# Operations Research in Finance:
## A Novice Introduction to Financial Engineering

Çağın Ararat
Bilkent University

April 10, 2018

# Finance

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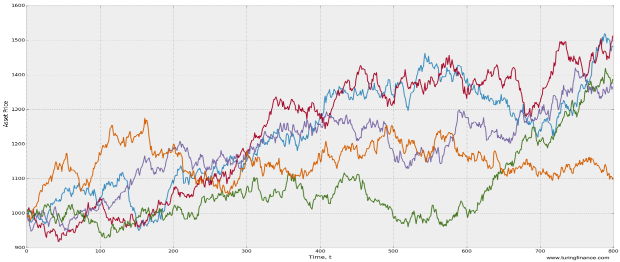
  - a lot of uncertainty (randomness)



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  - Stocks of companies
  - Gold
  - Oil
  - Foreign currencies (USD, EUR, etc.)
- More complicated securities such as options, forward and futures contracts have payoffs that are based on the price of an underlying security. These are called derivative securities.
  - Options
  - Forward contracts
  - Futures contracts

# Financial markets

- Where are these securities traded?
  - Exchanges: Regulated centralized markets where security buyers and sellers meet and negotiate on the prices. Participating companies must obey certain rules. All transactions are secured by the exchange. Many stocks and options are traded on exchanges.
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- The ten largest stock exchanges in the world:
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- Turkey
  - Exchange: Borsa İstanbul (stocks, gold, derivatives)
  - OTC market: Grand Bazaar Effective Market, Gold Market

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- Stock prices fluctuate a lot!
- Prices from April 9, 2017 to April 9, 2018:

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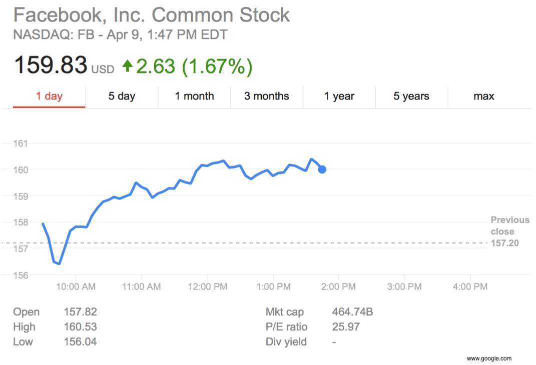
- The numbers of buyers and sellers are affected by many economic, political, social, technological factors, which, in turn, affect the prices as well.
- Instead of explaining all these factors, as financial engineers, we take the data as is and try to find a mathematical model that is consistent with data.

# Financial engineering: modeling

- Let us consider the prices during April 9, 2018 (yesterday):



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- There is still uncertainty in the closing price of the day.
- Considering a full day, we may classify time as
  - Past: 9:30 am - 1:47 pm
  - Present: 1:47 pm
  - Future: 1:47 pm - 4:00 pm

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- Derivatives pricing...

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- If the strike price is more expensive, that is, if $S_T \leq K$, then the holder does nothing and gets zero payoff.
- Therefore, the payoff from the call option is given by

$$h(S_T) = \max\{S_T - K, 0\} = (S_T - K)^+.$$

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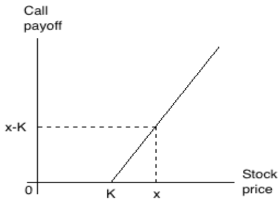


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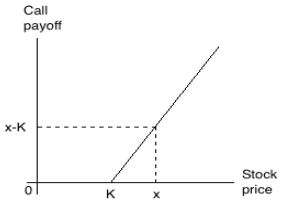


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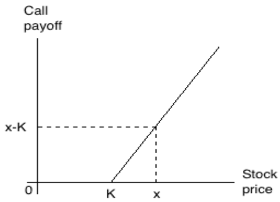


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- How to determine the price of this call option?

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  - In general, you need a more sophisticated procedure: run an algorithm, solve a system of equations, solve an optimization problem, etc. (More on optimization: IE 202, IE 303)
  - For more complex options, you can only calculate the price approximately via computational techniques.

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  - Suppose that the (unit) bond price is $A_0$ liras today so that the price will increase to $A_1 = A_0(1 + r)$ liras tomorrow.

bond price tree

today $\quad A_0 \xrightarrow{\;\;1\;\;} (1+r)A_0 \quad$ tomorrow

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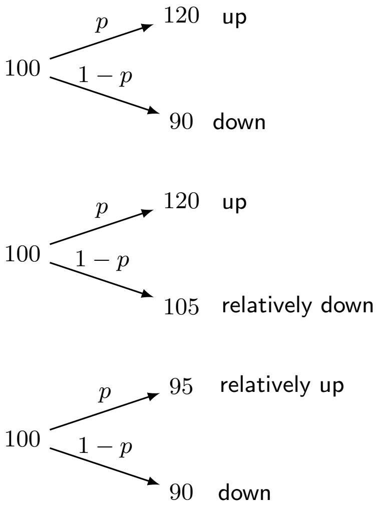
  - Just like flipping a coin in high school probability: up for heads, down for tails.
  - Suppose that the (unit) stock price is $S_0$ liras today. Let us denote by $S_T$ the price tomorrow. So

$$S_T = \begin{cases} uS_0 & \text{if the price goes up} \\ dS_0 & \text{if the price goes down} \end{cases}$$

stock price tree

$$\begin{tikzcd}[column sep=large]
 & & uS_0 & \\
\text{today} & S_0 \arrow[ur, "p"] \arrow[dr, "1-p"'] & & \text{tomorrow} \\
 & & dS_0 &
\end{tikzcd}$$

- The words *up* and *down* are used only in the relative sense.



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- Question: What should be the relationship among $u - 1, d - 1, r$ for the model to be *financially sound*?
- Possible cases:
  - Case 1: $d - 1 < u - 1 \leq r$. The bond always outperforms the stock.
  - Case 2: $r \leq d - 1 < u - 1$. The stock always outperforms the bond.
  - Case 3: $d - 1 < r < u - 1$. There is no clear winner. The performance of the bond is in between the good (up) and bad (down) performances of the stock.

Kahoot time!

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- The bond always outperforms the stock. As a rational investor, you would always invest money in the bond and never in the stock.
- An even better idea is to buy (long) bonds and sell (short) stocks. This way you can make free money.

# A very simple market model

- Example: Consider the following trees for the stock and the bond.

$$\begin{tikzcd}
 & 120 \\
100 \arrow[ur, "p"] \arrow[dr, "1-p"] & \\
 & 90
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
100 \arrow[r, "1"] & 120
\end{tikzcd}$$

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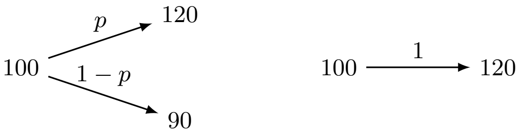


- Start with no money today. Borrow one stock to be given back tomorrow and immediately sell it stock for $S_0 = 100$. Deposit your $100$ liras into the bank by buying $\frac{S_0}{A_0} = \frac{100}{100} = 1$ bond.

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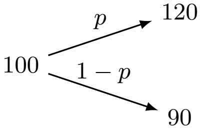


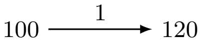


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- Wait until tomorrow! The new value of your bond is $A_1 = 120$ liras. Withdraw this amount from the bank.

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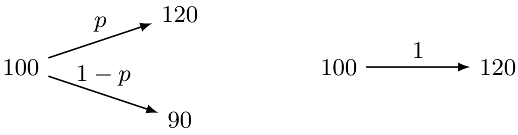


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- If the stock price goes up to $120$, then buy a stock with your $120$ liras. Give the stock back to its original holder (lender). Your net gain is zero in this case.

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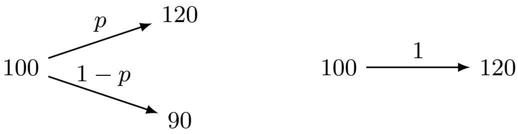


- Start with no money today. Borrow one stock to be given back tomorrow and immediately sell it stock for $S_0 = 100$. Deposit your 100 liras into the bank by buying $\frac{S_0}{A_0} = \frac{100}{100} = 1$ bond.
- Wait until tomorrow! The new value of your bond is $A_1 = 120$ liras. Withdraw this amount from the bank.
- If the stock price goes up to 120, then buy a stock with your 120 liras. Give the stock back to its original holder (lender). Your net gain is zero in this case.
- If the stock price goes down to 90, then buy a stock with 90 of your 120 liras. Give the stock back to its holder. Your net gain is $120 - 90 = 30$ liras in this case.

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- Case 2: $r \leq d - 1 < u - 1$ is symmetric, you can make free money by longing the stock and shorting the bond. Hence, (NA) is violated in Case 2 as well.

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- A sound market should be free of arbitrage opportunities. This principle is called the principle of no-arbitrage (NA).
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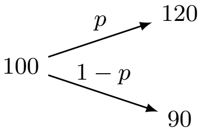
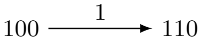
- Case 2: $r\leq d-1<u-1$ is symmetric, you can make free money by longing the stock and shorting the bond. Hence, (NA) is violated in Case 2 as well.
- Case 3: $d-1<r<u-1$ is the only case where (NA) is not violated.

# A very simple market model

- Example: Consider the following trees for the stock and the bond.



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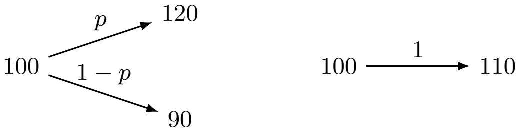


- Start with no money today. Borrow one stock to be given back tomorrow and immediately sell it stock for $S_0 = 100$. Deposit your $100$ liras into the bank by buying $\frac{S_0}{A_0} = \frac{100}{100} = 1$ bond.

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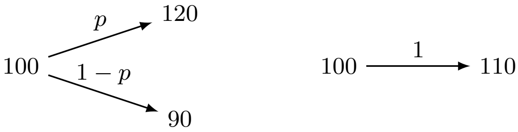


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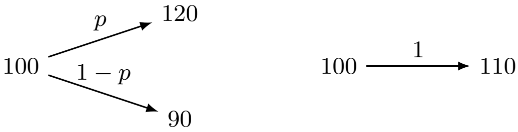


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- If the stock price goes up to $120$, then borrow $10$ liras from your friend and buy a stock all of your $120$ liras. Give the stock back to the lender. You still owe $10$ liras to your friend. Your net gain is $-10$ (loss) in this case.

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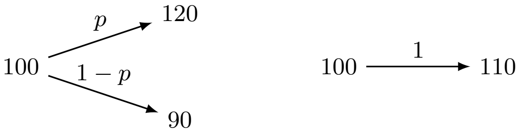


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- If the stock price goes down to $90$, then buy a stock with $90$ of your $110$ liras. Give the stock back to its holder. Your net gain is $110 - 90 = 20$ liras in this case.

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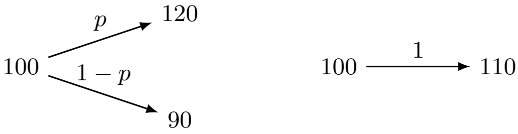


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- Hence, you make money with probability $1 - p > 0$ and lose money with probability $p > 0$. This is not an arbitrage!

- Indeed, you can prove that an arbitrage cannot exist under Case 3.

- Indeed, you can prove that an arbitrage cannot exist under Case 3.
- Hence,
$$\text{(NA) holds.} \quad \Leftrightarrow \quad d-1<r<u-1.$$
- This result is called the fundamental theorem of asset pricing (FTAP). We will see why this is related to pricing assets (derivatives).

## Pricing a call option

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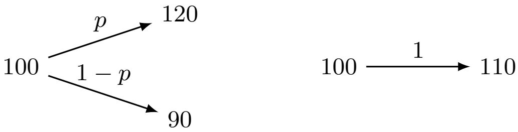

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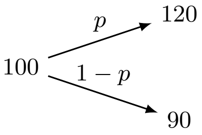


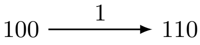


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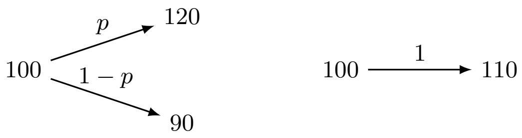


- Suppose that the strike price of the call is $K = 105$.
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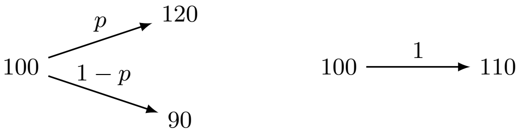


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- If the stock price goes down to $90$ tomorrow, then it is not profitable for the option holder to buy a stock for $105$. Hence, the holder does not exercise the option and gets no payoff.

# Pricing a call option

- The payoff structure of the call can be summarized by the following tree:

$$
\begin{tikzcd}
 & 15 \\
C_0 =? \arrow[ur, "p"] \arrow[dr, "1-p"'] & \\
 & 0
\end{tikzcd}
$$

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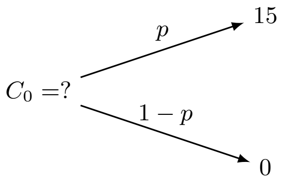


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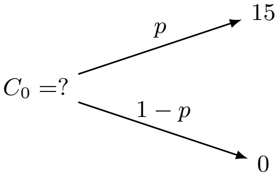


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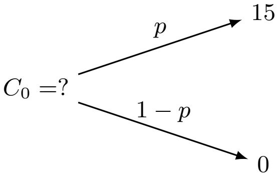


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- What should $C_0$ be under (NA)?
- If $C_0$ is chosen to be too low or too high, then an arbitrage may arise.
- To determine $C_0$, we will first replicate the call using stocks and bonds. The price of the replicating strategy will determine $C_0$.

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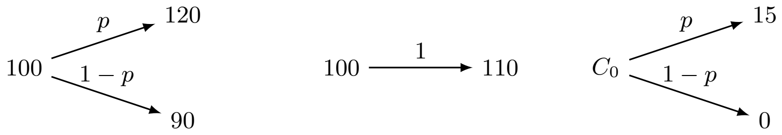


$x$ stocks $\quad + \quad$ $y$ bonds $\quad = \quad$ 1 call option

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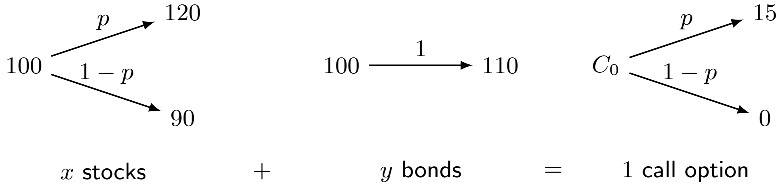


- Solve for $(x, y)$ the system

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120x + 110y &= 15 \\
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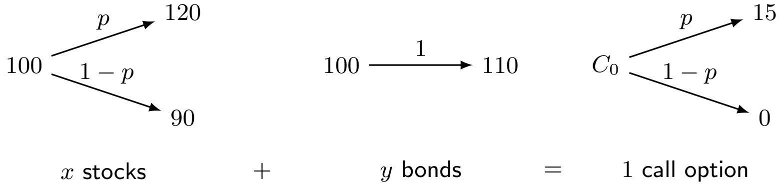


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120x + 110y &= 15 \\
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- There is a unique solution: $x = \frac{1}{2}, y = -\frac{9}{22}$. This is the replicating strategy.
- Hence, holding a call option is "equivalent" to holding $\frac{1}{2}$ stocks and owing $\frac{9}{22}$ bonds to somebody else.
- Hence, the price of the call should be given by $C_0 = 100x + 100y = \frac{100}{11}$. Done!

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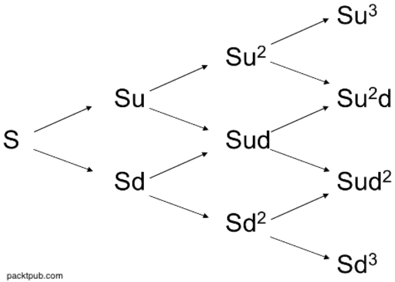
- Discrete time (multiple but finitely many periods): $N$-period binomial model



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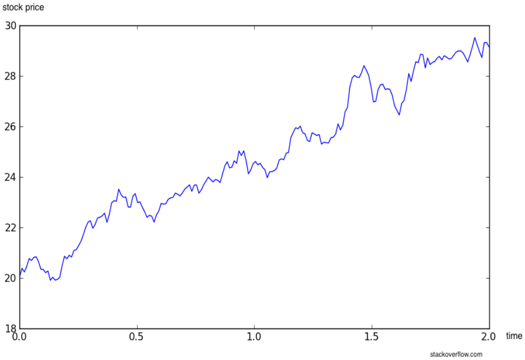
- Continuous time (real time): Geometric Brownian motion (Black-Scholes model)



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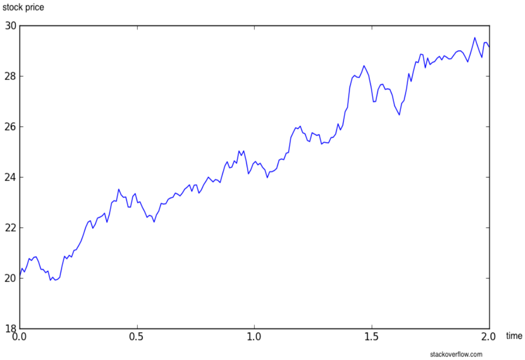


- For details and more: IE 440 Introduction to Financial Engineering ☺

Thank you! Time for kahoot again!