Lecture Notes-1

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1 Markov Inequality

$$X \geq 0,\ P[X \geq \epsilon] \leq \frac{E[X]}{\epsilon} \tag{1}$$

Why is this true?
$\begin{cases} 1, X \geq a \\ 0, \text{otherwise} \end{cases}$
What is E[X]? $E[I] \leq \frac{E[X]}{a} = P[X \geq a]$

2 Tchebysev Inequality

X is a random variable
$\underbrace{\text{P[—X-E[X]—} \geq \epsilon]}_{P[(X-E[X])^2 \geq \epsilon^2]} \leq \frac{Var(X)}{\epsilon^2}$

$$P[(X-E[X])^2 \geq \epsilon^2] \leq \frac{E[(X-E(X))^2]}{\epsilon} = \frac{Var(X)}{\epsilon^2} \tag{2}$$

3 Strong Law of Large Numbers (SLLN)

It talks about the distribution of the random variables (it does not have to be identical) $X_1, X_2, ..., X_n$ independent random variables (identical) $\sim$ X

What does independent random variable mean?
Consider X,Y $\Rightarrow P[X|Y] = P[X]$ and $P[Y|X] = \frac{P[XY]}{P[X]} = \frac{P[X|Y]P[Y]}{P[X]} = P[Y]$
Any subset of independent random variable sequence is independent. Take two random variable from $X_1, ..., X_n \Rightarrow X_1, X_2$ are independent. Then,

$$P[|\underbrace{\frac{X_1 + ... + X_n}{n}}_{F_n} - E[X]| \geq \epsilon] \leq \frac{Var(X)}{n\epsilon^2} \tag{3}$$

Basically says, if there is huge dataset, sample average will not deviate from the actual average more than ϵ (Consider $n \to \infty, \frac{Var(X)}{n\epsilon^2} \to 0$)

$$P[|F_n - E[X]| \geq \epsilon] = P[|F_n - E[X]^2| \geq \epsilon^2] \leq \frac{E[|F_n - E[X]^2|]}{\epsilon^2} = \tag{4}$$

$$\begin{aligned} P[|F_n - E[X]| \geq \epsilon] &= P[|F_n - E[X]^2| \geq \epsilon^2] \\ &= \frac{1}{n^2} \\ E[|X_1 + X_2 + ... + X_n - nE[X]|^2] & \\ &= \frac{1}{n^2} E[|(X_1 - E[X]) + (X_2 - E[X])...(X_n - E[X])|] \\ &= \frac{1}{n^2} \sum_{i=1}^{n} E[X_i - E[X]]^2 \\ &= \frac{1}{n} Var(X) \end{aligned} \tag{5}$$

(In case of no correlation: $E[(X_i - E[X])(X_j - E[X])] = 0$)

4 C^γ Diffeomorphism

continuously differentiable isomorphism

Why sample?

- To capture the essentials of the data source high dimensional spaces, volume index (which the mass of the data at) has to be calculated
- substitution (algebraic) as geometric operations (Substitution is mapping from real line (algebraically) and substitution is mapping from $(-2\pi, 2\pi)$(geometrically)

4.1 Isomorphism

A **continuous bijection** with continuous inverse Linear isomorphism: the isomorphism is linear

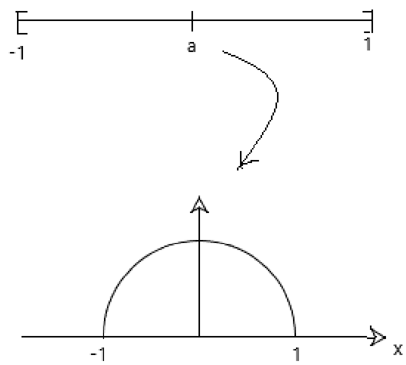
Diffeomorphism helps to take a set in certain region and maps it to another region preserving certain features

The aim of these substitution/calculation is to find $\int e^{x^2}$. Take $\phi(a) = (a, \sqrt{1-a^2})$,

$$\int_{-1}^{+1} (1 - a^2) da = \frac{4}{3} \tag{6}$$

$\phi(t) = (cost, sint) \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (cos^2 t) dt = \frac{\pi}{2} \tag{7}$$



The way the substitution is made is significant!

$$\int_a^b f(x)dx = lim_{n\to\infty} \sum f(x)\Delta x, \qquad \Delta x = \frac{b-a}{n} \tag{8}$$

Integration geometrically means dividing into grids and then summing up the grids.However, all the grids (increment size) do not need to be the same (geometrically).
What to do? Scaling is a way to handle it.

Riemann Integral (the actual definition of the integral)
It allows uneven increments

$$\lim_{m\times m(-)\downarrow 0} \sum_i f(x)(x_{i+1} - x_i) \tag{9}$$

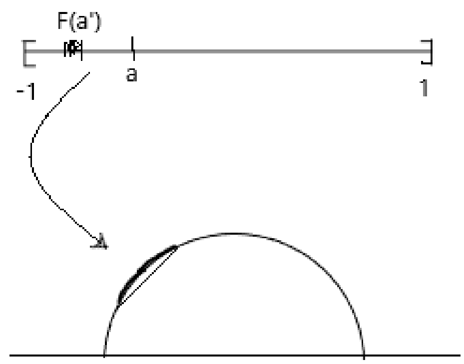
$$\sum_i F(a_i)\Delta a_i = \sum_i f \circ \phi(a_i) d(a_i) \tag{10}$$

Since $a_i \Rightarrow 0$, arc length and quad length will be approximately same. Therefore, we can change (10) to

$$\sum_i F(a_i)\Delta a_i = \sum_i f \circ \phi(a_i) \times \underbrace{L(a_i)}_{\phi(a_{i+1})-\phi(a_i)}, F(a_i) = \frac{f\circ\phi(a_i)|\phi(a_{i+1}) - \phi(a_i)|}{a_{i+1} - a_i} \tag{11}$$

$F(a) = f \circ \frac{d\phi}{da}$

For two variables, ϕ_1, ϕ_2, $\begin{vmatrix} \frac{\delta\phi_1}{\delta x} & \frac{\delta\phi_1}{\delta y} \\ \frac{\delta\phi_2}{\delta x} & \frac{\delta\phi_2}{\delta y} \end{vmatrix}$



Why diffeomorphism? It is required to calculate the derivative
$A, B \subseteq R^d$, A $\xrightarrow{\phi}$ B

$$\int_A f \circ \phi |D_\phi| = \int_B f \tag{12}$$

It is required to sample data points. Then, it has a distribution f and you can **pull back**$from"bad"domainto"nice"one.$

It is about pulling back maps of the data sets, not changing the essentials of the datasets. Therefore, the data which is sampled is same.

$$\int_{-\infty}^{\infty} e^{x^2} dx = I \tag{13}$$

In two dimension,

$$\int_{-\infty}^{\infty} e^{x^2+y^2} \underbrace{dxdy}_{\text{volume element}} = I \tag{14}$$

Take

$$(X = \gamma cos\theta \text{ and } Y = \gamma sin\theta) \Rightarrow \begin{vmatrix} cos\theta & -\gamma sin\theta \\ sin\theta & \gamma cos\theta \end{vmatrix} \tag{15}$$

$$I^2 = \int_0^{\infty} e^{\gamma^2} \gamma d\gamma d\theta = \pi \tag{16}$$

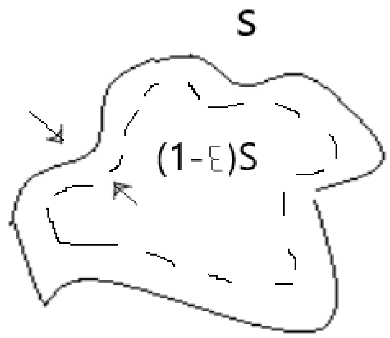


Fig. 1. The intuition behind high dimensional objects

5 High Dimensional Objects

Intuition: As in figure

$d >> 1$

$$\frac{Vol((1-\epsilon)S)}{Vol(S)} = (1-\epsilon)^d \underbrace{\leq}_{(1-x)\leq e^{-x}} e^{-\epsilon d} \to 0 \tag{17}$$

In large datasets, the mass is most likely lie on frontier.