The Economic Order Quantity for Freight Discount Costs

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Abstract: In this note we give algorithms to solve an important practical problem, the classical E.O.Q. model with set-up cost including a fixed cost and freight cost, where the freight cost has a quantity discount (economies of scale).

■ One of the important extensions of the classical Economic Order Quantity model is the quantity discount (Buffa and Miller [2], Hadley and Whitin [3], Johnson and Montgomery [5], McClain and Thomas [8], and Silver and Peterson [9]). The traditional quantity discount model analyzes solely the unit purchase price discount. However, in many practical problems, the discount may occur in the freight cost. This note analyzes and provides algorithms to solve such a problem.

The assumptions of this problem are essentially the same as the E.O.Q. model except for the set-up cost structure. Let D denote the deterministic stationary annual demand rate, h denote the unit inventory holding cost, c denote the unit purchase cost, where c is independent of the order quantity. The setup cost contains not only a fixed cost but also a freight cost, and is a function of the order size. Let Q denote the amount of each order; the setup cost is defined as

$$S(Q) = F_j + A, \text{ if } N_{j-1} < Q \leq N_j, \quad j = 1, 2, \ldots, J, \text{ where} \tag{1}$$

$$F_j < F_{j+1} \text{ and } F_{j+1}/N_{j+1} < F_j/N_j. \tag{2}$$

N_0 is the minimum quantity that can be freighted (usually zero), and N_J is the maximum quantity that can be freighted (usually unlimited). A is the fixed cost and F_j is the freight cost for order size Q if $N_{j-1} < Q \leq N_j$. The inequality $F_{j+1}/N_{j+1} < F_j/N_j$ implies that there is some quantity discount in the freight cost for changing the order size from N_j to N_{j+1}. Since c is constant we will exclude the annual purchase cost cD from our cost function. Let

$$K_j(Q) = (F_j + A)\, D/Q + 1/2\,(hQ). \tag{3}$$

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Then the total annual cost can be expressed as

$$K(Q) = S(Q)\, D/Q + 1/2\,(hQ) = K_j(Q) \quad \text{if } N_{j-1} < Q \leq N_j,\ j = 1, 2, \ldots, J. \tag{4}$$

Note that $K_j(Q)$ is a convex function of Q and has its minimum point at

$$Q_j = \sqrt{2(F_j + A)D/h}. \tag{5}$$

Hence $K(Q)$ is convex in each section $N_{j-1} < Q \leq N_j$. Our purpose is to find an ordering quantity Q^* that minimizes the total annual cost.

An important special case, say Special Model 1, is the model with

$$N_0 = 0,\ N_j = jP, \text{ and } F_j = R_0 + (j-1)R, \text{ where} \tag{6}$$

$$R_0 > R, \tag{7}$$

and P, R_0 and R are positive constants. Namely, there is a basic charge R_0 for the first P freight units and there is an incremental charge R for each P more freight units. (Note that the freight unit can be in terms of weight or volume.) The inequality (7) implies that there is some quantity discount in the freight cost for changing the order size from jP to $(j + 1)P$. This special case arises in many practical situations, for example, in postal service charges.

If we relax the inequality (7) to

$$R_0 \geq R, \tag{8}$$

then there is another important special case, say Special Model 2, the model with $R_0 = R$. In this special model, we have

$$N_0 = 0,\ N_j = jP, \text{ and } F_j = jR. \tag{9}$$

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Interpret P as the capacity of each cargo and R as the unit cargo cost. Hence, in this special model, the freight cost of each order equals jR when the order size satisfies $(j-1)P < Q \leq jP$. Namely, the freight cost is the cargo cost in each order. Since $F_{j+1}/N_{j+1} = F_j/N_j = R/P$ for $j = 1, \ldots, J-1$, there is no quantity discount for changing the order size from jP to $(j+1)P$. This special model has attracted much attention (See, for example, Aucamp [1], Iwaniec [4], Lippman [6], [7], Silver and Peterson [9].) Aucamp [1] has provided a solution to this special model. However we will see later that his method is not applicable to our model.

In this note, we will provide an algorithm to solve our general model. We will also provide a more efficient algorithm to solve Special Model 1 as well as Special Model 2.

Results

The following theorem provides an algorithm to solve our general model.

Algorithm I.

Step 1. Compute $Q_0 = \sqrt{2AD/h}$. Let i be the largest index such that $Q_0 > N_i$. If $i = J$, then $Q^* = N_J$ and stop. Otherwise, go to step 2.

Step 2. Compute Q_j by (5) and compare it with N_j for $j = i+1, i+2, \ldots$ until the first index $k \leq J$ is found such that $Q_k \leq N_k$, then go to step 3. If $Q_j > N_j$ for all $i+1 \leq j \leq J$, then set $k = J+1$ and go to step 4.

Step 3. Compute the cost $K(Q)$ by (4) for $Q = N_i$ (provided $N_i > 0$), $N_{i+1}, \ldots, N_{k-1}$ and Q_k. Select the one that yields the minimum cost as Q^* and stop.

Step 4. Compute the cost $K(Q)$ by (4) for $Q = N_i$ (provided $N_i > 0$), $N_{i+1}, \ldots, N_{k-1}$. Select the one that yields the minimum cost as Q^* and stop.

Theorem 1. The value of Q^* found in Algorithm I is the optimal order quantity for the general model.

(All proofs are in the appendix.)

Remarks:

(1) Since no shortages are allowed in the EOQ model, we assume $Q > 0$ in this note. Hence we will delete the case $Q = 0$ in our discussion. For example, if $i = 0$ in step 1, and if $N_0 = 0$ then we will delete $Q = N_0 = 0$ in steps 3 and 4.

(2) It is easy to see that it takes $0(J)$ operations in Algorithm I to find Q^*.

(3) Note that $\sqrt{2AD/h}$ is the original EOQ (without considering the freight cost). Hence, Algorithm I and Theorem 1 show that if the original EOQ is greater than N_i for some i, then any $Q < N_i$ cannot be the optimal order quantity. Theorem 1 also shows that for any $Q > N_{k-1}$, where k is found in step 2, only Q_k can possibly be an optimal solution. Also for any Q between N_i and N_{k-1} we only need to consider $\{N_i, \ldots, N_{k-1}\}$ as the candidates for the optimal solution.

(4) Aucamp [1] has shown that in Special Model 2 the optimal order quantity Q^* is either the standard EOQ or one of the two surrounding integer multiples of P. It is easy to see that we need to consider more candidates for Q^* in our more general model.

Example 1. Considering the problem with the following parameters;

$A = \$300$,
$D = 2{,}000$ units/year,
$h = \$2$/unit/year,
$F_j = jB(1 - 0.02(j-1))$, and
$N_0 = 0$, $N_j = 100\,j$, for $j = 1, \ldots, J$,
where $J = 30$ and $B = \$200$.

Note that N_j and F_j satisfy (2). Applying Algorithm I to this problem, we have $Q_0 = \sqrt{2AD/h} = 775$. Since $N_7 < 775 < N_8$, $i = 7$. Also the first index k such that $Q_k \leq N_k$ is $k = 25$ and $Q_{25} = 2408$. Comparing the cost corresponding to $Q = 700, 800, \ldots, 2400$ and $Q = 2408$, we see that $Q^* = 1700$ and $K(Q^*) = \$4773$ give the optimal solution. However, if the original EOQ of 775 were ordered instead of 1700, then the cost would be $K(775) = \$5100$.

Algorithm I can be used to solve our general model. For the above special cases, it can be shown that the following more efficient algorithm can be used to solve Special Model 1 (for $R_0 > R$) as well as Special Model 2 (for $R_0 = R$).

Algorithm II.

Step 1. Compute $Q_0 = \sqrt{2(A + R_0 - R)\,D/h}$. If $Q_0 \geq JP$, then $Q^* = JP$. Otherwise go to step 2.

Step 2. Let i be the index such that $iP < Q_0 \leq (i+1)P$ and compute $Q_{i+1} = \sqrt{2(F_{i+1} + A)\,D/h}$. If $Q_{i+1} > (i+1)P$, then go to step 3. Otherwise go to step 4.

Step 3. Compute the cost $K(Q)$ for $Q = iP$ (provided $i > 0$) and $Q = (i+1)P$. Select the one that yields the minimum cost as Q^* and stop.

Step 4. Compute the cost $K(Q)$ for $Q = iP$ (provided $i > 0$) and $Q = Q_{i+1}$. Select the one that yields the minimum cost as Q^* and stop.

APPENDIX:

Proof of Theorem 1. If $Q_0 \leq N_J$ in Step 1, we will consider three possible regions for the order quantity Q; $0 < Q \leq N_i$, $N_i < Q \leq N_{k-1}$, and $N_{k-1} < Q$, where i and k are found in steps 1 and 2.

For the first region, $0 < Q \leq N_i$, note that the function $AD/Q + 1/2(hQ)$ is convex in Q and has its minimum point at $Q_0 = \sqrt{2AD/h}$. Hence $AD/Q + 1/2(hQ)$ is decreasing in

Q for $0 < Q \le N_i < Q_0$, where i is found in step 1. Thus, for any $Q < N_i$, we have $N_{j-1} < Q \le N_j$ for some $1 \le j \le i$ and

$$\begin{aligned} K(Q) &= K_j(Q) \\ &= (F_j + A)D/Q + 1/2(hQ) \\ &= F_j/Q + [AD/Q + 1/2(hQ)] \\ &> F_j/Q + [AD/N_i + 1/2(hN_i)] \\ &\ge F_j/N_j + AD/N_i + 1/2(hN_i), \\ &\quad \text{(because } Q \le N_j) \\ &\ge F_i/N_i + AD/N_i + 1/2(hN_i), \\ &\quad \text{(because } F_j/N_j \ge F_i/N_i) \\ &= K_i(N_i) \\ &= K(N_i). \end{aligned} \tag{10}$$

Hence, we see that any $0 < Q < N_i$ cannot be an optimal solution. In other words, in the first region, $0 < Q \le N_i$, only $Q = N_i$ is possible to be an optimal solution.

For the second region, $N_i < Q \le N_{k-1}$, note that $Q_j > N_j$ for any $i < j \le k - 1$ by the definition of k. Since $K_j(Q)$ is a convex function of Q and Q_j is the minimum point of $K_j(Q)$, $K_j(Q)$ is a decreasing function of Q for $Q \le N_j$. Thus, for any $N_i < Q \le N_{k-1}$, *let* $N_{j-1} < Q < N_j$, where $i < j \le k - 1$. We have

$$K(Q) = K_j(Q) \ge K_j(N_j) = K(N_j). \tag{11}$$

Hence, we see that in the second region, $N_i < Q \le N_{k-1}$, we need only consider $Q = N_{i+1}, \ldots, N_{k-1}$ as candidates for the optimal solution.

Now consider the third region, $Q > N_{k-1}$. First note that $N_{k-1} < Q_k \le N_k$, where k is found in step 2, because $Q_k > Q_{k-1} > N_{k-1}$ and $Q_k \le N_k$ by the definition of k. Note also that for any $1 \le j \le J$, $K_j(Q) < K_{j+1}(Q)$ for all Q, because $F_j < F_{j+1}$. Hence, we have

$$\begin{aligned} K(Q_k) &= K_k(Q_k) \\ &\le K_k(Q) && \text{for all } Q \\ &\le K_j(Q) && \text{for all } j > k. \end{aligned} \tag{12}$$

Hence, if $k < J$, then in region 3 only Q_k can possibly be an optimal solution. Combining with the above discussion, we see that the value of Q^* found in step 3 is the optimal order quantity. If $k = J + 1$, then it is obvious than any $Q > N_J$ is not a feasible solution. Hence the value of Q^* found in step 4 is the optimal solution.

Finally, if $Q_0 > N_J$, then for any $0 < Q < N_J$, we can show that $K(Q) > K(N_J)$ by the same argument as that of (10). Hence the optimal order quantity is N_J.

Q.E.D.

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