

Cramer - Chernoff Theory

$$\lambda > 0, P[X \geq t] = P[e^{\lambda X} \geq e^{\lambda t}] \leq \inf_{\lambda > 0} \frac{E[e^{\lambda X}]}{e^{\lambda t}}$$

$$P[X \leq t] \leq e^{-\psi_X^*(t)}$$

$$\ln P[X \geq t] \leq \inf_{\lambda > 0} \ln E[e^{\lambda X}] - \lambda t$$

$$\equiv -\sup_{\lambda > 0} (\lambda t - \ln E[e^{\lambda X}]) = \psi_X^*(t)$$

Cramer - Chernoff Transformation

$$(\psi_X(t)) \equiv \sup_{\lambda > 0} (\lambda t - \psi_X(\lambda)) = \psi_X^*(\lambda)$$

↓ when $t \geq E[X]$

$$\sup_{\lambda \in \mathbb{R}} (\lambda t - \psi_X(\lambda)) \quad w$$

Fenchel Legendre Transformation

$$f_X(x) \equiv \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \quad N(0, \sigma^2)$$

$$\psi_X(t) \equiv \ln M_X(t) = \ln e^{\frac{1}{2}\sigma^2 t^2} = \frac{1}{2}\sigma^2 t^2$$

$$\psi_X^*(\lambda) = \sup_{t \in \mathbb{R}} \left(\lambda t - \frac{1}{2}\sigma^2 t^2 \right) \quad t > EX = 0$$

$$\lambda - \sigma^2 t = 0 \Rightarrow t = \frac{\lambda}{\sigma^2}$$

$$\psi_X^*(\lambda) = \frac{\lambda^2}{\sigma^2} - \frac{1}{2} \frac{\lambda^2}{\sigma^2} = \frac{\lambda^2}{2\sigma^2}$$

$$P[|X| \geq \lambda] \leq 2e^{-\frac{\lambda^2}{2\sigma^2}}$$

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Definition: A random variable X is "sub-gamma" with variance σ^2

if $\Psi_X(t) \leq \frac{1}{2} t^2 \sigma^2$ $M_X(t) \leq e^{\frac{1}{2} t^2 \sigma^2}$

Thm: Let X be a centered r.v (i.e., $E[X]=0$) and

$$P[X > x] \vee P[-X > x] \leq e^{-\frac{x^2}{2v}} \text{ for some } v > 0.$$

This result gives moment bound estimates of these kinds of random variables (in particular sub-gaussian)

Then $\forall$ integers $q \geq 1$, $E[X^{2q}] \leq 2 \times q! (2v)^q \leq q! (4v)^q$

$X \leftarrow \frac{X}{\sqrt{v}}$ (Scaling)

$$E[X^{2q}] = \int_0^\infty P[|X|^{2q} > x] dx$$

$$= 2q \int_0^\infty x^{2q-1} P[|X| > x] dx$$

$$\leq 4q \int_0^\infty x^{2q-1} e^{-x^2/2} dx$$

$\xrightarrow{X=\sqrt{2t}}$ $\leq 4q \int_0^\infty (2t)^{q-1} e^{-t} dt$

$$= 4 \cdot 2^{q-1} \int_0^\infty t^{q-1} e^{-t} dt = 2^{q+1} \cdot q!$$

?

claim: $E[X] = \sum_x P[X \geq x]$

$$P(A) = \int_A dP = \int_x I_A(\cdot) P(\cdot)$$

$$= E[I_A(\cdot)]$$

Then, $\sum_x P[X \geq x] = \sum_x E[I(X \geq x)]$

$$= E[\underbrace{\sum_x I(X \geq x)}_X] = E[X]$$

what happens if $X < x$?

$$\frac{2^q q!}{q!} = \prod_{j=1}^q (q+j) \geq \prod_{j=1}^q 2j = 2^q q!$$

$X \sim X'$ (copy of them)

$$Y = X - X'$$

What is the moment generating function of Y ? $E[e^{\lambda(Y)}] = ?$

Claim: $\lambda > 0$, X is centered r.v. $E[e^{-\lambda X}] \geq 1$.

Proof: Jensen inequality will be used.

Jensen inequality:

$$\phi: \mathbb{R}^n \supseteq G \xrightarrow{\text{convex}} \mathbb{R}$$

convex convex

$$X = (X_1, X_2, \dots, X_n) \in G.$$

$$E[\phi(X)] \geq \phi(E[X]) \rightarrow E[\phi(X)] \geq \phi(E[X_1], \dots, E[X_n])$$

- Real line is convex

$$E[e^{\lambda(-X)}] \geq e^{\lambda E[-X]} = 1$$

end of the proof

$$Y = X - X'$$

$$E[e^{tY}] = E[e^{tX}] \cdot \underbrace{E[e^{-tX}]}_{\geq 1}$$

$$\Rightarrow E[e^{tY}] \geq E[e^{tX}]$$

Generalized Jensen

↓
Minkovskii

↓
Cauchy-Schwarz

$$E[Y^{2q}] = E[(X-X')^{2q}]$$

- odd-powers are 0.

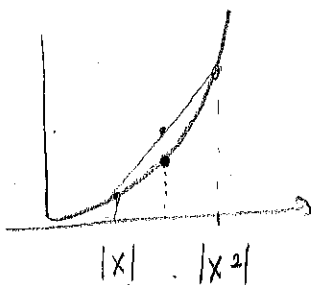
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$$E[Y^{2q+1}] = E[(X-X')(X-X')^{2q}] =$$

$$\leq E[(|X|+|X'|)^{2q}] \leq 2^{2q} E[|X|^{2q}]$$

$$\left\{ \begin{aligned} (|X|+|X'|)^{2q} &= 2^{2q} \left(\frac{1}{2}|X| + \frac{1}{2}|X'| \right)^{2q} \leq 2^{2q} \left(\frac{1}{2}|X|^{2q} + \frac{1}{2}|X'|^{2q} \right) \\ &\text{not correct!} \end{aligned} \right.$$

$x \rightarrow x^{2q}$ convex



$$E[Y^{2q}] \leq 2^{2q} E[X^{2q}]$$

$$E[e^{tX}] \leq E[e^{tY}] \leq \sum_{q=0}^{\infty} \frac{2^{2q} E[X^{2q}]}{(2q)!} = \sum_{q=0}^{\infty} \frac{2^q q!}{(2q)!} \left(\frac{t^2 (16v)}{2} \right)^q$$

$$\frac{2^q q!}{(2q)!} = \frac{2q(2q-2)(2q-4)\dots q!}{(2q)(2q-1)(2q-2)\dots(q+1)(q\cdot 2\cdot 1)} \leq \frac{1}{q!}$$

$$\leq \sum_{q=0}^{\infty} \left(\frac{t^2 (16v)}{2} \right)^q \frac{1}{q!} = e^{\frac{t^2 (16v)}{2}} = e^{t^2 (8v)} = e^{2t^2 (4v)} \downarrow C$$

Result: If for some constant $C > 0$,

$E[X^{2q}] \leq q! C^q$ then $X \sim$ subgaussian with $v = 4C$.

$$1. E[X^{2q}] \leq q! (4\sigma)^q$$

$$2. \text{ If } c > 0, E[X^{2q}] \leq q! c^q$$

— subgaussian $c = 4\sigma^2$

— the n -th order moment estimates: $E[X^n] \sim \left(\frac{n}{2}\right)!$ varies.

Johnson-Lindenstrauss Theorem: (query optimization — you're searching smt, then it goes to database and presents the most relevant)

— Take a suitable r.v.

random $\downarrow$ Projection.

— gaussian & sub-gaussian.

— recommendations
— search engines.

(nearest neighbour)

It allows to keep distances in original way in projection space.

— the beauty: the projection depends only number of data points not the dimension of vector space.

— Inf. dim space, we can also do projections: If we have a set, one way to measure is cardinality. Are there any other? Yes.

* We are still in finite domain.

Let $A \subset \mathbb{R}^d$ be finite set $|A| = N$. Assume $v \geq 1$.

$X_{ij} \in \underline{G}(v)$ $G(v)$ = subgaussian with var v .

Let $\varepsilon, \delta \in (0, 1)$. Then if $K \geq 100 N^2 \varepsilon^{-2} \ln\left(\frac{N}{\sqrt{\delta}}\right)$ then.

w.p $1 - \delta$ (at least) W is an ε -isometry on A .

$$S_d \supset T \equiv \left\{ \frac{v-v'}{\|v-v'\|} : v \neq v', v, v' \in A \right\}$$

— What's the cardinality of T ?

$$|T| \leq \underbrace{\left(\frac{n}{2}\right)}_{\text{why?}}$$

$$- \sup_{v \in T} \left| \|\Pi(v)\|^2 - 1 \right| \leq \varepsilon$$

ε -isometry

$$v \in T \Rightarrow \|v\| = 1$$

that's why not the $\|v\|$ but 1

subgaussian

$$\underbrace{\mathbb{E} \left[e^{\lambda \Pi_i(v)} \right]}_{\substack{\text{dict projections} \\ ? \\ \text{are also sub-gaussians}}} = \mathbb{E} \left[e^{\lambda \sum_{j=1}^d v_j X_{ij}} \right] = \prod_{j=1}^d \mathbb{E} \left[e^{\lambda v_j X_{ij}} \right]$$

$$\leq e^{\lambda^2 \sum_{j=1}^d v_j^2 v/2}$$

$$= e^{\lambda^2 v/2}$$

subgaussian

$$\text{Therefore, } \Pi_i(v) \in \overline{\mathcal{G}(v)} \Rightarrow \mathbb{E}[\Pi_i(v)^{2q}] \leq \frac{q!}{2} (4v)^2$$

Pf of the first generalization of Johnson-Lindenstrauss Then