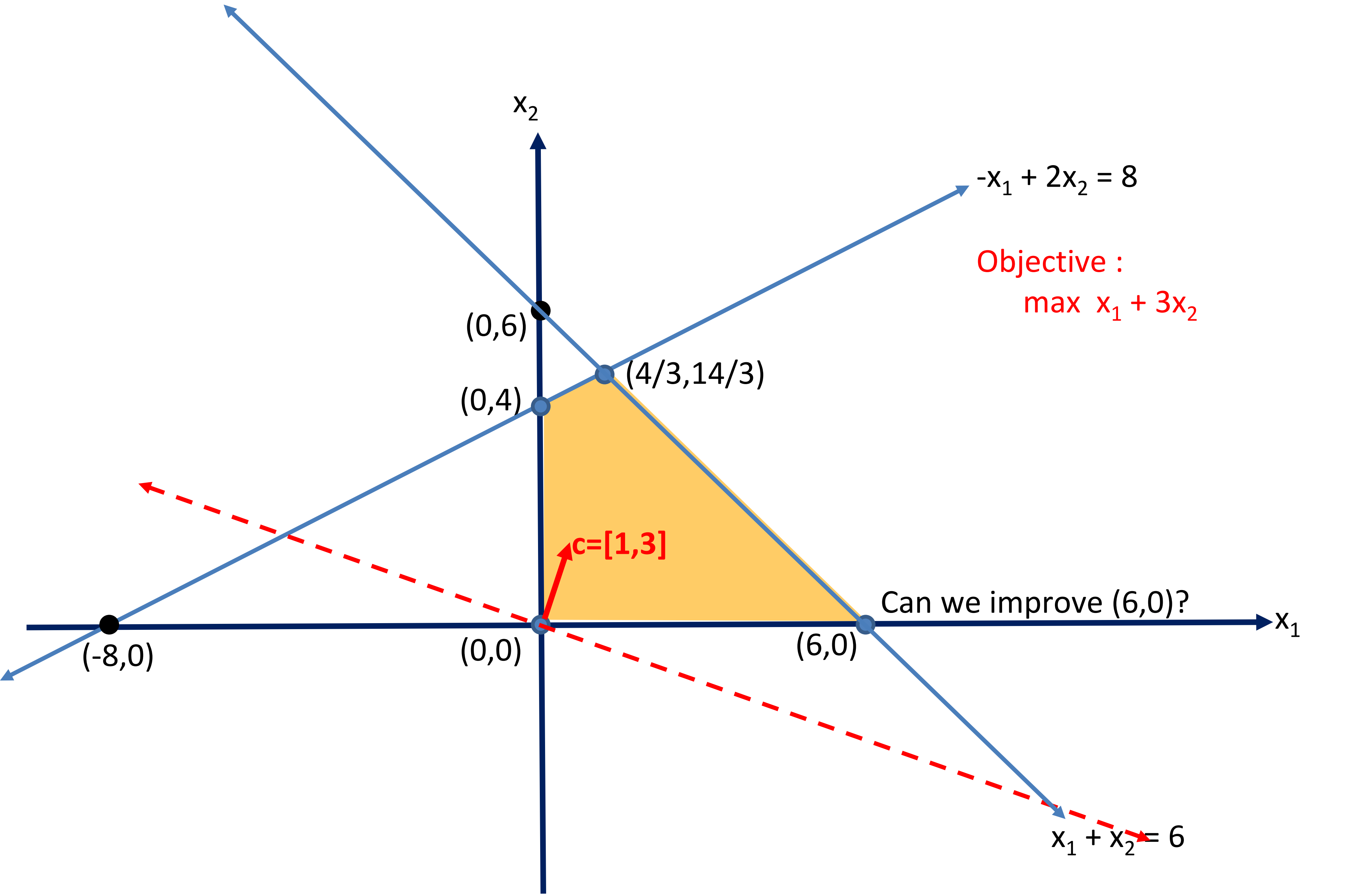
x_2
$-x_1 + 2x_2 = 8$
Objective :
max $x_1 + 3x_2$
(0,6)
(4/3,14/3)
(0,4)
c=[1 3]
(0,3)
c=[1 3]
x_1
(-8,0)
(0,0)
(6,0)
(9,0)
$x_1 + 3x_2 = 9$
$x_1 + x_2 = 6$



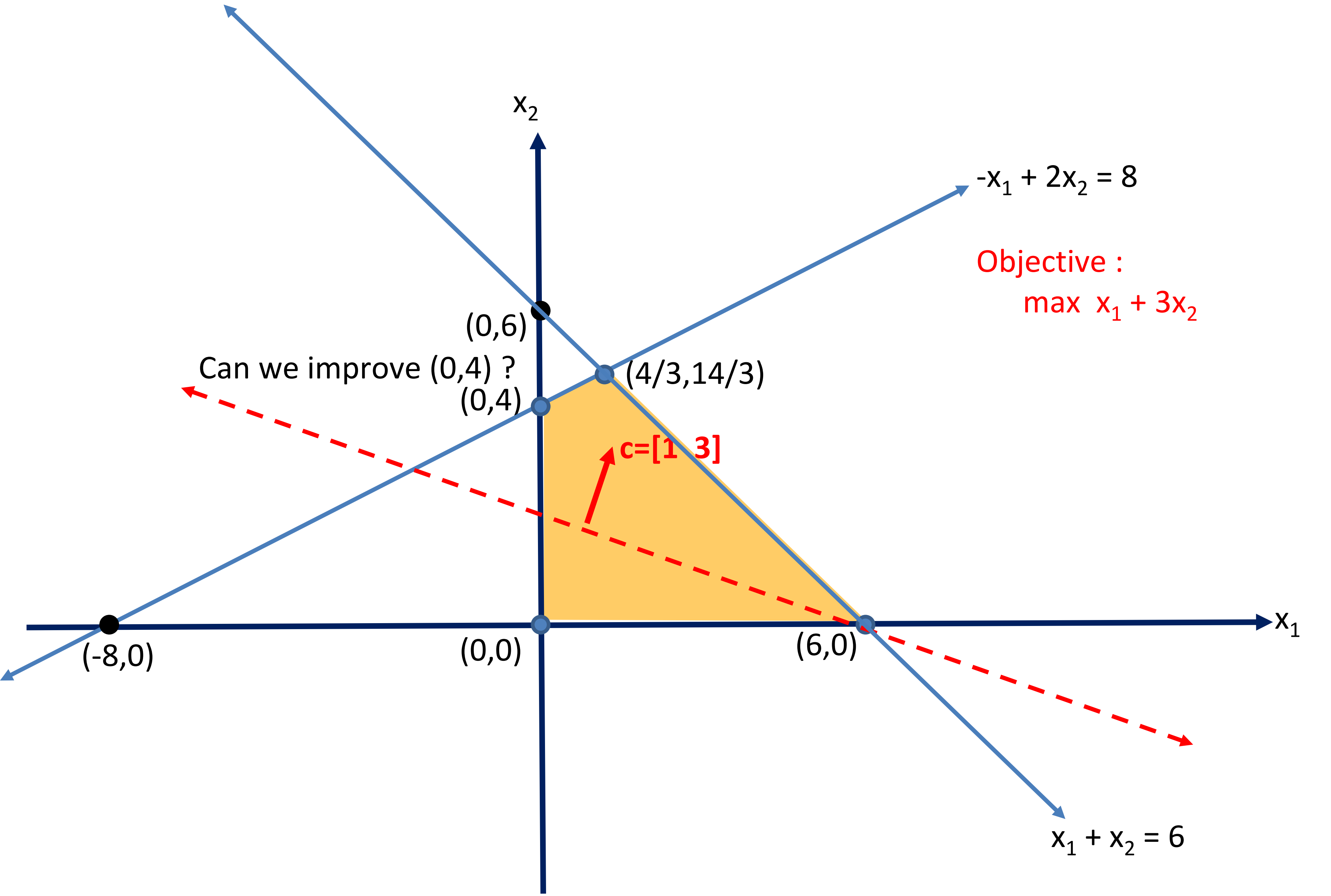
x_2
$-x_1 + 2x_2 = 8$
Objective :
max $x_1 + 3x_2$
(0,6)
(4/3,14/3)
(0,4)
c=[1 3]
c=[1 3]
(0,3)
x_1
(-8,0)
(0,0)
(6,0)
(9,0)
(12,0)
$x_1 + 3x_2 = 12$
$x_1 + 3x_2 = 9$
$x_1 + x_2 = 6$



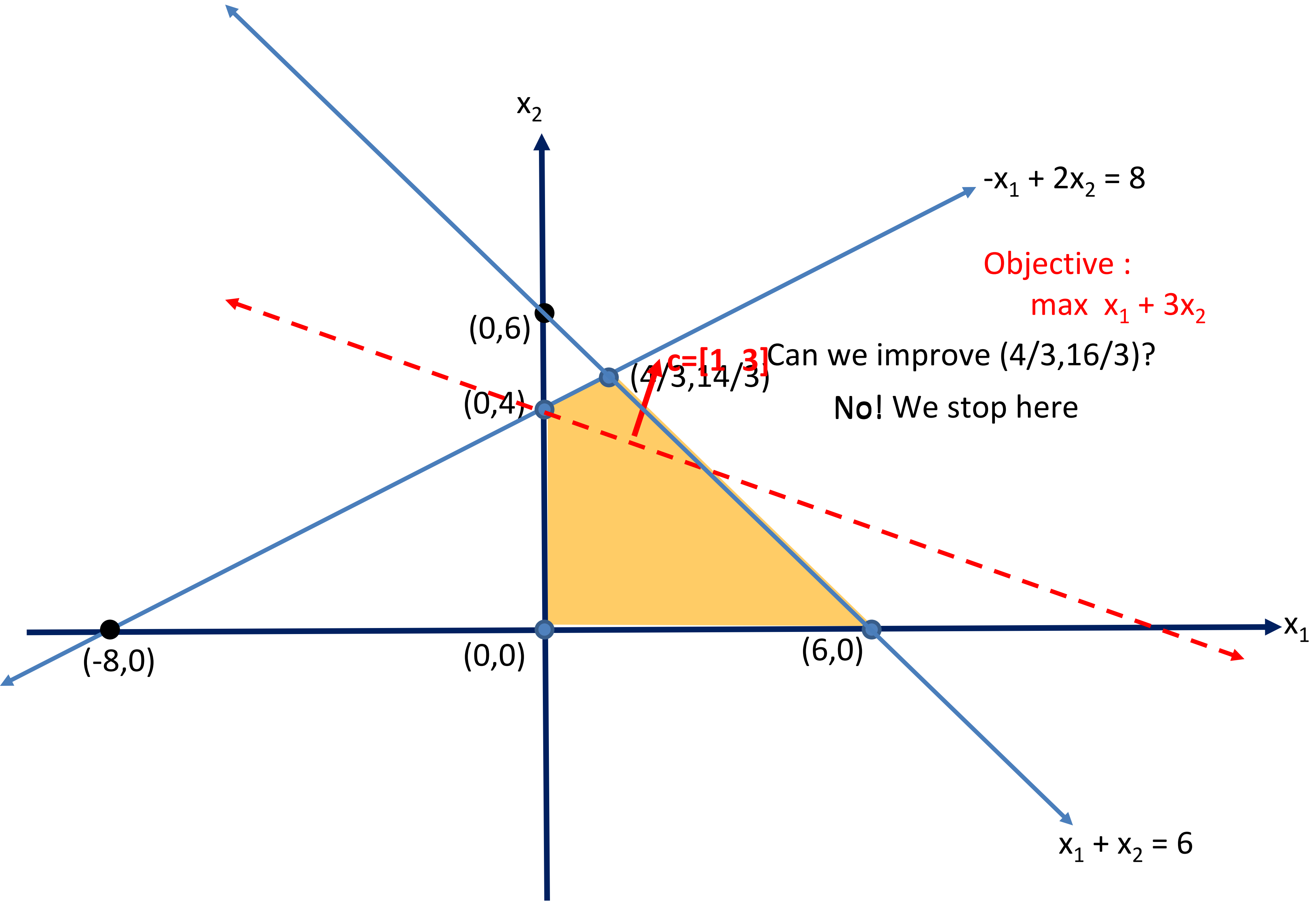
x_2
$-x_1 + 2x_2 = 8$
Objective :
max $x_1 + 3x_2$
c=[1,3]
(0,6)
(4/3,14/3)
(0,4)
$x_1+3x_2= 46/3$
x_1
(-8,0)
(0,0)
(6,0)
$x_1 + x_2 = 6$



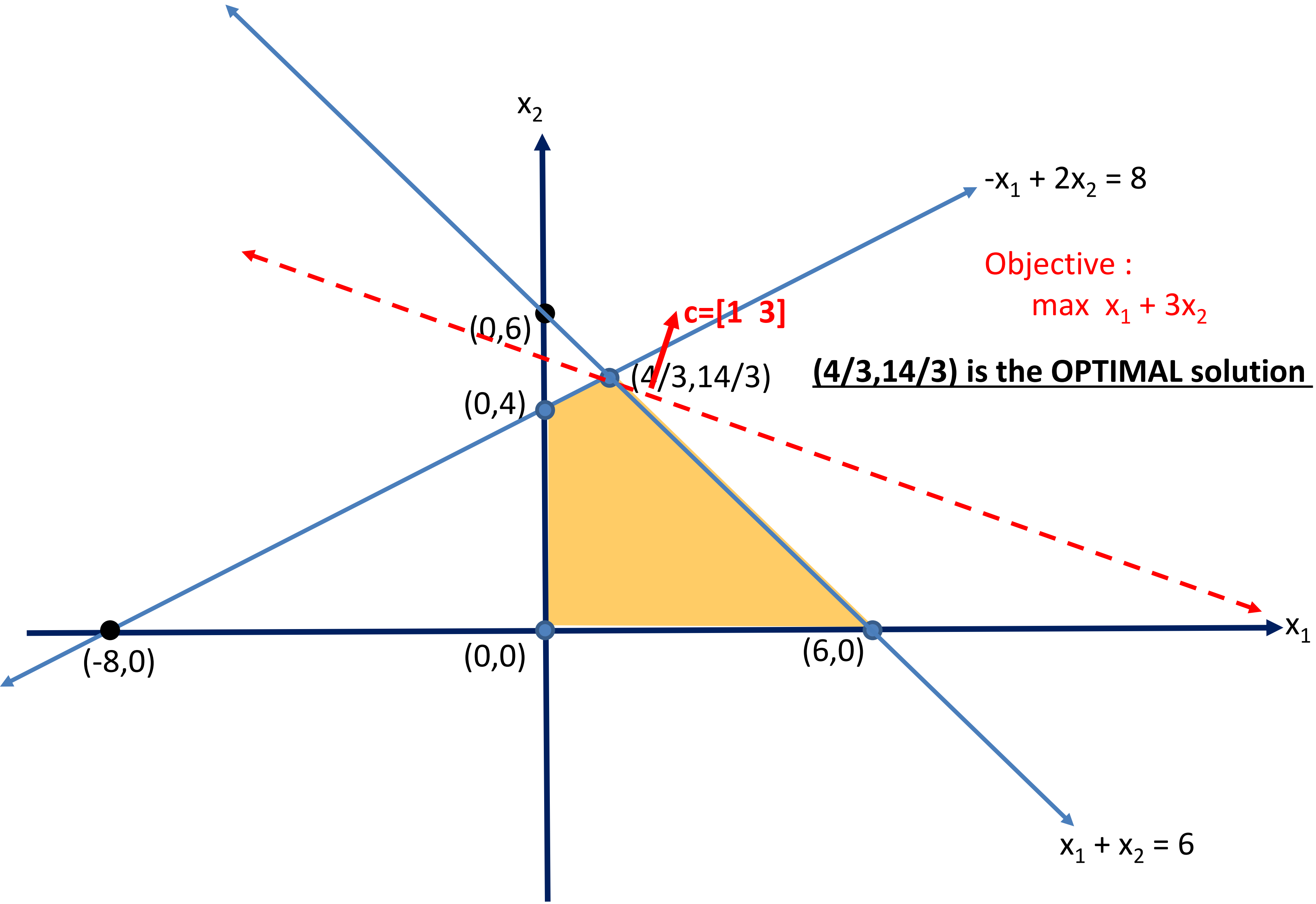
x_2
$-x_1 + 2x_2 = 8$
Objective :
max $x_1 + 3x_2$
(0,6)
(4/3,14/3)
(0,4)
c=[1,3]
Can we improve (6,0)?
x_1
(-8,0)
(0,0)
(6,0)
$x_1 + x_2 = 6$



x_2
$-x_1 + 2x_2 = 8$
Objective :
max $x_1 + 3x_2$
(0,6)
Can we improve (0,4) ?
(4/3,14/3)
(0,4)
c=[1 3]
x_1
(-8,0)
(0,0)
(6,0)
$x_1 + x_2 = 6$



x_2
$-x_1 + 2x_2 = 8$
Objective :
max $x_1 + 3x_2$
c=[1, 3]
Can we improve (4/3,16/3)?
No! We stop here
(0,6)
(4/3,14/3)
(0,4)
x_1
(-8,0)
(0,0)
(6,0)
$x_1 + x_2 = 6$


x_2
$-x_1 + 2x_2 = 8$
Objective :
max $x_1 + 3x_2$
c=[1 3]
(0,6)
(4/3,14/3)
(4/3,14/3) is the OPTIMAL solution
(0,4)
x_1
(-8,0)
(0,0)
(6,0)
$x_1 + x_2 = 6$

Graphical Solution of 2-variable LP Problems

Point (4/3, 14/3) is the last point in the feasible region when we move in the improving direction. So, if we move in the same direction any further, there is no feasible point.

Therefore, point z*= (4/3, 14/3) is the maximizing point. The **optimal value** of the LP problem is 46/3.

Graphical Solution of 2-variable LP Problems:

We may graphically solve an LP with two decision variables as follows:

Step 1: Graph the feasible region.

Step 2: Draw an isoprofit line (for max problem) or an isocost line (for min problem).

Step 3: Move parallel to the isoprofit/isocost line in the improving direction. The last point in the feasible region that contacts an isoprofit/isocost line is an optimal solution to the LP.

More examples

Graphical Solution of 2-variable LP Problems

Let us modify the objective function while keeping the constraints unchanged:

Ex 1.a)

$$\max \quad x_1 + 3\,x_2$$

s.t.

$$x_1 + x_2 \le 6 \qquad (1)$$

$$-x_1 + 2x_2 \le 8 \qquad (2)$$

$$x_1, x_2 \ge 0 \qquad (3), (4)$$

$\Rightarrow$

Ex 1.b)

$$\max \quad x_1 - 2\,x_2$$

s.t.

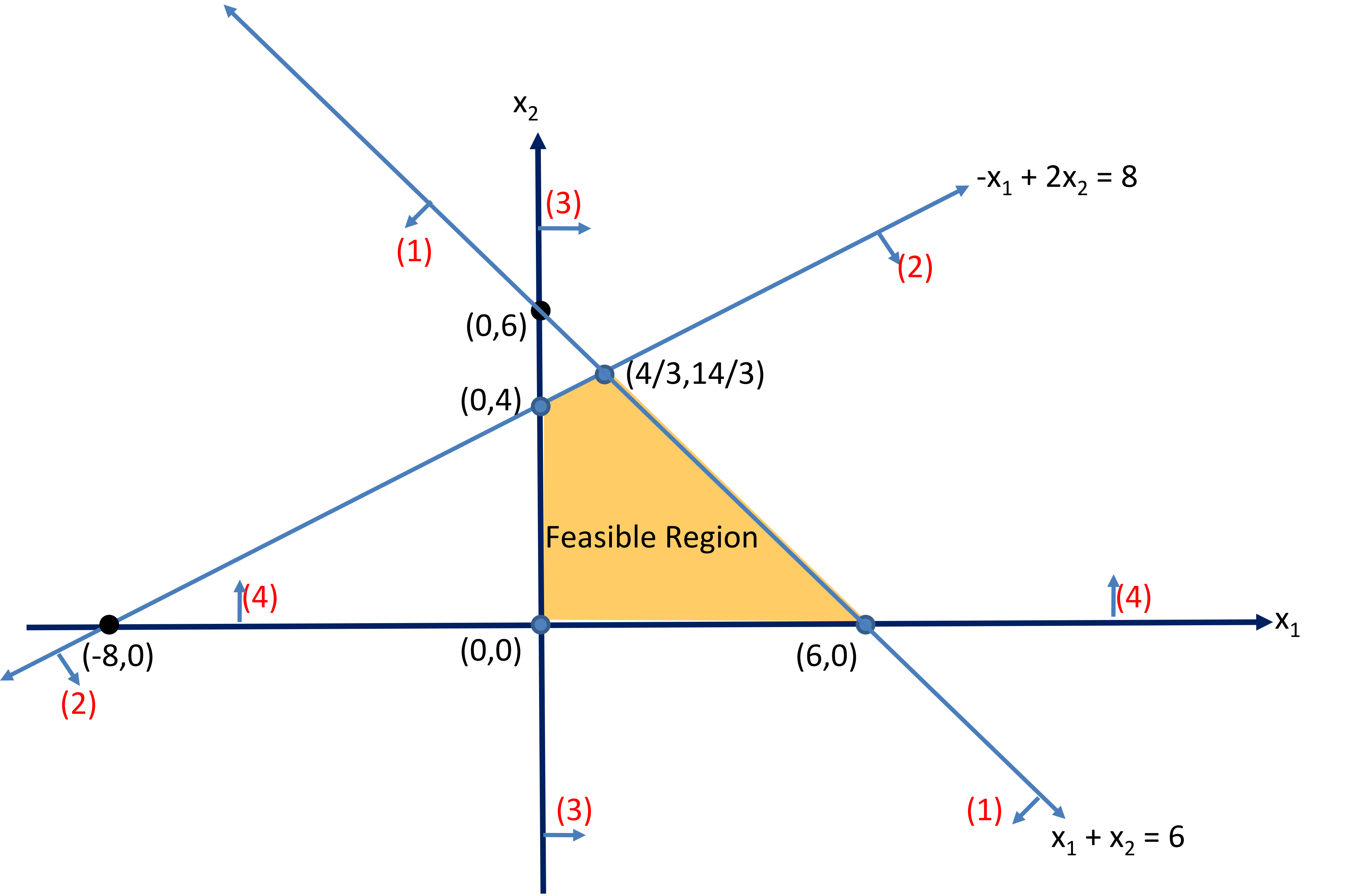
$$x_1 + x_2 \le 6 \qquad (1)$$

$$-x_1 + 2x_2 \le 8 \qquad (2)$$

$$x_1, x_2 \ge 0 \qquad (3), (4)$$

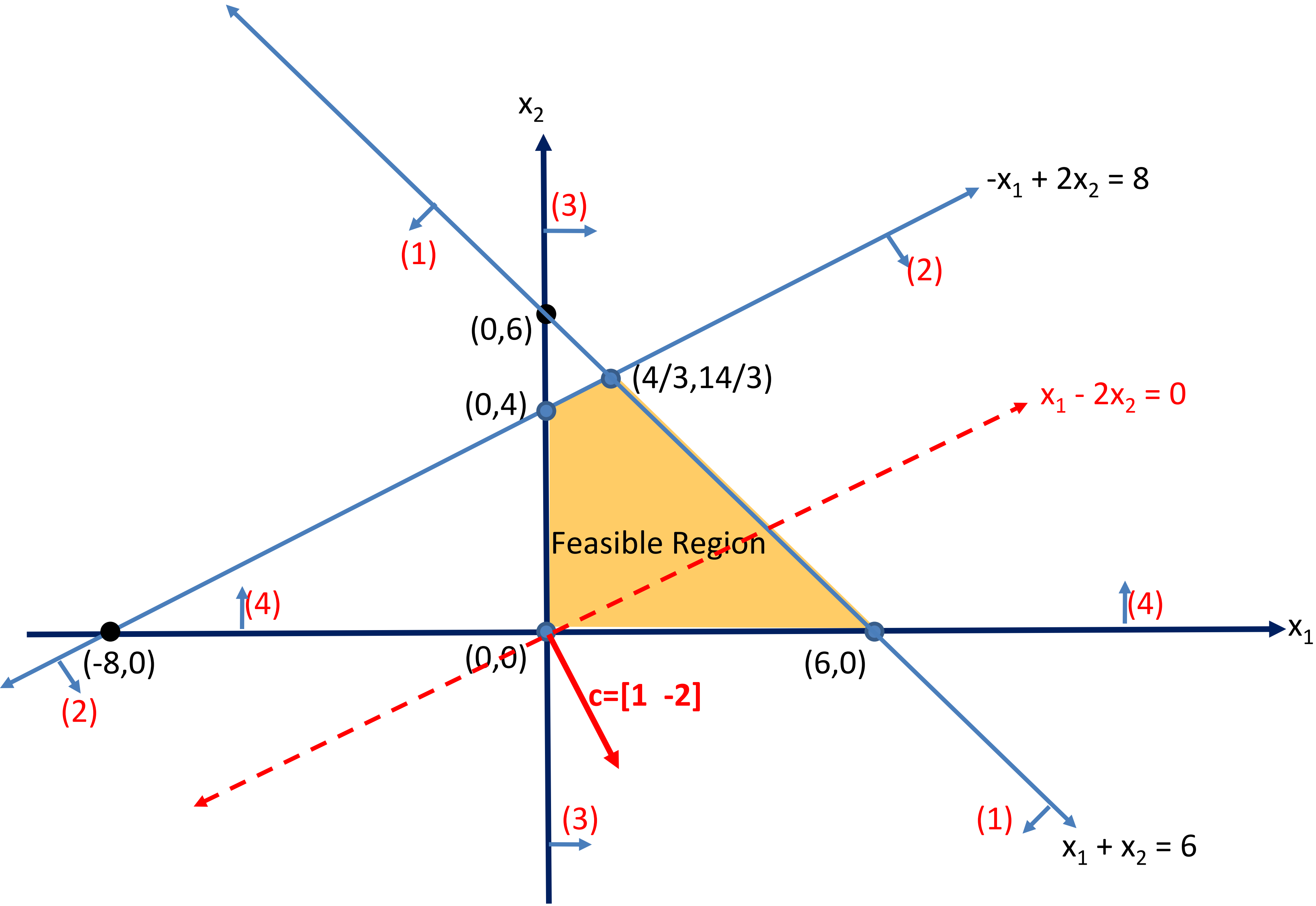
Graphical Solution of 2-variable LP Problems

Does feasible region change?

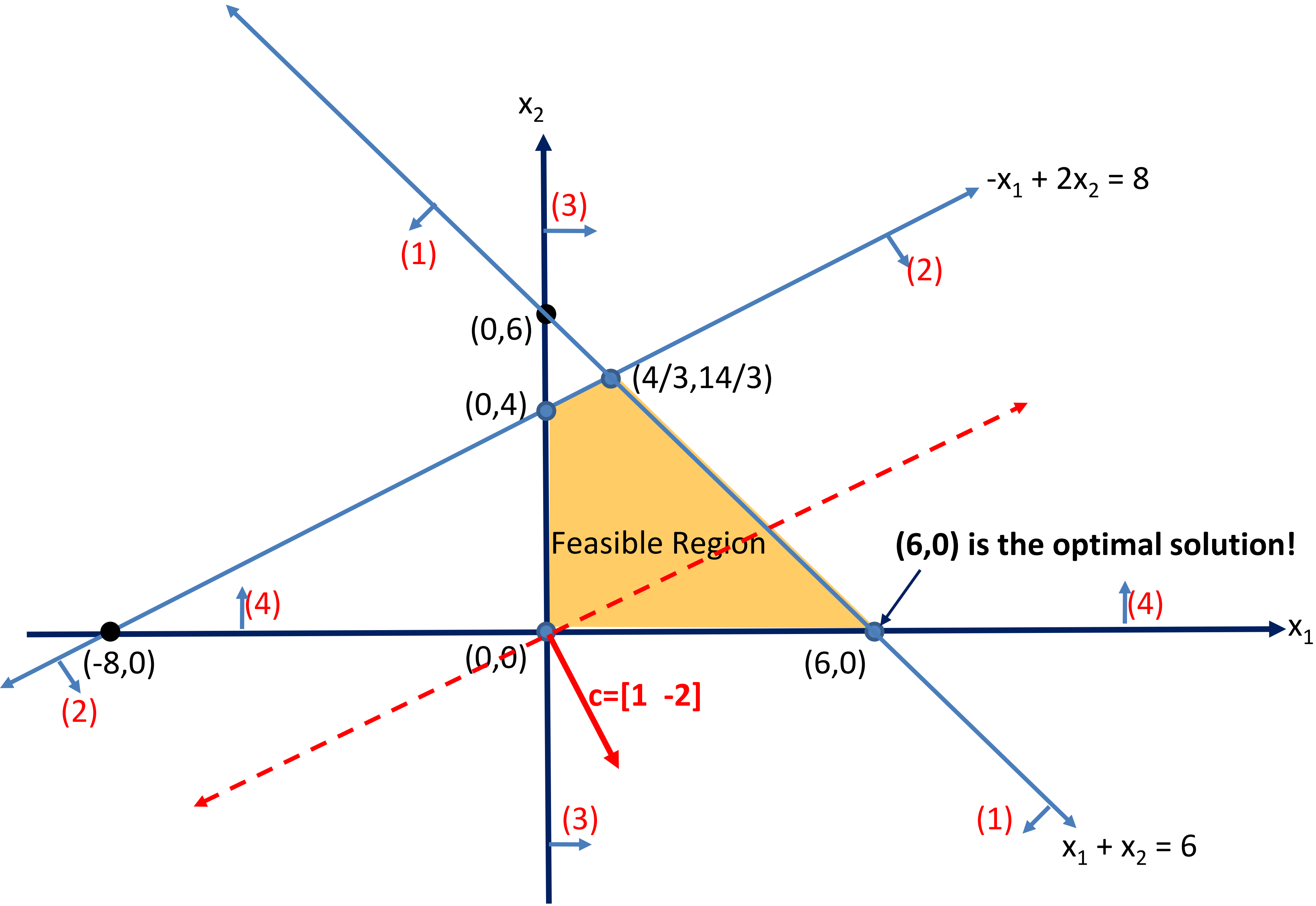

x_2
$-x_1 + 2x_2 = 8$
(3)
(1)
(2)
(0,6)
(4/3,14/3)
(0,4)
Feasible Region
(4)
(4)
x_1
(-8,0)
(0,0)
(6,0)
(2)
(3)
(1)
$x_1 + x_2 = 6$

Graphical Solution of 2-variable LP Problems

What about the improving direction?



x2
-x1 + 2x2 = 8
(3)
(1)
(2)
(0,6)
(4/3,14/3)
(0,4)
x1 - 2x2 = 0
Feasible Region
(4)
(4)
x1
(-8,0)
(0,0)
(6,0)
c=[1 -2]
(2)
(3)
(1)
x1 + x2 = 6



x_2
$-x_1 + 2x_2 = 8$
(3)
(1)
(2)
(0,6)
(4/3,14/3)
(0,4)
Feasible Region
(6,0) is the optimal solution!
(4)
(4)
x_1
(-8,0)
(0,0)
(6,0)
(2)
c=[1 -2]
(3)
(1)
$x_1 + x_2 = 6$

Graphical Solution of 2-variable LP Problems

Now let us consider:

Ex 1.c)

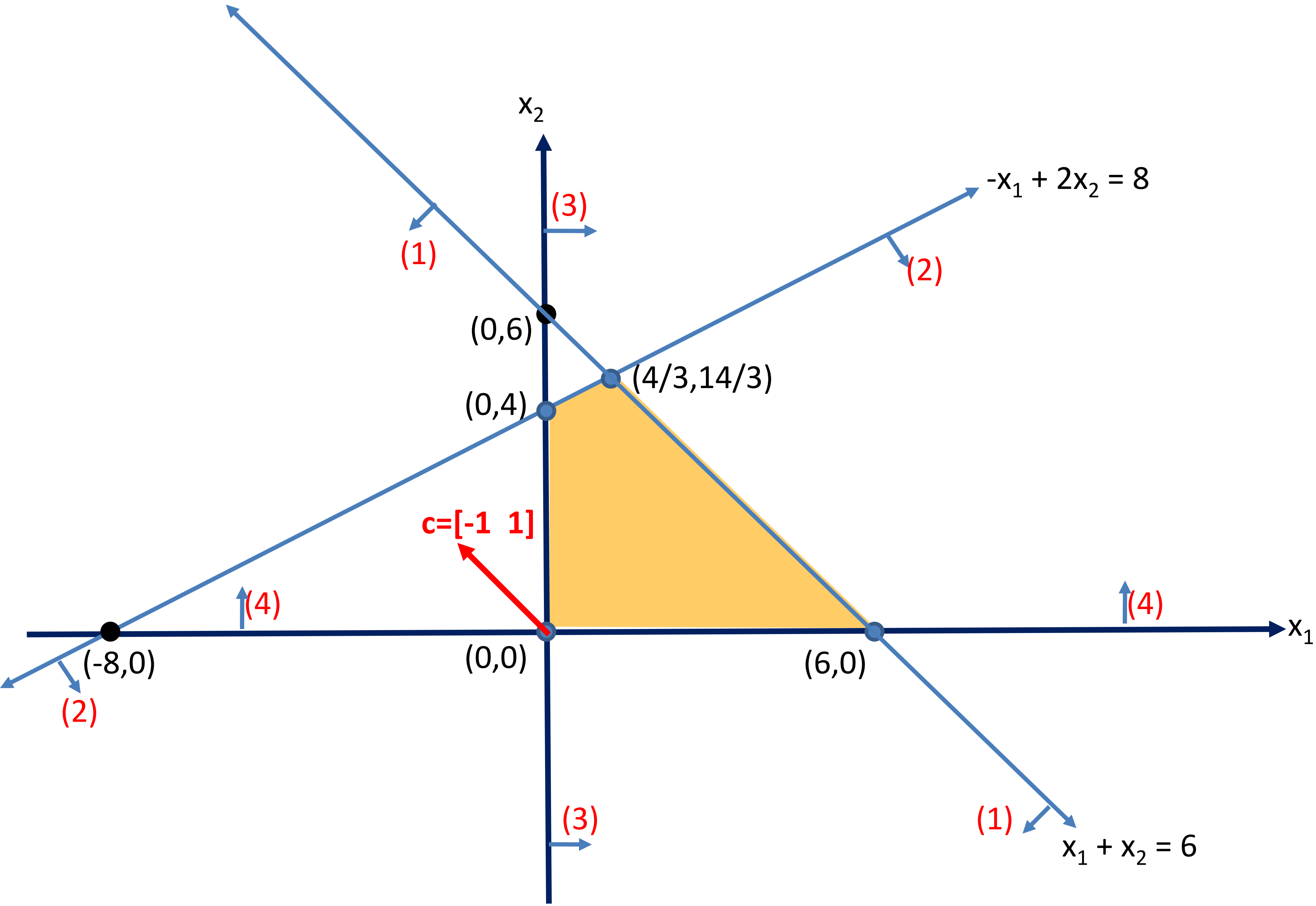
max $-x_1 + x_2$

s.t.

$x_1 + x_2 \leq 6$ (1)

$-x_1 + 2x_2 \leq 8$ (2)

$x_1, x_2 \geq 0$ (3), (4)



x_2
$-x_1 + 2x_2 = 8$
(3)
(1)
(2)
(0,6)
(4/3,14/3)
(0,4)
c=[-1 1]
(4)
(4)
x_1
(-8,0)
(0,0)
(6,0)
(2)
(3)
(1)
$x_1 + x_2 = 6$

Graphical Solution of 2-variable LP Problems

Now, let us consider:

Ex 1.c)

$$\max \quad -x_1 + x_2$$

s.t.

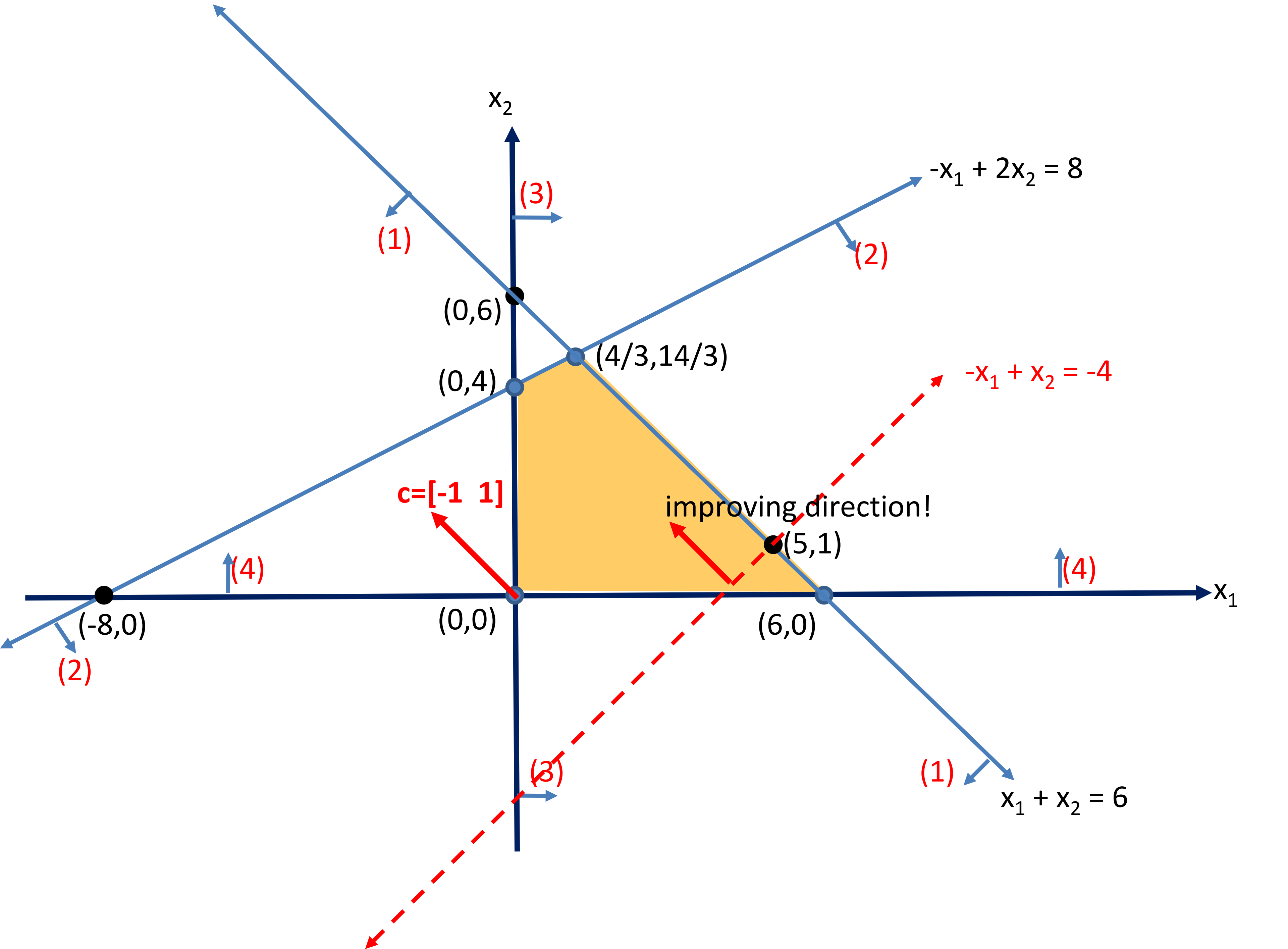
$$x_1 + x_2 \leq 6 \qquad (1)$$

$$-x_1 + 2x_2 \leq 8 \qquad (2)$$

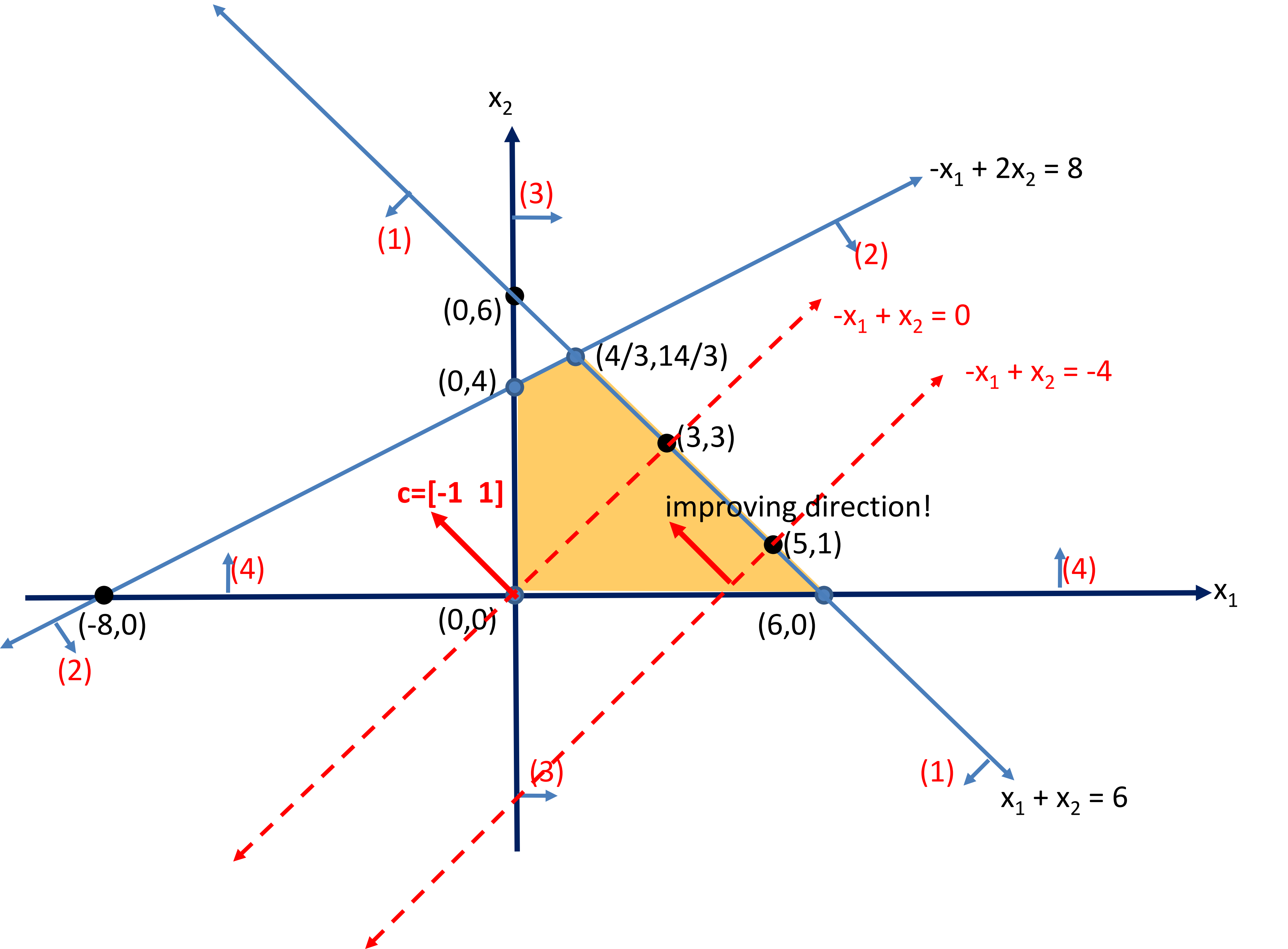
$$x_1, x_2 \geq 0 \qquad (3), (4)$$

Question:

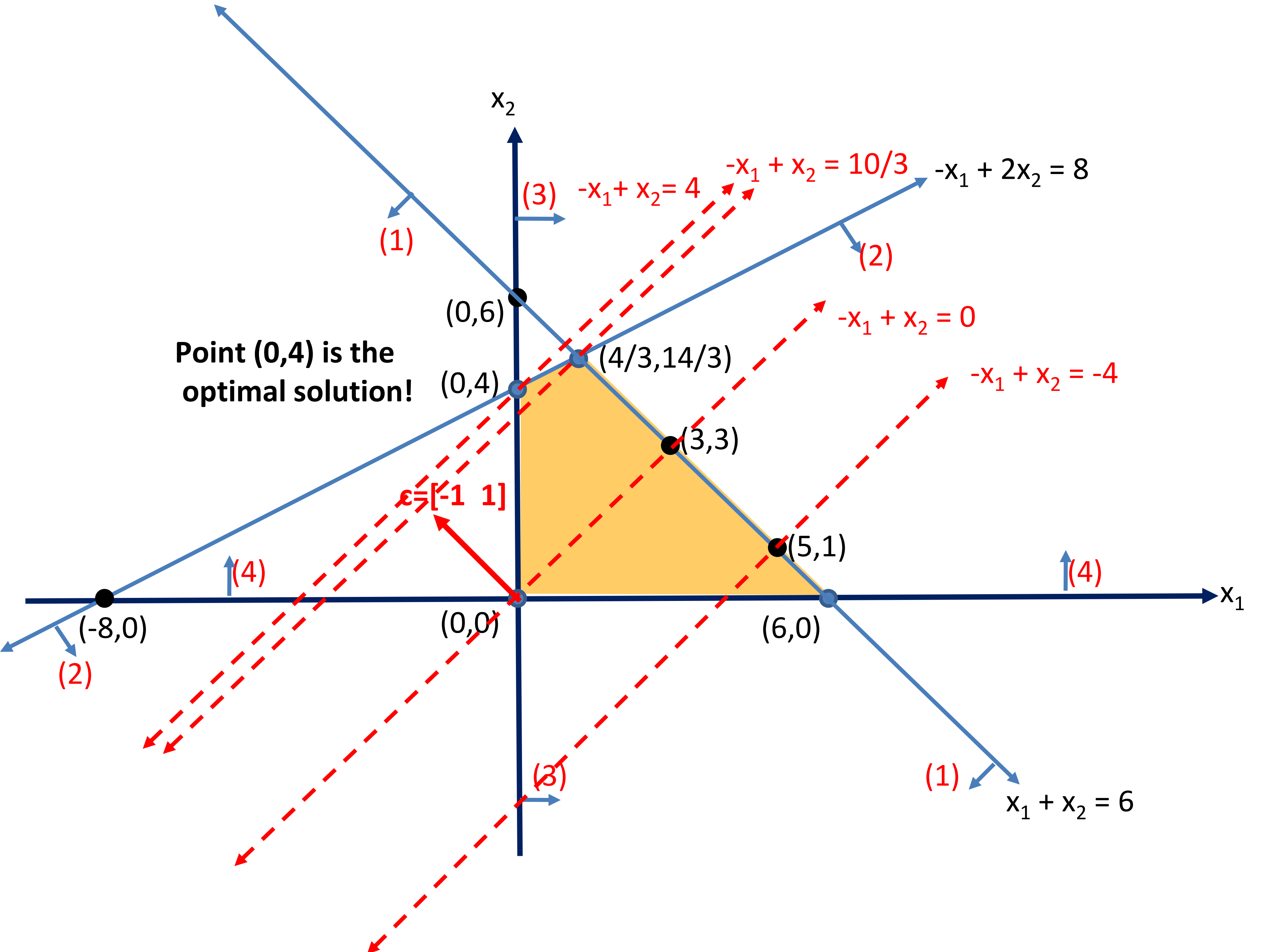
Is it possible for (5,1) be the optimal?



x_2
$-x_1 + 2x_2 = 8$
(3)
(1)
(2)
(0,6)
(4/3,14/3)
(0,4)
$-x_1 + x_2 = -4$
c=[-1 1]
improving direction!
(5,1)
(4)
(4)
x_1
(-8,0)
(0,0)
(6,0)
(2)
(3)
(1)
$x_1 + x_2 = 6$



x_2
x_1
$-x_1 + 2x_2 = 8$
$x_1 + x_2 = 6$
$-x_1 + x_2 = 0$
$-x_1 + x_2 = -4$
(1)
(2)
(3)
(4)
(0,6)
(0,4)
(4/3,14/3)
(3,3)
(5,1)
(0,0)
(6,0)
(-8,0)
c=[-1 1]
improving direction!



x2
x1
-x1 + x2 = 10/3
-x1 + 2x2 = 8
-x1+ x2= 4
(3)
(1)
(2)
(0,6)
-x1 + x2 = 0
Point (0,4) is the optimal solution!
(4/3,14/3)
(0,4)
-x1 + x2 = -4
(3,3)
c=[-1 1]
(5,1)
(4)
(4)
(0,0)
(-8,0)
(6,0)
(2)
(3)
(1)
x1 + x2 = 6

Graphical Solution of 2-variable LP Problems

Now, let us consider:

Ex 1.d)

$$\max \quad -x_1 - x_2$$

s.t.

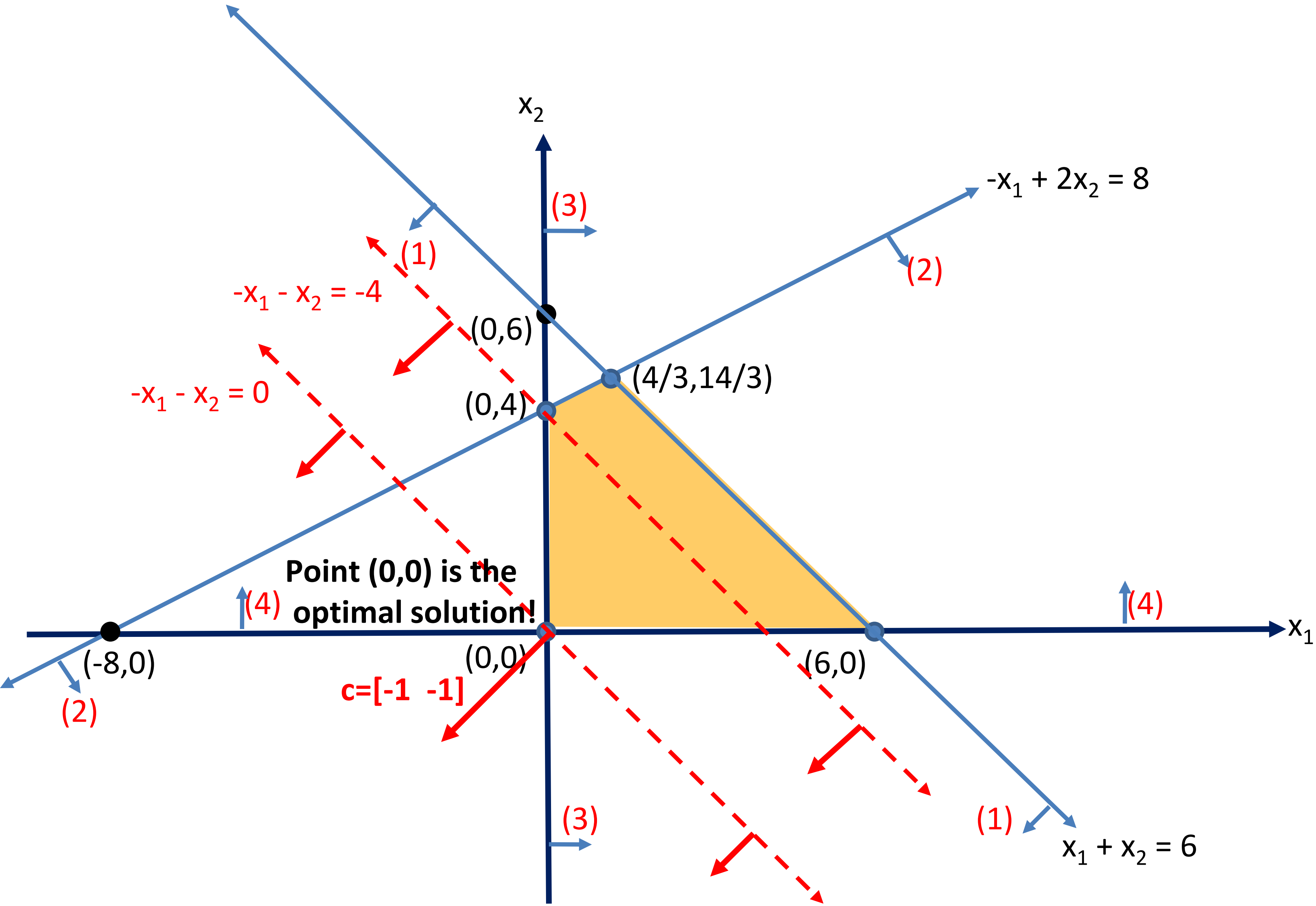
$$x_1 + x_2 \leq 6 \qquad (1)$$

$$-x_1 + 2x_2 \leq 8 \qquad (2)$$

$$x_1, x_2 \geq 0 \qquad (3), (4)$$

Question:

Is it possible for (0,4) be the optimal?



x_2
x_1
$-x_1 + 2x_2 = 8$
$x_1 + x_2 = 6$
$-x_1 - x_2 = -4$
$-x_1 - x_2 = 0$
(1)
(2)
(3)
(4)
(0,6)
(0,4)
(4/3,14/3)
(-8,0)
(0,0)
(6,0)
Point (0,0) is the optimal solution!
c=[-1 -1]

Another Example

Graphical Solution of 2-variable LP Problems

Ex 2)

A company wishes to increase the demand for its product through advertising. Each minute of radio ad costs \$1 and each minute of TV ad costs \$2.

Each minute of radio ad increases the daily demand by 2 units and each minute of TV ad by 7 units.

The company would wish to place at least 9 minutes of daily ad in total. It wishes to increase daily demand by at least 28 units.

How can the company meet its advertising requirementsat minimum total cost?

Graphical Solution of 2-variable LP Problems

x_1: # of minutes of radio ads purchased

x_2: # of minutes of TV ads purchased

Graphical Solution of 2-variable LP Problems

x_1: # of minutes of radio ads purchased

x_2: # of minutes of TV ads purchased

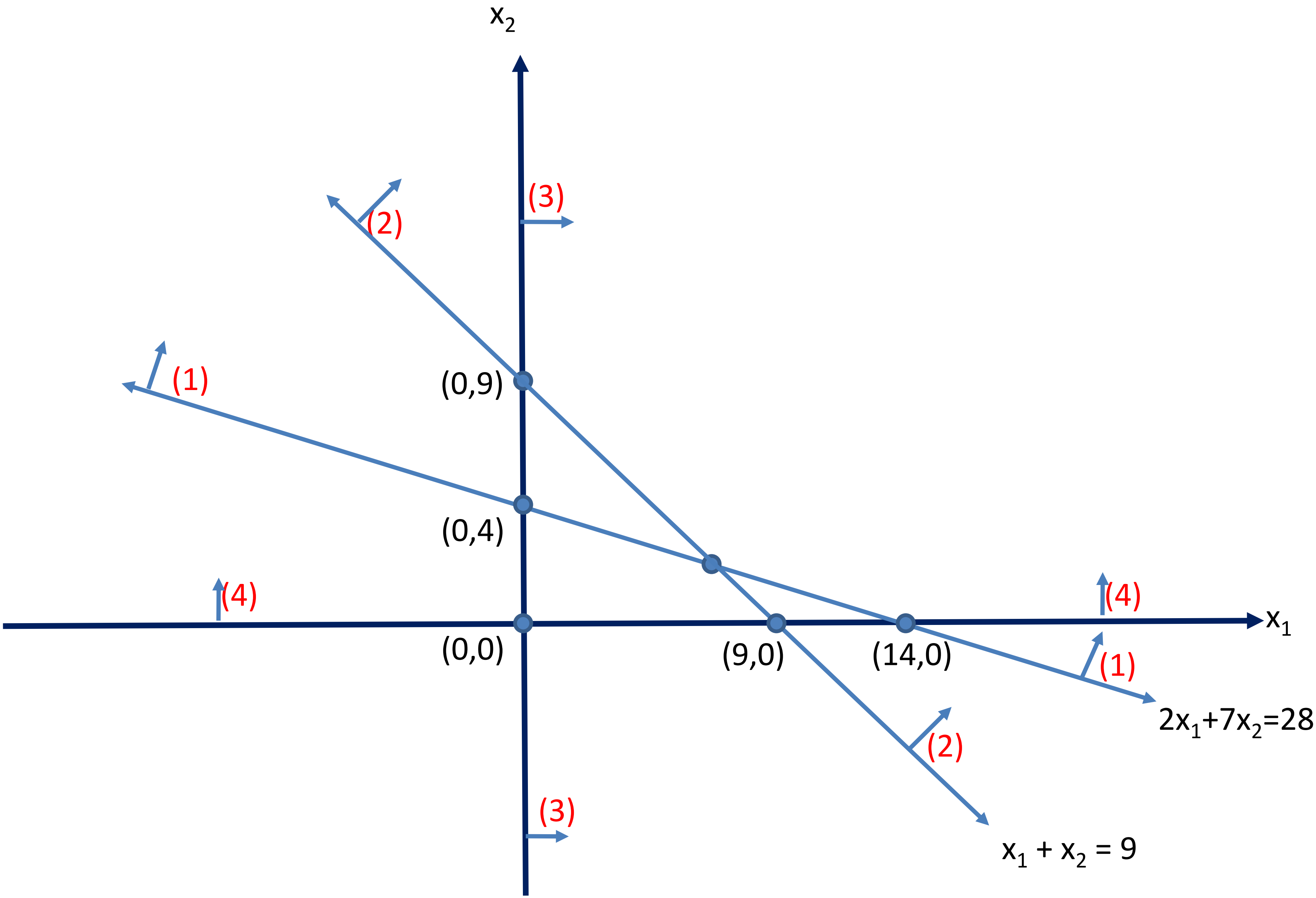
$$\min \quad x_1 + 2x_2$$

s.t.

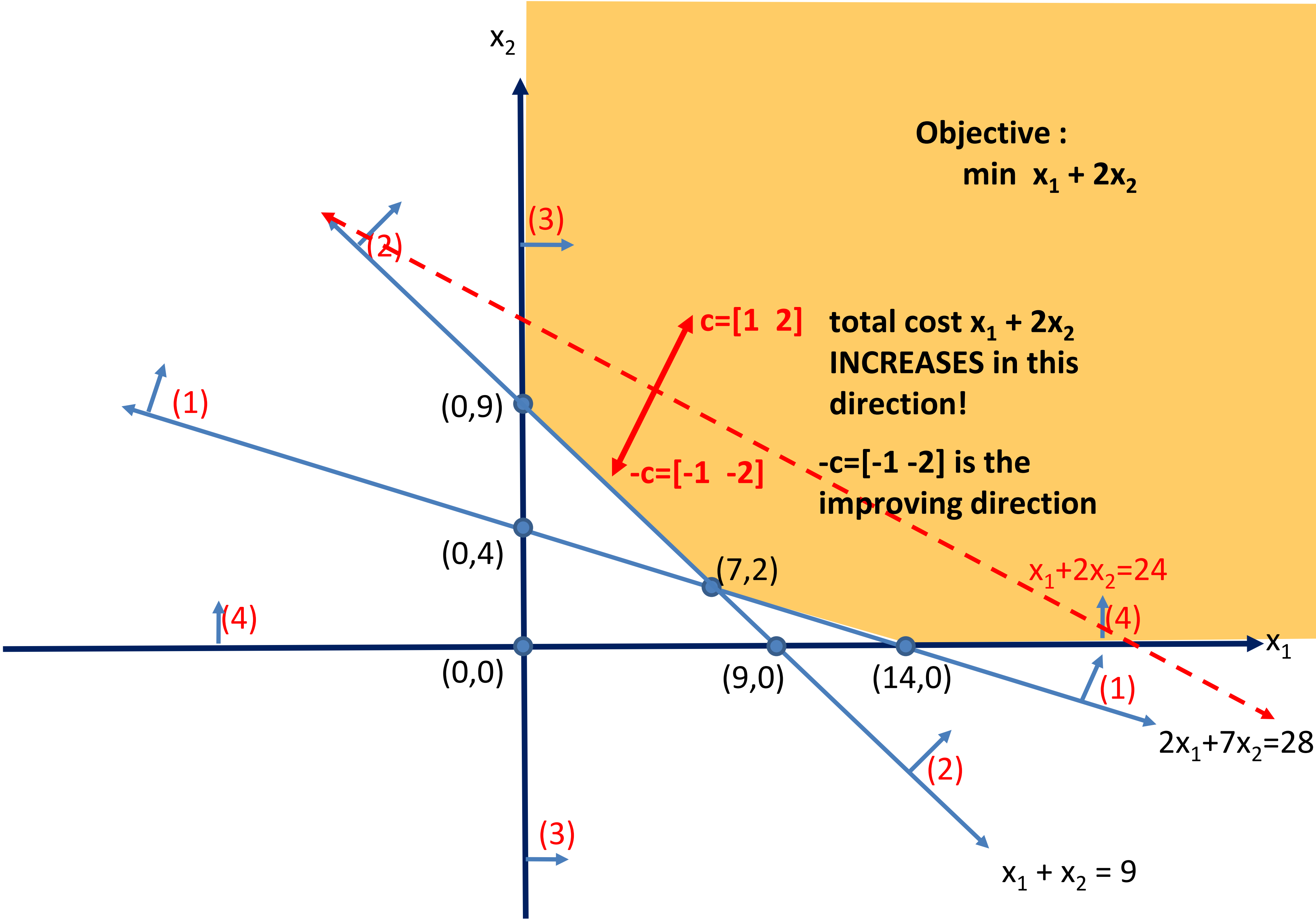
$$2x_1 + 7x_2 \geq 28 \qquad (1)$$

$$x_1 + x_2 \geq 9 \qquad (2)$$

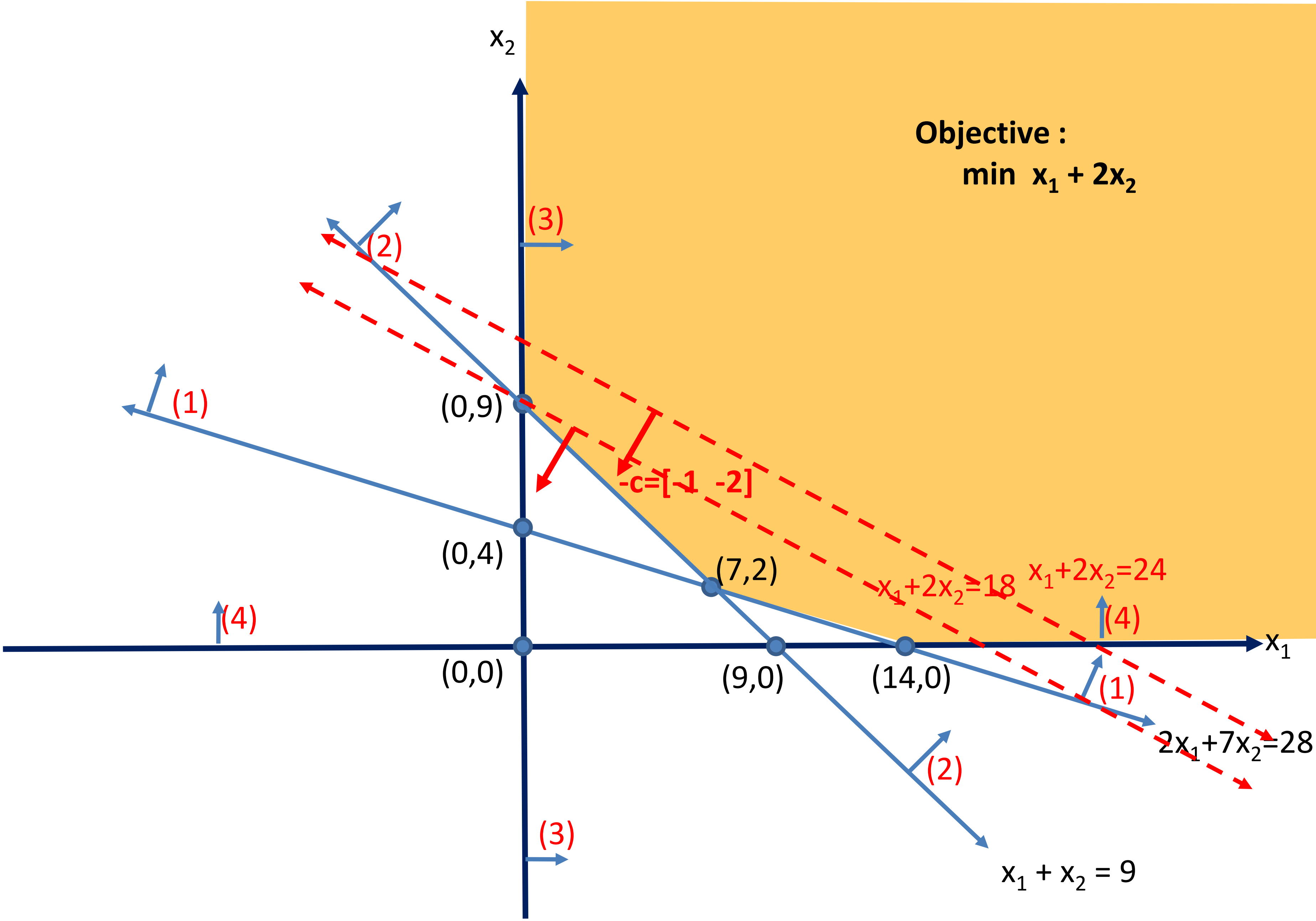
$$x_1, x_2 \geq 0 \qquad (3), (4)$$



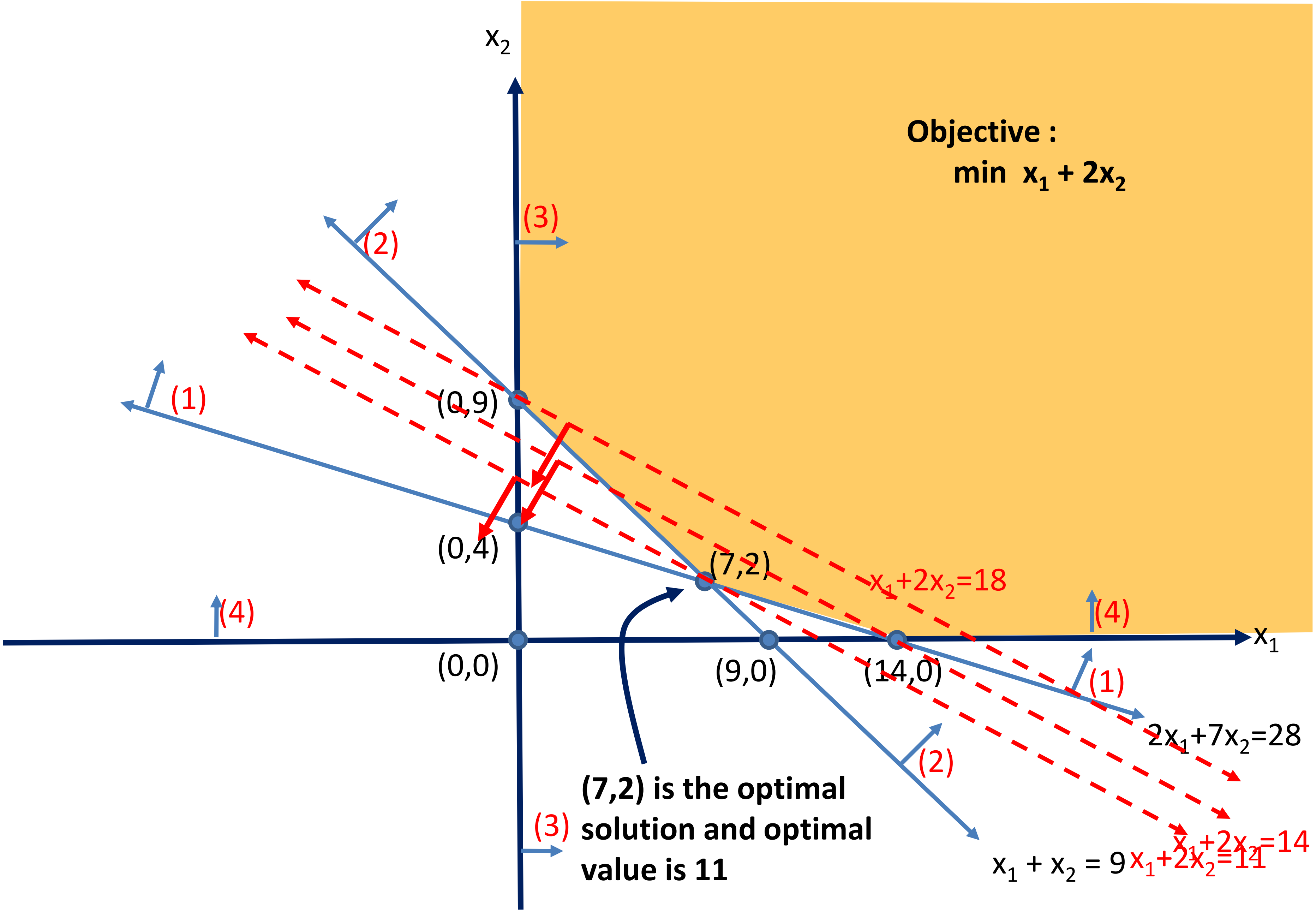
x_2
x_1
(3)
(2)
(1)
(0,9)
(0,4)
(4)
(0,0)
(9,0)
(14,0)
(4)
(1)
$2x_1+7x_2=28$
(2)
(3)
$x_1 + x_2 = 9$



x_2
x_1
Objective :
min $x_1 + 2x_2$
(3)
(2)
c=[1 2]
total cost $x_1 + 2x_2$
INCREASES in this
direction!
(1)
(0,9)
-c=[-1 -2]
-c=[-1 -2] is the
improving direction
(0,4)
(7,2)
$x_1+2x_2=24$
(4)
(4)
(0,0)
(9,0)
(14,0)
(1)
$2x_1+7x_2=28$
(2)
(3)
$x_1 + x_2 = 9$



Objective :
min $x_1 + 2x_2$
x_2
x_1
(3)
(2)
(1)
(4)
(0,9)
(0,4)
(0,0)
(7,2)
(9,0)
(14,0)
-c=[-1 -2]
$x_1+2x_2=18$
$x_1+2x_2=24$
$2x_1+7x_2=28$
$x_1 + x_2 = 9$



Objective :
min $x_1 + 2x_2$
x_2
x_1
(1)
(2)
(3)
(4)
(0,9)
(0,4)
(7,2)
(0,0)
(9,0)
(14,0)
$x_1+2x_2=18$
$2x_1+7x_2=28$
$x_1 + x_2 = 9$
$x_1+2x_2=14$
$x_1+2x_2=11$
(7,2) is the optimal solution and optimal value is 11

Graphical Solution of 2-variable LP Problems:

We may graphically solve an LP with two decision variables as follows:

Step 1: Graph the feasible region.

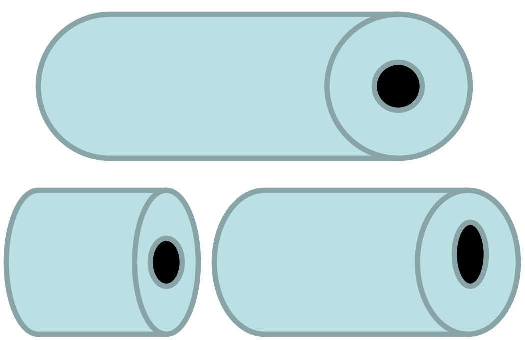
Step 2: Draw an isoprofit line (for max problem) or an isocost line (for min problem).

Step 3: Move parallel to the isoprofit/isocost line in the improving direction. The last point in the feasible region that contacts an isoprofit/isocost line is an optimal solution to the LP.

More challenging modelling examples

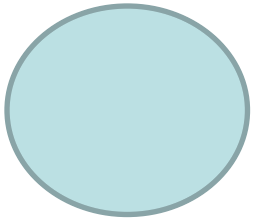
Trim Loss Problem

- Certain products are manufactured in wide rolls.
- Wide rolls are later cut into rolls of smaller strips to meet specific demand.
- Examples: Paper, photographic film, sheet metal, textile fabric.

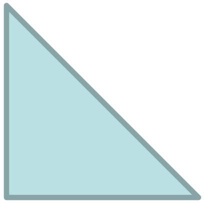


Another Example

100 of this shape



7 of this shape



22 of this shape and size

- Available stock

How to cut depends on the demand

Example: Wrapping Paper

- Paper rolls are manufactured as 60–inch (wideness).
- Demand is for 10-inch, 20-inch, 25-inch and 30-inch rolls.

customer	10-inch	20-inch	25-inch	30-inch
X	107	109		110
Y	80		25	
Z	89			29

Possible patterns?

Examples:

- 6 10-inch
- 3 20-inch
- 2 30-inch
- 4 10-inch + 1 20-inch
- ....

What do you think our concern will be?

- Waste – remains of the original roll
- Total number of original rolls used

A simpler example to work-out

- Wider rolls with 48-inch
- Demand is for 21-inch and 25-inch
- Patterns

	25-inch	21-inch	Waste
Pattern A	**0**	**2**	**6**
Pattern B	**1**	**1**	**2**

- Demand: 20 25-inch rolls, 50 21-inch rolls

Example

	25-inch	21-inch	Waste
Pattern A	0	2	6
Pattern B	1	1	2

Decision variables:

- x : the number rolls cut into Pattern A
- y : the number rolls cut into Pattern B

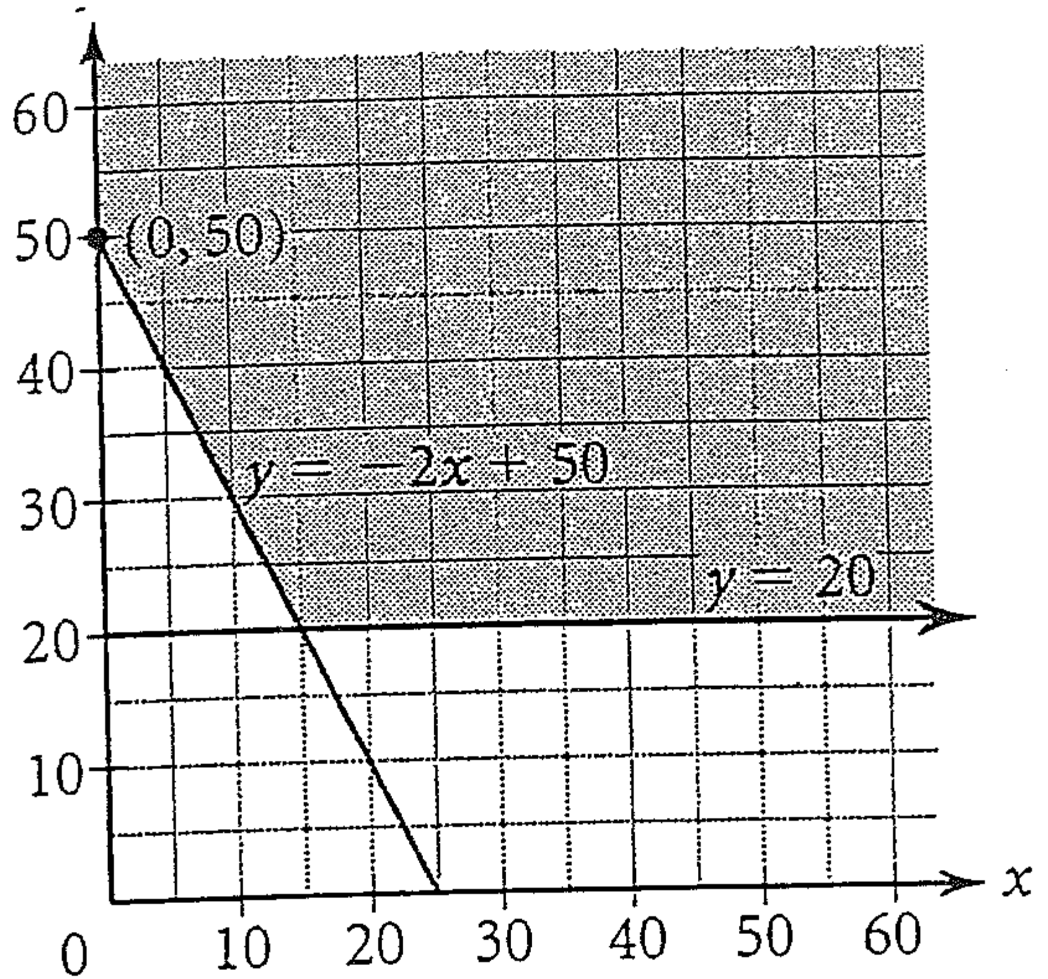
Constraints: We want to satisfy demand

- $y \geq 20$; $2x + y \geq 50$
- $x \geq 0$, $y \geq 0$

Objective function: Minimize waste

where waste = $w = 6x + 2y$

Example continued



Example continued

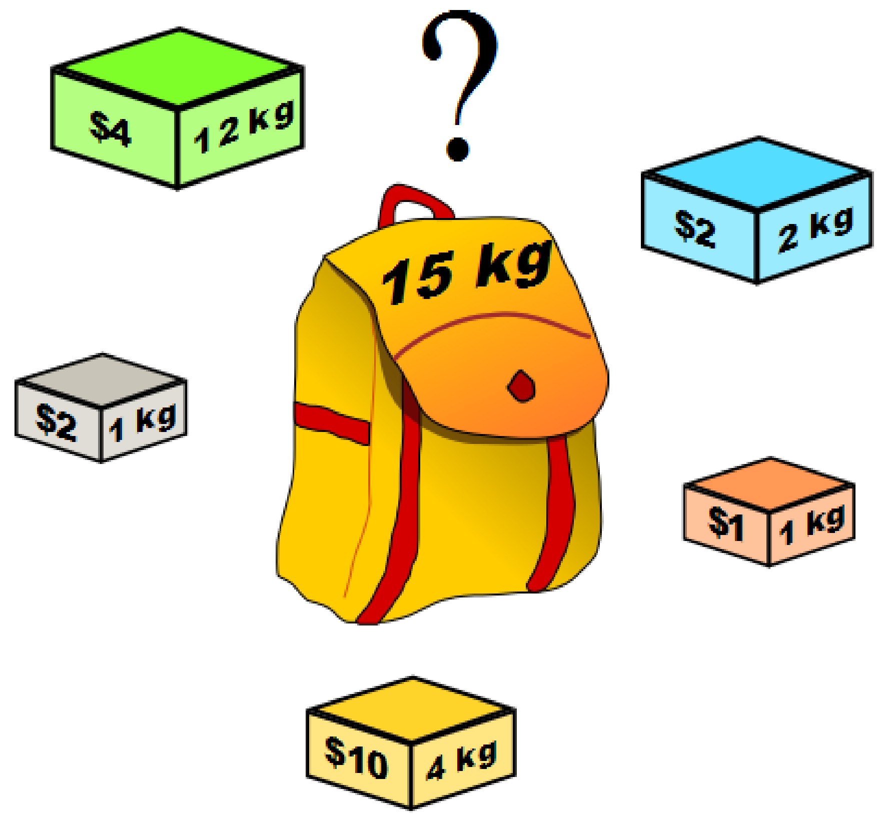
- Solution? Corner solution:
- (0,50), w = 100
- What is the problem with the solution?
- Total produced: 50 21-inch + 50 25-inch
- Demand? 50 21-inch + 20 25-inch
- You need to be able to store the additional 25-inch rolls and use later. Any additional cost even if you can?

Knapsack problem

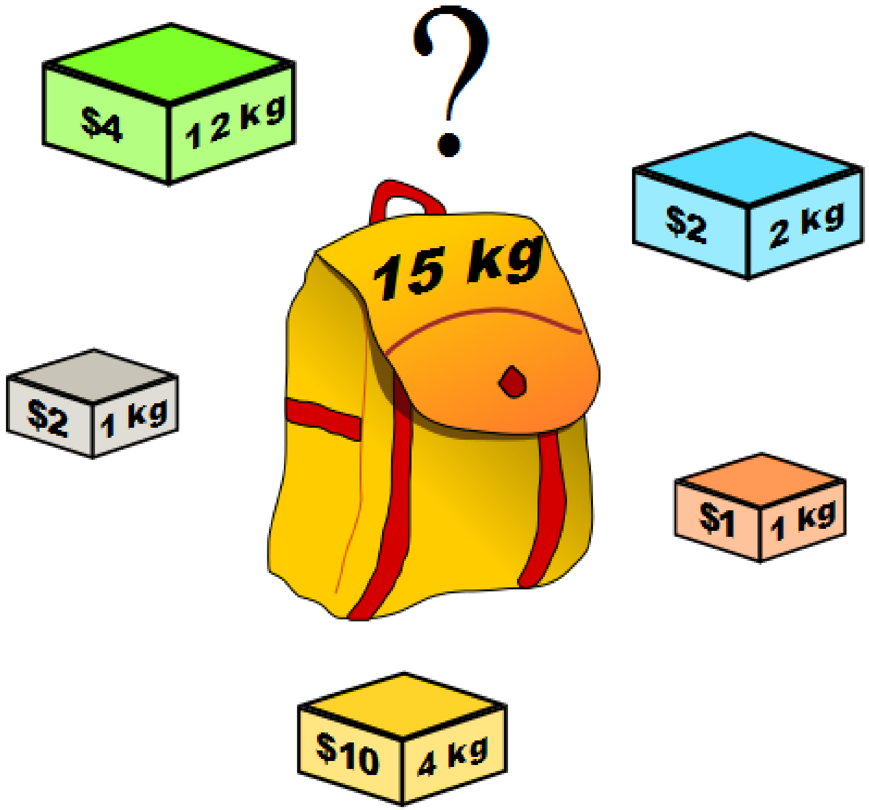
You are going on a trip. Your **Suitcase has a capacity of b kg's**.

- You have **n different items** that you can pack into your suitcase 1, . . . , n.
- **Item i** has a weight of $\mathbf{a_i}$ and gives you a utility of $\mathbf{c_i}$.

How should you pack your suitcase to maximize total utility?



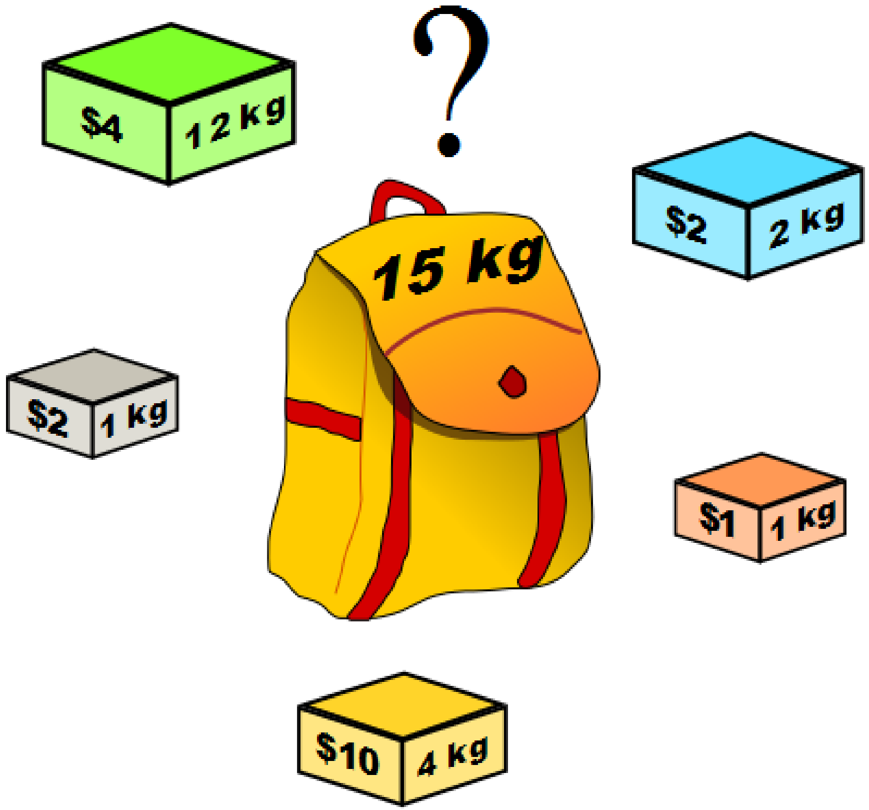
Knapsack problem



Knapsack problem

$$x_i = \begin{cases} 1, & \text{if item } i \text{ is packed in the suitcase} \\ 0, & \text{otherwise} \end{cases}$$

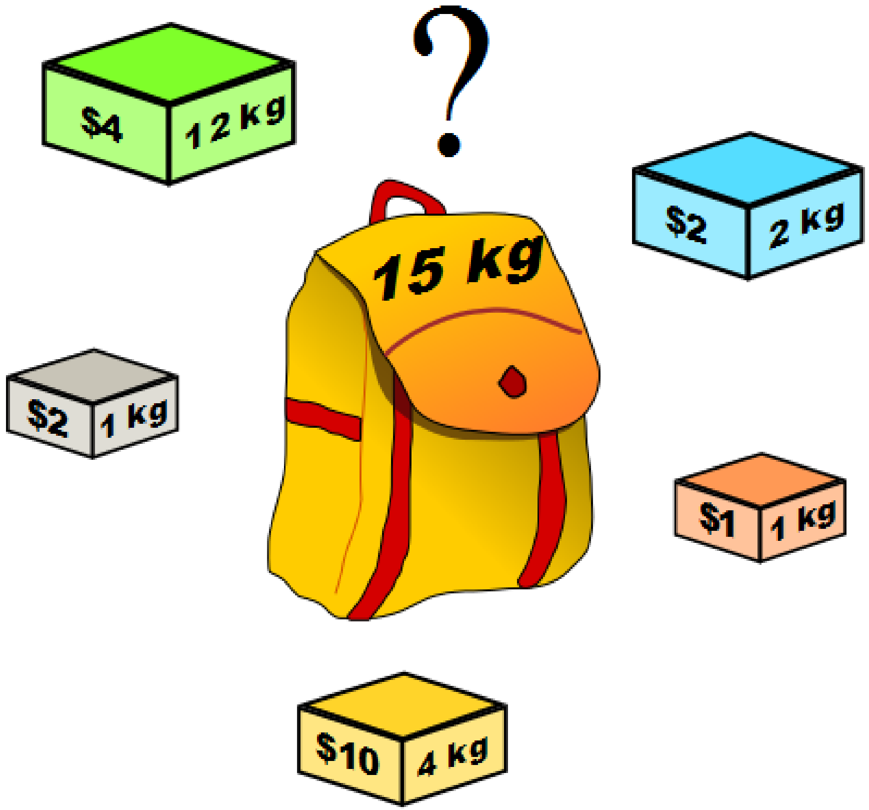
$$i = 1, \ldots, n$$



Knapsack problem

$$x_i = \begin{cases} 1, & \text{if item } i \text{ is packed in the suitcase} \\ 0, & \text{otherwise} \end{cases}$$

$$i = 1, \ldots, n$$



So the model formulation is:

$$\max \sum_{i=1}^{n} c_i x_i$$

$$s.t.$$

$$\sum_{i=1}^{n} a_i x_i \le b$$

$$x_{ij} \in \{0,1\}, \; i = 1, \ldots, n$$

This is also an example of an IP problem since x_i can only take one of the two integer values 0 and 1.

KAHOOT