Lecture Notes-2

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$$
\begin{aligned}
\overrightarrow{X} &= (X_1, X_2, ..., X_d)|X|^2 \\
&= \sum_{i=1}^{d} {X_i}^2 E[|X|^2] \\
&= E[X_1^2] + E[X_2^2] + ... + E[X_d^2] \\
&= |X| \ O(\sqrt{d})
\end{aligned}
$$

1 Strong Law of Large Numbers (SLLN)

n points P[—-O($\sqrt{d}$)—$\geq \epsilon$]$\leq$ $\frac{Var(X)}{n\epsilon^2}$

If we chose two points among n points, what kind of configuration will they have on average?

$$
\begin{aligned}
|Y - Z|^2 = \sum_i |Y_i - Z_i|^2 E[|Y - Z|^2] \\
&= E[\sum_i |Y_i - Z_i|^2] \\
&= E(\sum_i ({Y_i}^2 + {Y_i}^2 + 2Y_i \times Z_i) \\
&= d + d = 2d \\
&\qquad E[Y^2] = d \\
&\qquad E[Z^2] = d
\end{aligned}
$$

2 Volume/Area of Unit Ball in d-dimensions

$\frac{V(d)}{A(d)}$ represents the unit ball. In case of having a ball with radius r, the ratio becomes $\frac{V(r)}{A(r)}$

$$
V(d) = \int_{x_1=-1}^{x_1=1} ... \int_{-\sqrt{1-x_1^2-...-x_{d-2}^2}}^{\sqrt{1-x_1^2-...-x_{d-2}^2}} \int_{-\sqrt{1-x_1^2-...-x_{d-1}^2}}^{\sqrt{1-x_1^2-...-x_{d-1}^2}} dx_d dx_{d-1} ... dx_1 \tag{1}
$$

When d=2, d$\Omega\gamma^{d-1}d\gamma$

$$\int_0^{2\pi} d\Omega \int_0^1 \gamma d\gamma = \pi \tag{2}$$

$$V(d) = \int_{S_d} d\Omega \int_0^1 \gamma^{d-1} d\gamma = \frac{A(d)}{d} \tag{3}$$

$$\begin{aligned} I(d) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} e^{-(x_1^2+x_2^2+\cdots+x_d^2)} dx_1 dx_2 \dots dx_d \\ &= \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)^d \\ &= \sqrt{\pi}^d \\ &= \pi^{d/2} \end{aligned} \tag{4}$$

2.1 Spherical Coordinates

$$\begin{aligned} I(d) &= \int_{S_d} d\Omega \int_0^{\infty} e^{-\gamma^2} \gamma^{d-1} d\gamma \\ &= A(d) \int_0^{-\infty} e^{-\gamma^2} \gamma^{d-1} d\gamma \\ &= A(d) \int_0^{\infty} e^{-t} t^{\frac{d-1}{2}} \frac{t^{-\frac{1}{2}}}{2} dt \\ &= A(d) \int_0^{\infty} e^{-t} t^{\frac{d}{2}-1} \\ &= \frac{1}{2} \Omega(\tfrac{d}{2}) A(d) \end{aligned}$$

Then,

$$A(d) = \frac{\pi^{\frac{d}{2}}}{\frac{\Gamma(\frac{d}{2})}{2}} \tag{5}$$

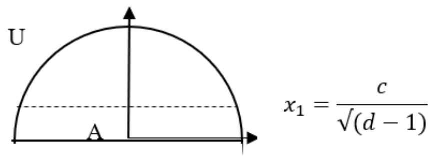
and

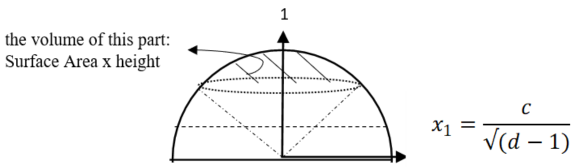
$$V(d) = \frac{\pi^{\frac{d}{2}}}{\frac{d \times \Gamma(\frac{d}{2})}{2}} \tag{6}$$

Theorem 1. *Given $c \geq 1$ and $d \geq 3$, at least $1 - \frac{2e^{\frac{-c^2}{2}}}{c}$ fraction of the volume of d-dimensional unit ball has $|X_1| \leq \frac{c}{\sqrt{d-1}}$*

$\frac{Vol(U/A)}{Vol(U)} \leq \frac{\texttt{upper bound of Vol(U/A)}}{\texttt{lower bound of Vol(U)}} \approx \frac{2e^{\frac{-c^2}{2}}}{c}$

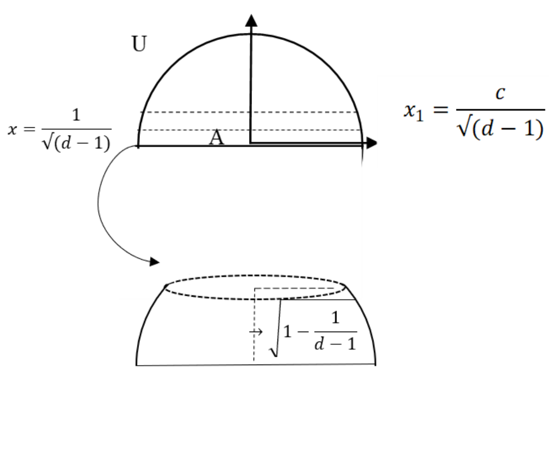
How to find an upper bound on Vol(A)?





$$\begin{aligned}
Vol(U/A) &= V(d-1)\int_{\frac{c}{\sqrt{d-1}}}^{1} (1-x_1^2)^{\frac{d-1}{2}}\,dx_1 \\
&\le V(d-1)\int_{\frac{c}{\sqrt{d-1}}}^{1} \frac{x_1\sqrt{d-1}}{c}(1-x_1^2)^{\frac{d-1}{2}}\,dx_1 \\
&\le \frac{(d-1)\sqrt{d-1}}{c}\int_{\frac{c}{\sqrt{d-1}}}^{\infty} x_1 e^{\frac{-(d-1)x_1^2}{2}}\,dx_1 \\
&= \frac{e^{\frac{-c^2}{2}}}{d-1}
\end{aligned}$$

$$Vol(U/A) \le \frac{Vol(d-1)}{c\sqrt{d-1}}e^{\frac{-c^2}{2}} \tag{7}$$



Then the volume is lower bound on U

$$\left(1-\frac{1}{d-1}\right)^{\frac{d-1}{2}} V(d-1)\frac{1}{\sqrt{d-1}} \tag{8}$$

Consider the formula $(1-\mathtt{a}x) \leq (1-x)^{\mathtt{a}}$, $\mathtt{a} \geq 1$

Replace $\mathtt{a}=\frac{d-1}{2}$ and x=$\frac{1}{d-1}$, then (8) becomes

$$1-\left(\frac{d-1}{2}\right)\times\left(\frac{1}{d-1}\right)=\frac{1}{2} \tag{9}$$

Theorem 2. *N points $X_1,...,X_N$ from d-dimensional unit ball with probability* $1-O(\frac{1}{n})$

1. • $|X_i| \geq 1-\frac{2lnN}{d}$, $P[|X_i|<1-\frac{2lnN}{d}] \leq e^{-d\frac{2lnN}{d}} \leq e^{-\epsilon d}$
2. • $|X_i - X_j| \leq \frac{\sqrt{6lnN}}{\sqrt{d-1}}$, $P[|X_i| > \frac{c}{\sqrt{d-1}}] \leq \frac{2e^{-\frac{c^2}{2}}}{c}$

Take N points, among all these points, there are $\binom{N}{2}$ pairs
X_i- "North $X_j \rightarrow X_i$ (X_j projected on X_i)
Projection larger than $\frac{\sqrt{6lnN}}{d-1}$ has order O$\left(\frac{1}{N^3}\right)$ and O$\left(\binom{N}{2}\frac{1}{N^3}\right)=O(\frac{1}{N})$

3 Chernoff Bounds

(Sums of Poisson Trials,Extension of Tcheybshev Bounds) X is a random variable and t is given. Then Moment Generating Function $E[e^{tX}]=M_X(t)$ and $M_X^n(0)=E[X^n]$

From Markov Inequality,
t>0, $P[X\geq \mathtt{a}]=P[e^{tX}\geq e^{ta}] \leq min_{t>0}\frac{E[e^{tX}]}{e^{t\mathtt{a}}}$
t<0, $P[X\leq \mathtt{a}] \leq min_{t<0}\frac{E[e^{tX}]}{e^{t\mathtt{a}}}$

3.1 Poisson Trials

$X_1, X_2, ..., X_N$ are independent and X_i= $\begin{cases} 1\mathtt{wp}p_i \\ 0wp(1-p_i) \end{cases}$
X=$\sum_{i=1}^{N} X_i$, $\mu=E[X]=\sum_{i=1}^{N} b_i$

$$M_X(t)=E[e^{tX}]=\left(E[e^{tX_i}]\right)^N \tag{10}$$

$$\begin{aligned} M_{X_i}(t) &= E[e^{tX_i}] \\ &= p_i e^t + (1-p_i)e^0 \\ &= \underbrace{1 + p_i(e^t - 1) \leq e^{p_i(e^t-1)}}_{(1+x)\leq e^x} \end{aligned} \tag{11}$$

Then, equation (10)

$$\begin{aligned} E[e^{tX_i}]^N &\leq \prod_{i=1}^{N} e^{p_i(e^t-1)} \\ &= e^{\sum_{i=1}^{N} p_i(e^t-1)} \\ &= e^{\mu(e^t-1)} \end{aligned} \tag{12}$$

Result:

$$\delta > 0, P[X \geq (1+\delta)\mu] \leq \left[\frac{e^\delta}{(1+\delta)^{1+\delta}}\right] \tag{13}$$