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Analysis of a deterministic demand production/inventory system under nonstationary supply uncertainty

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In this article we investigate a periodic review inventory model under deterministic dynamic demand and supply unavailability. In a given period, supply is either available or completely unavailable with given probabilities. Supply unavailability probabilities are nonstationary over time. We show the optimality of an order-up-to level policy, and obtain a newsboy-like formula that determines the optimal order-up-to levels. Our formula would provide guidance as to the appropriate amount of inventory to stock in the face of uncertainties in the supply process.

1. Introduction

In the classical formulations of the inventory models it is generally assumed that supply or production capacity is either unlimited over the planning horizon, or it is equal to a deterministic level. In most of the production/inventory models that involve uncertainties in the environment, attention has been focused on the probabilistic modeling of the customer demand side. Therefore, until recent years uncertainties in the supply side have not received the amount of treatment they deserved. In particular, in the past decade widespread application of just-in-time (JIT) practices in supply situations has increased the importance of modeling the supplier uncertainties. Several factors, such as shortages in material availability, unexpected machine breakdowns, process adjustments and strikes, make the treatment of supply uncertainty an important issue in the analysis of inventory problems. For a recent review of supply/yield uncertainty models, see Yano and Lee [1].

We observed in a real-life shipment problem a similar situation, where to consider *sure* delivery times was not a proper assumption. In the system observed, daily due-dates were set ahead of time for the deliveries and only financial actions were taken if a delivery due date is not satisfied. The aim of this article is to consider revised ordering quantities as another possible action. The policy may not be applicable unless real-time delivery monitoring is achieved, but still it gives a benchmark to any other solution proposed.

In his pioneering work, Silver [2] considers an EOQ model where the quantity received is a random proportion of the quantity requisitioned. He obtains modified EOQ formulas under such a supply uncertainty model. In the spirit of proportional supply uncertainty, Shih [3] and Ehrhardt and Taube [4] consider single-period models where the demand in the period is a random variable. Henig and Gerchak [5] analyze a periodic review model where the quantity received is a random multiple of the order size, and show the optimality of an order-up-to type policy.

Ciarallo *et al.* [6] analyze a stochastic demand production/inventory model where the available capacity in a given period is a random variable. They show the optimality of order-up-to type policies, but the generality of their model leads to intractibility in computing the order-up-to levels. Under a similar capacity uncertainty model and average cost per period criterion, Güllü [7] proposes a solution for the optimal order-up-to level by constructing an analogy with queueing systems. Wang and Gerchak [8] extend the model of Ciarallo *et al.* to incorporate random yield as well as random capacity.

Parlar and Berkin [9] propose an EOQ type formulation where the supply is available or disrupted for random durations in the planning horizon. Karaesmen *et al.* [10] extend their model to incorporate correlations between supply availability and disruption durations. Parlar *et al.* [11] consider a periodic review model with setup costs using a Markovian supply availability structure in which the supply is either available or completely unavailable. They show the optimality of (S,s) policies where s depends on the supply state in the previous period.

Anupindi and Akella [12] consider three supply models based on the type of delivery contracts that a buyer may have with the suppliers. The first model that they study is a one-delivery contract with all the order quantity deliv-

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ered either in the current period with a probability, or in the next period. We propose a similar model, where all the order quantity is either delivered or cancelled. This situation is very similar to JIT supply chains, where the supplier has deliveries every period (periods are usually days). If the supplier cannot meet the order on time for one reason, the supplier informs the buyer that the order is cancelled. Consequently, the buyer's new order takes this information into account.

In this article we consider a single-item, periodic review inventory system. We assume that the demand sequence over the planning horizon is deterministic and dynamic. We assume linear procurement, holding and backorder costs and the objective is to minimize total costs over the planning horizon. The supply uncertainty structure we employ is similar to that considered by Parlar *et al.* We assume that in any period the supply is either fully available or unavailable. The supply is assumed to follow a Bernoulli process. The availability and unavailability probabilities are nonstationary and independent from one period to another. The assumption on the supply structure may not directly follow some real practices. However, for the real-life case the authors of this article observed, the buyer can be in a position of revising the quantity ordered at the end of each period. Hence, if a delivery is not made in a certain period, the total quantity to be delivered next period is computed taking the supply information into account, rather than leaving that late shipment in transit.

In this article we make three contributions. First, under the above assumptions we present a nonstationary supply uncertainty model integrated with a dynamic demand structure. Secondly, we demonstrate the optimality of an order-up-to policy and explicitly characterize the optimal order-up-to levels. Finally, as the main contribution that makes our work different from the previous research, we provide a simple newsboy-like formula for computing the optimal order-up-to levels over the planning horizon. The rest of the article is organized as follows. In Section 2 we present our cost and supply model and analyze the form of the optimal ordering policy. In Section 3, the optimal parameters of the policy, the order-up-to levels are explicitly characterized and a one-pass algorithm is presented for computing the levels. It turns out that the optimal order-up-to level for any period n is equal to the cumulative demand of the following K_n periods. These levels can be determined by simply checking a newsboy-like ratio. Non-zero fixed ordering cost is discussed in Section 4. We conclude the study and discuss some extensions in Section 5.

2. The model and the optimal ordering policy

Notation

- N length of the planning horizon;
- D_n demand in period n for $n = 1, 2, \ldots, N$;
- p_n probability that the supply is unavailable in period n, $0 \leq p_n < 1$;
- q_n probability that the supply is available in period n, $q_n = 1 - p_n$;
- $S_n(u)$ realized supply in period n if an amount u is requisitioned;
- h holding cost per unit per period;
- b backorder cost per unit per period;
- c purchasing or production cost per unit; charged for the amount received (we assume that $b > c$).

Suppose an amount of u is requisitioned in period n. The realized supply is $S_n(u) = 0$ with probability p_n, and $S_n(u) = u$ with probability q_n. Let $L_n(y)$ denote the single-period inventory costs incurred at the end of period n, where the inventory level after the realization of supply is y:

$$L_n(y) = h \max(0, y - D_n) + b \max(0, D_n - y). \quad (1)$$

For $n = 1, 2, \ldots, N+1$ we define $C_n(I)$ as the expected cost of operating the inventory model through periods $n, n+1, \ldots, N+1$ with the application of the best ordering strategy when the inventory level at the beginning of period n is I. We impose the end condition that $C_{N+1}(I) = 0 \; \forall \, I$. The dynamic programming recursion for this problem is

$$C_n(I) = \min_{u \geq 0} \{ E[L_n(I + S_n(u)) + cS_n(u)] + E[C_{n+1}(I + S_n(u) - D_n)] \} \quad (2)$$

for $n = 1, 2, \ldots, N$. In period n, an amount of u units is ordered, and an amount of $S_n(u)$ is received. The first term in (2) stands for the expected cost for period n. After the realization of demand, the inventory level drops to $I + S_n(u) - D_n$, and the expected costs under applying the best ordering strategy through periods $n+1, \ldots, N+1$ are incurred. The expectations are taken over the supply variable.

The expected one-period inventory related cost, $E[L_n(I + S_n(u))]$, can be written as

$$E[L_n(I + S_n(u))] = p_n L_n(I) + q_n L_n(I + u). \quad (3)$$

Using (3) we can write (2) as

$$C_n(I) = \min_{u \geq 0} \{ p_n \{ L_n(I) + C_{n+1}(I - D_n) \} + q_n \{ L_n(I + u) + cu + C_{n+1}(I + u - D_n) \} \}. \quad (4)$$

Note that the first two terms in the minimization bracket in (4) do not depend on u. For ease of notation we define the auxiliary function $G_n(y) := L_n(y) + cy + C_{n+1}(y - D_n)$. By making the transformation $y = u + I$, (4) reduces to

$$C_n(I) = p_n G_n(I) - cI + q_n \min_{y \geq I} G_n(y). \quad (5)$$

We first observe that the basic ingredient of our formulation, the single-period cost function $L_n(y) + cy$ is convex in y. This observation leads to the following series of results.

For ease of exposition we provide the proofs of the theorems in Appendix A.

Theorem 1. *For* $n = 1, 2, \ldots, N$:
(i) $G_n(y)$ *is convex in* y;
(ii) $C_n(I)$ *is convex in* I;
(iii) *the optimal ordering policy that minimizes the inventory problem under supply uncertainty is of order-up-to type. There exists* y_n^* *that minimizes* $G_n(y)$, *and an amount of* $y_n^* - I_n$ *is ordered whenever the inventory level at the beginning of period n,* I_n *is less than or equal to* y_n^*. *Otherwise, no order is given.*

3. Characterization of the optimal order-up-to levels

In this section we explicitly characterize the order-up-to levels of the optimal policy. Passing through various steps we will show that the optimal order-up-to level for period n is equal to the cumulative demand of K_n periods (including the demand for period n):

$$y_n^* = \sum_{i=n}^{n+K_n-1} D_i \quad \text{for some } 1 \le K_n \le N - n + 1.$$

Moreover, progressing backwards, given that the optimal order-up-to level for period $n+1$ is equal to K-period demand, the optimal order-up-to level for period n can be determined by checking the inequality

$$\left(\sum_{i=0}^{m-1} \prod_{j=1}^{i} p_{n+j} \Big/ \sum_{i=0}^{N-n} \prod_{j=1}^{i} p_{n+j} \right) \le \frac{b-c}{h+b}$$

for $m = 1, 2, \ldots, K$.

The steps of our exposition can be summarized as follows. First, we show that the order-up-to level for period n is bounded below by the demand for this period, and it is bounded above by the demand for period n plus the order-up-to level for period $n+1$. Next, we explicitly write $G_n(y)$, and using this representation we obtain the characterization for the order-up-to level for period n.

Proposition 1. $y_n^* \ge D_n \ \forall\, n = 1, 2, \ldots, N$.

Proposition 2. $y_n^* \le D_n + y_{n+1}^* \ \forall\, n = 1, 2, \ldots, N$, *where* $y_{N+1}^* = 0$.

For ease of notation define $D(j,k) := \sum_{i=j}^{k} D_i$, the cumulative demand over periods j through k. In the next result we demonstrate that the cost-to-go function $G_n(y)$ for period n can be written explicitly in terms of cost, demand, and supply parameters. This representation will be used to obtain the optimal order-up-to levels.

Proposition 3. *For* $y \le y_{n+1}^* + D_n$,

$$\begin{aligned} G_n(y) = {} & L_n(y) + cy + \sum_{i=1}^{N-n} \left(\prod_{j=1}^{i} p_{n+j} \right) \\ & \times \{ L_{n+i}(y - D(n, n+i-1)) + c(y - D(n, n+i-1)) \} \\ & + \sum_{i=1}^{N-n} \left(\prod_{j=1}^{i-1} p_{n+j} \right) c(y - D(n, n+i-1)) \\ & + \sum_{i=1}^{N-n} q_{n+i} \left(\prod_{j=1}^{i-1} p_{n+j} \right) G_{n+i}(y_{n+i}^*), \end{aligned} \tag{6}$$

where $\sum_1^0 = 0$ *and* $\prod_1^0 = 1$ *by convention.*

For the rest of the exposition we define $Q_n(k) = \sum_{i=0}^{k} \prod_{j=1}^{i} p_{n+j}$, $k = 0, 1, \ldots$. Note that $Q_n(k)$ is increasing in k.

Proposition 4. *Assume that* $y_{n+1}^* = D(n+1, n+K)$ *for some* $n \in \{1, 2, \ldots, N-1\}$ and $1 \le K \le N - n$. *Then, for* $i = 1, 2, \ldots, K$,

$$G_n(D(n, n+i)) \le G_n(D(n, n+i) - \eta) \quad \forall\, 0 \le \eta \le D_{n+i} \tag{7}$$

if and only if

$$\frac{Q_n(i-1)}{Q_n(N-n)} \le \frac{b-c}{b+h}. \tag{8}$$

Notice that the inequality established in Proposition 4 does not depend on the demand sequence, only on the supply unavailability probabilities from period $n+1$ through period N, and the cost parameters. Proposition 4 also guarantees that the optimal order-up-to levels always occur in one of the cumulative demand points. If there is a benefit of increasing (decreasing) the order-up-to level a small amount η, it should be increased (decreased) up to the next cumulative demand point. This observation is stated in the next corollary.

Corollary 1. $y_n^* \in \{D_n, D(n, n+1), \ldots, D(n, N)\}$ *for* $n = 1, 2, \ldots, N$.

We now state the central result of this article.

Theorem 2. *The optimal order-up-to level for period* $n \in \{1, 2, \ldots, N\}$, y_n^* *is equal to the* K_n*-period demand* $D(n, n+K_n-1)$ *for some* $1 \le K_n \le N - n + 1$, *with* $K_N = 1$. *Given that* $y_{n+1}^* = D(n+1, n+K)$ *(K-period demand) for some* $1 \le K \le N - n$ *and* $n \le N - 1$, *let*

$$K' = \max\left\{ i = 1, 2, \ldots, K : \frac{Q_n(i-1)}{Q_n(N-n)} \le \frac{b-c}{h+b} \right\}. \tag{9}$$

Then, $y_n^* = D(n, n+K')$. *If no such* K' *exists, then* $y_n^* = D_n$.

The following algorithm determines the optimal order-up-to levels.

Algorithm:
Step 0. $K = 1$ $(y_N^* = D_N)$.
Step 1. For $n = N - 1$ to 1:
Find K' satisfying (9). $y_n^* = D(n, n + K')$, set $K = K' + 1$. If no such K' exists, set $y_n^* = D_n$ and $K = 1$.

Observe that both $Q_n(i-1)/Q_n(N-n)$ and $(b-c)/(h+b)$ lie in $[0,1]$. Therefore, the inequality given by (8) provides consistent comparisons for determining the optimal order-up-to levels.

An immediate special case is when the supply availability is stationary over time, $p_n = p < 1 \; \forall \, n = 1, 2, \ldots, N$. In this case, the inequality in (9) reduces to $(1-p^i)/>(1-p^{N-n+1}) \le (b-c)/(h+b)$ (since $Q_n(i-1) = \sum_{k=0}^{i-1} \prod_{j=1}^{k} p = (1-p^i)/(1-p)$ and $Q_n(N-n) = (1-p^{N-n+1})/(1-p)$). The intuitive reasoning behind the optimality inequality in this case is as follows. $(1-p^i)$ is the probability that there exists at least one supply period over the following i periods, and $(1-p^{N-n+1})$ is the probability that there exists at least one supply period over the remaining periods until the end of the planning horizon. If the former probability is sufficiently smaller than the latter, then an order for an additional period's demand is given. The term sufficiently smaller is clarified by the ratio of the unit backorder cost over the unit holding cost.

As an extension of the above, consider the situation where there is a long-term agreement between the supplier and the buyer. In other words, N, the horizon length, can be a very large number. So the inequality becomes approximately $1 - p^i \le (b-c)/(h+b)$. For the buyer to order only up to the current period's demand and carry very low inventory to justify JIT operation (that is, for no K' satisfying (9) to exist) the probability of the supplier's not satisfying the order should be small. Actually, in this case if $p < (h+c)/(h+b)$, the buyer will consider only the current period's demand in finding the order-up-to quantity. In the long run the amount of inventory carried by the supplier will be very small for such low values of p.

We conclude the section by noting some other special cases. If $p_n = 0 \; \forall \, n$ (deterministic supply availability), then the inequality given by (8) never holds, and therefore $y_n^* = D_n \; \forall \, n = 1, 2, \ldots, N$. If $h = c = 0$, then the inequality given by (8) always holds $y_n^* = D(n, N) \; \forall \, n = 1, 2, \ldots, N$. Another interesting special case, formalized in Corollary 2 below, nicely demonstrates the impact of nonstationarity in the supply process on the order-up-to levels.

Corollary 2. *Suppose that $p_{n+1} = 0$ for some $n = 1, 2, \ldots, N-1$. Then $y_n^* = D_n$.*

Proof: In this case $Q_n(i-1) = Q_n(N-n) = 1$, and no K' satisfying (9) can be found. Hence, $y_n^* = D_n$. The intuitive explanation is as follows. If the requested supply is going to be realized with certainty in the next period, then there is no reason for taking extra precautions in this period. ■

4. Non-zero fixed ordering cost

In our analysis the fixed ordering costs are assumed to be zero. In this section we investigate the possible extensions for the case where the fixed ordering cost takes positive values. It should be noted that whenever fixed costs are incorporated into such a supply uncertainty model, there are two possibilities in accounting for the fixed costs: a fixed cost is incurred every time an order is made (independent of whether or not it is realized), or the fixed cost is incurred only if the order is realized. Here, we sketch the optimal policy for the latter case (see [11] for the investigation of the former case under a different demand/supply structure). We define the random ordering cost function as

$$J(S_n(u)) = \begin{cases} k + cS_n(u) & \text{if } S_n(u) > 0; \\ 0 & \text{if } S_n(u) = 0, \end{cases}$$

where k is the fixed cost. The dynamic programming recursion becomes

$$C_n(I) = \min_{u \ge 0} \{ E[J(S_n(u)) + L_n(I + S_n(u))] + E[C_{n+1}(I + S_n(u) - D_n)] \}.$$

Observe that

$$E[J(S_n(u))] = k \Pr\{S_n(u) > 0\} + cE[S_n(u)] = k \Pr\{S_n(u) > 0\} + cq_n u = (k 1_{(u>0)} + cu) q_n.$$

Following similar steps to those in Section 2, we rewrite $C_n(I)$ as

$$C_n(I) = p_n G_n(I) - cI + q_n \min_{u \ge 0} \{ k 1_{(u>0)} + G_n(I + u) \},$$

where $G_n(y) = cy + L_n(y) + C_{n+1}(y - D_n)$. It can be proved that the functions $C_n(I)$ and $G_n(y)$ are k-convex $\forall \, n = 1, 2, \ldots, N$. The proof goes along the conventional lines [13]. k-convexity of $G_n(y)$ establishes that the optimal policy is of (s_n, S_n) type, where S_n minimizes $G_n(y)$ and s_n is such that $G_n(s_n) = G_n(S_n) + k$. It is not difficult to characterize S_n as the cumulative demand points (a similar characterization with the zero fixed-cost case). However, this need not be true for s_n's. For example, consider the case where $n = N$. In this instance one can show that $S_N = D_N$ and $s_N = D_N - (k/(b-c))$.

5. Extensions and conclusion

In this article we analyzed a deterministic demand inventory problem under supply unavailability. We obtained a simple formula that determines the policy parameters of an optimal order-up-to level policy. Our formula would provide guidance as to the appropriate amount of inventory to stock in the face of uncertainties in the supply process.

The fact that order-up-to levels are the sum of K-period demands may raise the possibility of obtaining a Wagner–Whitin type planning horizon result. For the case where p_n's are stationary over time and under linear ordering costs, one can show that the order-up-to levels in terms of periods covered (i.e., K) are non-increasing (Güllü *et al.* [14]). For this case, if we consider a realization of the supply process where each order arrives, the order-up-to levels found will either correspond to ordering for one period demand in the future (Kth period) or not ordering. This realization implies a planning horizon type result; however, it seems hard to generalize it, as orders are not always realized. Hence, the most important aspect of the planning horizon theorem, the existence of regenerative points, is no longer valid here; the periods covered by an order may be overlapping with another, eliminating the possibility of a regeneration point.

A more realistic supply uncertainty model is to have intermediate supply levels such as $Q_1, Q_2, \ldots$ in addition to complete availability or unavailability in supply, with respective probabilities $p_1, p_2, \ldots$. Even one such intermediate level $0 < Q < \infty$ leads us to the domain of a completely different problem, and our analysis cannot be directly extended to incorporate this case. Another possible extension is to operate with more than one supplier, each with a different fixed ordering cost. In the case of a single supplier one may expect to see an (s_n, S_n) type policy, but again for multiple suppliers our analysis cannot be directly extended to incorporate this case.

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Appendix A. Proofs of the results

Proof of Theorem 1: The proof of the theorem is performed by induction. For $n = N$, $G_N(y) = L_N(y) + cy + C_{N+1}(y - D_N) = L_N(y) + cy$, which is convex in y. The minimizer of $G_N(y)$ is $y_N^* = D_N$ (since $b > c$). $C_N(I) = p_N G_N(I) - cI + q_n \min_{y \geq I} G_N(y)$; therefore $C_N(I) = p_N G_N(I) - cI + q_N G_N(y_N^*) = p_N L_N(I) + q_N \{c(y_N^* - I) + L_N(y_N^*)\}$ if $I \leq y_N^*$, and $C_N(I) = p_N G_N(I) - cI + q_N G_N(I) = G_N(I) - cI = L_N(I)$ otherwise. This yields the form of the optimal ordering policy for $n = N$. The convexity of $C_N(I)$ follows from the convexity of $L_N(y) + cy$. Assume that the assertions hold for $n + 1$. The convexity of $G_n(y) = L_n(y) + cy + C_{n+1}(y - D_n)$ follows from the convexity of $L_n(y) + cy$ and $C_{n+1}(y - D_n)$. Let y_n^* be the minimizer of $G_n(y)$. It follows that

$$\begin{aligned} C_n(I) &= p_n G_n(I) - cI + q_n G_n(y_n^*) \\ &= p_n\{L_n(I) + C_{n+1}(I - D_n)\} + q_n\{c(y_n^* - I) \\ &\quad + L_n(y_n^*) + C_{n+1}(y_n^* - D_n)\} \end{aligned}$$

for $I \leq y_n^*$, and

$$\begin{aligned} C_n(I) &= p_n G_n(I) - cI + q_n G_n(I) = G_n(I) - cI \\ &= L_n(I) + C_{n+1}(I - D_n) \end{aligned}$$

for $I > y_n^*$. Hence, the form of the optimal ordering policy follows. The convexity of $C_n(I)$ is evident from the convexity of $C_{n+1}(I)$ and $L_n(y)$, which completes the proof. ■

Proof of Proposition 1: For $n = N$, $y_N^* = D_N$ and the assertion holds. Assume that the statement is true for $n+1$. It is sufficient to show that $G_n(y) \geq G_n(D_n) \forall > y \leq D_n$ (since $G_n(y)$ is convex). First note that $L_n(y) + cy \geq L_n(D_n) + cD_n \forall y \leq D_n$. Next, we need to show that $C_{n+1}(y - D_n) \geq C_{n+1}(0)$, since, if this is the case,

$$\begin{aligned} G_n(y) &= L_n(y) + cy + C_{n+1}(y - D_n) \\ &\geq L_n(D_n) + cD_n + C_{n+1}(D_n - D_n) = G_n(D_n). \end{aligned}$$

For $y - D_n \leq 0 \leq y_{n+1}^*$,

$$\begin{aligned} C_{n+1}(y - D_n) &= q_{n+1}G_{n+1}(y_{n+1}^*) + p_{n+1}G_{n+1}(y - D_n) \\ &\quad - c(y - D_n), \\ C_{n+1}(0) &= q_{n+1}G_{n+1}(y_{n+1}^*) + p_{n+1}G_{n+1}(0). \end{aligned}$$

But since $G_{n+1}(y - D_n) \geq G_{n+1}(0)$ ($y_{n+1}^* \geq 0$ minimizes $G_n(y)$) and $-c(y - D_n) \geq 0$, it follows that $C_{n+1}(y - D_n) \geq C_{n+1}(0)$, which proves the assertion. ∎

Proof of Proposition 2: The claim is intuitively obvious. Formally, it suffices to demonstrate that $\forall y \geq y_{n+1}^* + D_n$, $G_n(y) \geq G_n(y_{n+1}^* + D_n)$. Let $y = y_{n+1}^* + D_n + \eta$, for $\eta \geq 0$. We have

$$\begin{aligned} G_n(y) &- G_n(y_{n+1}^* + D_n) \\ &= c\eta + L(y) - L(y_{n+1}^* + D_n) + C_{n+1}(y - D_n) \\ &\quad - C_{n+1}(y_{n+1}^* + D_n - D_n). \end{aligned}$$

Note that

$$\begin{aligned} C_{n+1}&(y - D_n) - C_{n+1}(y_{n+1}^*) \\ &= G_{n+1}(y_{n+1}^* + \eta) - G_{n+1}(y_{n+1}^*) - c\eta \geq -c\eta, \end{aligned}$$

and $L(y) - L(y_{n+1}^* + D_n) = h\eta$. Therefore,

$$G_n(y) - G_n(y_{n+1}^* + D_n) \geq h\eta \geq 0. \quad ∎$$

Proof of Proposition 3: Equation (6) holds for $n = N$. Suppose that it holds for $n+1$. Note that $G_n(y)$ can be written as

$$\begin{aligned} G_n(y) &= L_n(y) + cy + C_{n+1}(y - D_n) \\ &= L_n(y) + cy + p_{n+1}G_{n+1}(y - D_n) \\ &\quad - c(y - D_n) + q_nG_{n+1}(y_{n+1}^*). \end{aligned} \tag{10}$$

The second equality follows from the fact that $y - D_n \leq y_{n+1}^*$. By Proposition 2, $y_{n+1}^* \leq y_{n+2}^* + D_{n+1}$. Therefore, for $y \leq y_{n+1}^* + D_n$, we have $y - D_n \leq y_{n+1}^* \leq y_{n+2}^* + D_{n+1}$. Hence, the induction assumption enables us to replace $G_{n+1}(y - D_n)$ with

$$\begin{aligned} &L_{n+1}(y - D_n) + c(y - D_n) \\ &\quad + \sum_{i=1}^{N-n-1} \prod_{j=1}^{i} p_{n+1+j}\{L_{n+1+i}(y - D_n - D(n+1, n+i)) \\ &\quad + c(y - D_n - D(n+1, n+i))\} \\ &\quad - \sum_{i=1}^{N-n-1} \prod_{j=1}^{i-1} p_{n+1+j}c(y - D_n - D(n+1, n+i)) \\ &\quad + \sum_{i=1}^{N-n-1} q_{n+1+i} \prod_{j=1}^{i-1} p_{n+1+j}G_{n+1+i}(y_{n+1+i}^*). \end{aligned} \tag{11}$$

Inserting (11) in (10) yields that

$$\begin{aligned} G_n(y) &= L_n(y) + cy + p_{n+1}\{L_{n+1}(y - D_n) + c(y - D_n)\} \\ &\quad + \sum_{i=1}^{N-n-1} \prod_{j=0}^{i} p_{n+1+j}\{L_{n+1+i}(y - D_n - D(n+1, n+i)) \\ &\quad + c(y - D_n - D(n+1, n+i-1))\} \\ &\quad - \sum_{i=1}^{N-n-1} \prod_{j=0}^{i-1} p_{n+1+j}c(y - D_n - D(n+1, n+i)) \\ &\quad + \sum_{i=1}^{N-n-1} q_{n+1+i} \prod_{j=0}^{i-1} p_{n+1+j}G_{n+1+i}(y_{n+1+i}^*) - c(y - D_n) \\ &\quad + q_{n+1}G_{n+1}(y_{n+1}^*). \end{aligned}$$

By the change of indices and rearranging terms we obtain (6). ∎

Proof of Proposition 4: First notice that for any $i = 1, 2, \ldots, K$,

$$\begin{aligned} D(n, n+i) \leq D(n, n+K) &= D_n + D(n+1, n+K) \\ &= D_n + y_{n+1}^*. \end{aligned}$$

Hence, we can use the characterization provided in Proposition 3 to write $G_n(D(n, n+i) - \eta) - G_n(D(n, n+i))$. It is tedious but straightforward algebra to verify that

$$\begin{aligned} G_n&(D(n, n+i) - \eta) - G_n(D(n, n+i)) \\ &= \eta\{-hQ_n(i-1) + bQ_n(N-n) - bQ_n(i-1) \\ &\quad - cQ_n(N-n)\} \geq 0 \end{aligned}$$

if and only if (8) is satisfied (since $\eta \geq 0$). ∎

Proof of Theorem 2: Since $G_N(y) = L_N(y) + cy$, $y_N^* = D_N$ and therefore $K_N = 1$ as desired. Assume that the assertions hold for $n+1$, and in particular $y_{n+1}^* = D(n+1, n+K)$. Notice that $y_n^* \geq D_n$ by Proposition 1 and $y_n^* \leq y_{n+1}^* + D_n = D(n, n+K)$ by Proposition 2. These observations, together with Corollary 1, assure that the minimum of $G_n(y)$ is achieved on $\{D_n, D(n, n+1), \ldots, D(n, n+K)\}$. Since $G_n(y)$ is convex, the minimum of it is equal to $D(n, n+K')$, where K' is the greatest number satisfying

$$G(D(n, n+K')) \leq G(D(n, n+K'-1)),$$

which is found by (9). If $K' = K$, then $G_n(y)$ is a decreasing convex function and the order-up-to level for period n is set to its highest possible value ($K+1$-period demand); however, if no such K' exists then $G_n(y)$ is an increasing convex function and hence $y_n^* = D_n$. ∎

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