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A QUANTITY DISCOUNT PRICING MODEL TO INCREASE VENDOR PROFITS*

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In this paper, we analyze how a supplier can structure the terms of an optimal quantity discount schedule. The vendor's challenge is to adjust his present pricing schedule to entice his major customer to increase his present order size by a factor of "K". Optimal levels for "K" and the corresponding price discount are determined in order to maximize the supplier's incremental net profit and cash flow. Implementation issues are discussed and future research needs identified.
(INVENTORY/PRODUCTION—POLICIES, PRICING; MARKETING—PRICING; INVENTORY/PRODUCTION—DETERMINISTIC MODELS)

1. Introduction

One of the better known models of inventory control involves lot sizing with quantity discount possibilities (Buffa and Miller 1979, Chase and Aquilano 1981, Hadley and Whitin 1963, Peterson and Silver 1979, Starr and Miller 1962). Such familiarity is not surprising, given the widespread use of quantity discounts in industry today.

The traditional Quantity Discount Model, however, analyzes things solely from one perspective: that of the buyer. It is the buyer whose total inventory related cost is minimized. It is the buyer who is advised how to best respond to the fixed discount terms of the supplier.

But who speaks on behalf of the seller? How does he/she know, for instance, just what terms should be offered? How many discounts should be proposed? At what quantity break locations should they take effect? And how big should any given price discount be?

This paper presents an economic analysis of these important questions. Its viewpoint is decidedly that of the supplier. It develops the terms of an "optimal" quantity discount pricing schedule, to be used by the supplier in his efforts to obtain larger orders from his sole or major customer.

The terms of the discount schedule are determined by a mathematical model which anticipates the buyer's likely reaction to any vendor proposal. The ultimate schedule is one which maximizes the supplier's resultant economic gain, but does so at absolutely no added cost to the buyer.

2. Price Discounts: The Vendor's Leverage

Many vendors (suppliers) would doubtlessly like to see their customers (buyers) increase the size of their current orders. Larger individual orders, if maintained with the same order frequency, obviously would mean higher total annual sales. But, even if accompanied by declining order frequency rates, larger individual orders may still be quite lucrative to the vendor.

A number of specific economic advantages come to mind. First, larger individual

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orders mean the vendor will need to process fewer orders per year from this customer. Thus, with larger orders the vendor should be able to reduce his yearly order processing costs.

Potentially even more important, however, are the manufacturing cost savings made possible by larger customer orders. This is especially true for the vendor who produces the item himself. Larger orders, if produced to order, will mean longer production runs and fewer manufacturing set-ups per year.

Shipping costs represent a third potentially important influence upon the development of an optimal quantity discount schedule. Larger orders may allow the vendor to capture transportation discounts currently unavailable to him.

Finally, larger individual orders will cause a change in the current pattern of orders placed throughout the year. This will mean a shift in both the magnitude and timing of order payments from the buyer. Hopefully, the vendor will then find he has the use of more of the buyer's money, earlier in the year. This may be very important to the vendor, depending upon the size of his particular opportunity cost of capital.

Unfortunately for the vendor, however, the buyer will tend not to change his present ordering strategy. In fact, if the buyer is currently an "optimal" decision maker, he will dare not change. For to deviate from his current plan will necessarily mean his total overall inventory related costs must increase.

One common operational way to entice the buyer to order more is for the vendor to offer an attractive price discount to offset these extra costs. Starr and Miller (1962) present a model which shows the buyer how many "extra" units he can "afford" to order, assuming the vendor is prepared to offer a compensating price discount of $X\%$.

As indicated earlier, however, the Starr and Miller model, like others before and after, assumes that the price schedule already exists, and then presents algorithms which guide the buyer in responding to it. Not shown, however, is specifically how the vendor ought to develop his best price structure: that is, the product discount plan which will then encourage his customer to order precisely what the vendor desires.

3. Keeping the Buyer Satisfied

We begin our analysis with a review of the situation from the buyer's point of view. We shall often refer to him as party number one (1) in our distribution channel, and hence generally use subscript "1" to designate his set of parameters or decisions.

Let us assume a deterministic world in which we let:

D_1 = the total yearly number of units demanded by party 1 (equal to that demanded by his customers);

S_1 = the buyer's fixed order processing cost ($/order);

P_1 = the current delivered unit price paid by the buyer ($/unit);

H_1 = the buyer's yearly inventory holding cost, expressed as a percentage of the value of the item (%/year);

Q_1 = the buyer's current order size (units/order).

Then the buyer's total annual inventory related cost is:

$$\mathrm{TIC}_1\,(Q_1) = P_1 D_1 + (D_1/Q_1)S_1 + (Q_1/2)(H_1 P_1), \tag{1}$$

where the individual terms reflect the total yearly purchasing cost (TPC), total yearly order processing costs (TSC), and total yearly holding costs (THC) respectively.

With no price breaks offered, only the last two terms in (1) fluctuate with the buyer's choice of Q. Application of the well-known Wilson lot sizing formula (Harris 1915) will therefore minimize the buyer's overall total cost. Specifically, we can assume that with no price discounts available, the buyer initially orders

$$Q_1 = Q_1^* = \sqrt{2D_1S_1/H_1P_1}\,. \tag{2}$$

By substituting (2) into (1) we an also infer that the buyer's total inventory related expenses

$$\text{TIC}(Q_1^*) = D_1P_1 + D_1S_1\sqrt{H_1P_1/2D_1S_1} + (H_1P_1/2)\sqrt{2D_1S_1/H_1P_1}\,. \quad \text{(3A)}$$

Simplification of (3A) leads to the more manageable expression:

$$\text{TIC}(Q_1^*) = D_1P_1 + \sqrt{2D_1S_1H_1P_1}\,. \quad \text{(3B)}$$

Recalling our assumption of constant, uniform demand, we expect the buyer is initially now placing $N^* = (D_1/Q_1^*)$ orders each year, evenly spaced throughout that time. Suppose now that the vendor (party number two (2)) requests the buyer to boost his current order size by a factor "K".

Then with no other compensating changes, adoption of this request will raise the buyer's yearly carrying cost but reduce his annual order processing expense. Substitution of the new lot size into (1) and comparisons with (3A) and (3B) reveal that the overall effect of the new plan can ultimately be expressed as

$$\text{TIC}(KQ_1^*) = P_1D_1 + \sqrt{2D_1S_1H_1P_1}\left(1 + \left((K-1)^2/2K\right)\right). \quad (4)$$

Since TIC $(KQ_1^*) >$ TIC (Q_1^*) for all $K > 1$, it seems reasonable to assume that the buyer will demand that any changes in order size be accompanied with a corresponding price discount. This discount, as far as the buyer is concerned, must be sufficient to *at least* compensate him for his additional inventory expenses.

Mathematically, his yearly purchasing savings must be greater than or equal to these extra charges. Consequently,

$$d_KD_1 \geqslant \sqrt{2D_1S_1H_1P_1}\,(K-1)^2/2K, \quad (5)$$

where d_K is the per unit dollar discount offered when the buyer orders K times his current order size.

Rearrangement of terms, therefore, leads to a relatively simple expression for the "break even price discount." Specifically,

$$d_K(BE) = \left(\sqrt{2S_1H_1P_1/D_1}\,\right)(K-1)^2/2K. \quad (6)$$

We refer to this value as the "practical" break even discount. In point of fact, it is not fully correct, for price discounts will also tend to have the additional benefit for reducing the buyer's *per unit* holding cost. Hence, (6) overstates somewhat the amount that the buyer would demand in a price discount. A more precise version of (6) is

$$d_K(BE) = \left(\sqrt{2D_1S_1H_1P_1}\,\right)(K-1)^2/(2D_1K + K^2Q_1H_1). \quad (7)$$

For analytical convenience, however, we shall continue to use the "practical discount" rather than the "theoretical discount." Numerical differences in the two tend to be minor. But even when they are not, we can look at the "error" as representing an added incentive to the buyer (in the form of reduced costs) to go along with the larger order policy.

4. The Vendor's Viewpoint

We turn now to the vendor's point of view. The relevant parameters are expanded to include:

S_2 = the vendor's order processing and manufacturing set up cost (\$/order received);

H_2 = the vendor's nominal opportunity cost of capital, expressed as an annual percentage (%/year);

M_2 = the vendor's gross profit on sales, expressed as a percent;

P_1 = the pre-discount unit price received by the vendor.

Under the current no-discount policy, the vendor's yearly net accounting profits are given by

$$YNP_2 = D_1(M_2P_1) - (N_1S_2); \tag{8}$$

where $N_1 = N_1^* = D_1/Q_1$.

Adoption of a new quantity discount arrangement leads to the revised annual net profits of

$$YNP_2^1 = D_1(M_2P_1 - d_K) - (N_1/K)S_2. \tag{9}$$

Substituting (6) for the required discount into (9) and setting the first derivative of the resultant expression with respect to K equal to zero, we find that the vendor's "optimal" value of K is

$$K^* = \sqrt{\left((2D_1S_2)/\left(Q_1\sqrt{(2D_1S_1H_1P_1)}\right)\right) + 1.} \tag{10}$$

Further analysis of (10) leads to the surprisingly simple expression

$$K^* = \sqrt{(S_2/S_1) + 1.} \tag{11}$$

Because the discount $d_K(BE)$ is sufficient to entice the buyer to boost his order by a factor K, equation (11) indicates that the profit maximizing vendor should inform this buyer that the *only* quantity discount available is for orders K^* times as large as his current order. The corresponding price discount is that value found by substituting K^* into (6).

Equation (11) indicates that current ordering patterns will change, except in the unusual case where the vendor's order processing and manufacturing set-up costs are zero. Somewhat surprisingly, for the situation in which the two parties have similar costs ($S_2 \equiv S_1$), the model can entail order size revisions of more than 40 percent.

Clearly, maximum "action" occurs when S_2 is substantially larger than S_1. This is most likely to occur in circumstances where the vendor produces the item internally, and hence must include manufacturing set-up costs, as well as order processing costs, in his determination of the size of S_2.

The model's "financial" impact can be distinguished from its "action" impact. The former is measured by examining the economic consequences of varying K from its initial level ($K = 1$) towards K^*. The derivative of (9) with respect to K shows that the incremental profit rate decreases with increasing K. Hence, the biggest financial gains to the vendor occur early. Subsequent increases in K yield fewer and fewer economic benefits. Knowledge of such benefit behavior could be of value to the uncertain vendor who, feeling that institutional considerations realistically prohibit him from moving his customer all the way to K^*, might otherwise abandon the basic upward thrust recommended by the model.

It should also be clear that while the proposed price discount schedule may be mutually beneficial to both buyer and seller, it is certainly not equally beneficial. As now designed, the present plan earmarks nearly all economic benefits for the vendor. Other "more equitable" benefit sharing arrangements could be considered. It is possible to show, for instance, that a complete reversal in the present profit distribution rule calls for the buyer to receive the revised unit discount,

$$d_K^1 = (S_2/Q)((K-1)/K). \tag{6^1}$$

Interestingly, if granted this higher discount, the buyer would now find it in his/her best interest to increase the present order size by the same factor (11).

In other words, the size of K^* does not depend upon who will receive the fruits of the larger ordering strategy. Hence, since K^* is invariant to the sharing formula, we can expect that the *location* of the single price break should always be the same, regardless of who sets the terms.

5. Accounting for Transportation Cost Savings

In some cases, the vendor may be responsible for paying the product shipping cost for his buyer (e.g., F.O.B. destination price policy). Transportation cost savings may be generated, therefore, if the buyer boosts his present order to qualify for shipping rate breaks.

We, therefore, assume in this section the existence of a transportation shipping schedule with "m" price break quantities: $q_0, q_1, q_2, \ldots, q_m$, and corresponding per unit transportation costs: $t_0, t_1, t_2, \ldots, t_m$.

The vendor is informed that for shipping an order of size Q_0, such that

$$q_j \leqslant Q_0 < q_{j+1}, \tag{12}$$

the relevant transportation rate is "t_j". Naturally,

$t_{j+1} < t_j$, for all j, including

t_0 = initial, present "no discount" transportation rate.

For convenience and compatibility with earlier analysis, we express the locations of the break quantities q_j as "multiples" of the buyer's original order size, Q_1. Thus we define

$$k_j = q_j/Q_1, \qquad j = 0, 1, 2, \ldots, m, \tag{13}$$

and note that the transportation rate t_j will apply, provided the buyer revises his present order size by a factor "K", where

$$k_j \leqslant K < k_{j+1}. \tag{14}$$

Thus, the relevant t_j is a function of the level of K selected and can be expressed henceforth as "$t_j(K)$". We use similar notation below for "d_K."

The vendor's revised annual yearly profit is

$$YNP_2^1 = D_1(M_2P_1 - d(K)) - (N_1/K)S_2 - D_1t_j(K), \tag{9^1}$$

where the final term accounts for the vendor's total annual transportation cost.

Transportation cost curves will typically behave in a nonlinear, discontinuous fashion. Such behavior makes optimization of (9^1) by classical differentiation techniques impossible. The special structure of the situation, however, makes location of the revised optimal level of K (i.e., K^* (revised)) reasonably simple. The fact that $d(K)$ is monotonically increasing while $t_j(K)$ is a never increasing step function leads us to conclude that

$$K^* \text{ (revised)} = \text{either} \begin{cases} K^* \text{ (orginal)} = \sqrt{S_2/S_1 + 1} \\ \text{or} \\ \text{some } k_j > K^* \text{ (original).} \end{cases} \tag{11^1}$$

In other words, the revised answer will always be the original answer or some one of the *higher* price break multiples referred to in (13).

Location of the revised K^*, therefore, becomes a matter of examining the full economic consequences via (9^1) of each candidate order multiple, including K^* original. Once identified, the buyer is again told that the only price break available (at K^* revised) is an amount indicated by (6).

It is easy to understand how the availability of transportation rate breaks might entice the supplier to push K well above the initial K^* level. The shrewd vendor, however, tempers his enthusiasm for the higher ordering factor because he knows that transportation and set up savings must ultimately be "bought" from the buyer via forever increasing product discounts.

The $j + 1$ transportation discount is profitable to the vendor provided

$$D_1(t_j - t_{j+1}) + D_1S_2/(Q_1(k_{j+1} - k_j)) > D_1(d(k_{j+1}) - d(k_j)). \tag{15}$$

6. Expanding the Basic Model: Cash Flow Considerations

The initial model can also be expanded in other important ways. One limitation, for instance, is its current focus upon accounting profits rather than discounted cash flows. This is a potentially important flaw because the buyer's order size can dramatically affect the size and timing of cash payments available to the vendor.

It is assumed that each new order will be accompanied by a check from the buyer as payment in full. It is useful to view such a payment stream as a "quasi annuity." Since we assume the first check is received immediately, rather than at the end of the first period, we have an arrangement called an "annuity due." (Hummel and Seebeck 1971)

Assuming that the vendor's yearly nominal opportunity cost of capital, H_2, is compounded continuously, the net present value to the vendor can be written

$$NPV_2 = (((M_2P_1 - d_K)KQ_1 - S_2)((1 - \exp^{-H_2})/(1 - \exp^{-H_2K/N_1^*})). \tag{16}$$

Note when $K = 1$, no discount applies and the equation measures only the present value of the original pre-discount situation.

In theory, the optimal value for K can be found by applying standard methods. In practice, however, calculus leads to a very complicated nonlinear expression whose positive roots ae quite difficult to obtain, except by numerical search techniques of one sort or another (see Powell 1970). It is usually just as easy to apply such techniques toward maximizing (15) directly.

7. Discussion/Summary

Quantity discounts are a common occurrence in many industries today. Both buyers and sellers potentially benefit from such arrangements. Even in today's high interest economy, many managers seem quite ready to trade off real price savings against more intangible carrying costs.

This paper presents an analytical approach to designing the terms of a vendor oriented, optimal quantity discount pricing schedule. It describes how a seller can improve his financial position by properly anticipating his major customer's reaction to any proposed discount plan. As such, it formalizes procedures which up to now have been done largely in less rigorous ways.

The supplier may find the paper's results attractively simple to understand and convey. Before implementing them, however, a number of important practical issues should also be considered. These include questions of data estimation, legal litigation, and competitive reaction.

In order to operationalize (6) and (11) the supplier must obtain numerical estimates of D_1, S_1, and H_1. While errors can occur in any of these values, H_1 is likely to be the most difficult to accurately determine. It is doubtful, for instance, that the supplier's own holding cost percentage, H_2, will closely match that of his buyer.

There probably is no one best method to estimate H_1. One plausible approach, however, is to use the earlier, pre-discount, actual lot size decisions of the buyer to impute H_1's past level. Important subsequent changes in interest rates, tax rates, etc. could then be included in order to calculate an improved, updated estimate. Such ad hoc procedures are not likely to yield perfectly accurate data estimates. Thankfully, however, perfection is not necessary. As a practical matter, only reasonably accurate data inputs are required. The relative insensitivity of K^*, $d(K^*)$ and YNP_2 to measurement errors permits the establishment of "good" discount plans which accomplish much of what the supplier wants even without access to perfect input data.

The supplier should normally expect few legal problems from implementing the recommended plan. In the United States the Robinson-Patman Act permits quantity discounts to be offered provided: 1. the discount terms are made available to all interested customers; and 2. the specific price concessions can be justified in terms of associated reductions in the supplier's manufacturing or distribution cost (Howard 1964). Both conditions can readily be satisfied here.

Management must be more cautious, however, in attempting to gauge likely competitive reaction. Other suppliers may counter this plan with matching or superior discount terms. The former has already been planned for in the paper's present (no sales growth) assumptions. But other more aggressive responses could possibly occur, and this risk must be evaluated before implementing the recommended strategy.

Equal care must be given to the applicability of the paper's one-customer assumption. Certainly such suppliers do exist. But they tend to be nearly invisible to the public; for they often are small, closely-held or privately owned companies, producing exclusively for other larger manufacturers or distributors. Sometimes these suppliers are common job-shops, making customized items for an individual buyer (e.g. Raytheon and the U.S. Defense Department). At other times they are manufacturing arms or divisions of independently run parent companies (e.g. Western Electric and AT&T).

Despite such variety of application, management should know that there are sizable risks in using this model within a multiple-customer situation. Once set, the specific discount terms legally must be made available to all qualified buyers (i.e. all customers willing to order at or above the established price break). System-wide supplier net profits can theoretically go up, down or remain the same, when discount terms are set solely upon the economic input data of only one customer. Additional research is needed to further investigate this discount problem on behalf of multiple-customer suppliers.

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