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Chernoff Bound

Poisson r.v.

$$X_i = \begin{cases} 1 & \text{wp } p_i \\ 0 & \text{wp } (1-p_i) \end{cases}$$

If we take a sequence $X_1, X_2, \dots X_n$ $\quad X = \sum X_i$

$$E[X_i] = p_i \quad \dots \quad E[X] = \sum p_i = \mu$$

$$* \; P[X \geq (1+\delta)\mu] \leq \left( \frac{e^{\delta}}{(1+\delta)^{1+\delta}} \right)^{\mu} \text{ — best tighter bound } (*)$$

(wlog $0 < \delta < 1$)

Proof: $$P[X \geq (1+\delta)\mu] \leq \underbrace{\frac{e^{(e^t-1)\mu}}{e^{t(1+\delta)\mu}}}_{\text{(RHS)}} \quad \forall t > 0 \qquad (1)$$

mon. fn $x \geq$ mon. fn $y$

$\phi(x) \geq \phi(y)$ $\qquad \phi = e^t$

$t = \ln(1+\delta)$ RHS becomes $\dfrac{e^{\delta\mu}}{(1+\delta)^{(1+\delta)\mu}}$

(?)

High tolerance — I need to go high dimensionality space but I don't need more data.

Even if your target is infinite dim, you still get finite projection.

$*$ for $0 < \delta < 1$

$$P[X \geq (1+\delta)\mu] \leq e^{-\mu\delta^2/3} \quad (**)$$

(stronger bound compared to (1))

$-\left(-\frac{\delta^2}{3}\right)$

[By taking logarithm] $$f(\delta) \equiv \delta - (1+\delta)\ln(1+\delta) + \frac{\delta^2}{3}$$

taking derivative of $(*)$ and $(**)$ - upper bounds.

for $0 < \delta < 1$ $$\underbrace{\left( \frac{e^{\delta}}{(1+\delta)^{1+\delta}} \right)^{\mu}}_{\text{tightest u b}} \leq e^{-\mu\delta^2/3}$$

If I can prove $f(\delta) \leq 0$, it will complete the proof.

$f(0) = 0$ $\quad f'(\delta) = -\ln(1+\delta) + \frac{2}{3}\delta$ $\quad f'(0) = 0 \quad f'(1) < 0$

$$f''(\delta) = -\frac{1}{1+\delta} + \frac{2}{3}$$

$\downarrow$ $\quad\quad \Downarrow$

$f(0) \leq 0$ $\quad\quad f'(\delta) < 0$

$$f''\left(\frac{1}{2}\right) = 0 \qquad f''\left(>\frac{1}{2}\right) > 0$$

$$f''\left(<\frac{1}{2}\right) < 0$$

$* \; R \geq 6\mu$

$$P[X \geq R] \leq 2^{-R}$$

$0 < \delta < 1$: (*) $P[X \leq (1-\delta)\mu] \leq \left[\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right]^{\mu}$

(*) $\Rightarrow P[X \leq (1-\delta)\mu] \leq e^{-\delta\mu^2/2}$

$$P[|X-\mu| \geq \delta\mu] \leq 2e^{-\mu\delta^2/3}$$

$\mu = \frac{n}{2}$

$\delta = \frac{1}{2}$

$P\left[|X-\mu| \geq \frac{n}{4}\right] \leq \frac{2}{3}$ (markov)

$\leq \frac{4}{n}$ (chebyshev Bounds)

$\leq 2e^{-\frac{n}{2}\cdot\frac{1}{4}\cdot\frac{1}{3}} = 2e^{-\frac{n}{24}}$ (chernoff)

Rademacher r.v. $X_i = \begin{cases} +1 & \text{wp } p \\ -1 & \text{wp } (1-p) \end{cases}$

$Y_i = \frac{1+X_i}{2}$ , $X_i = 2Y_i - 1$ — (HW)

Master Tail Bound Thm : → In high dimensional, mass are located around the border

$X = X_1 + \dots + X_N$ $X_i \underset{\text{indep}}{\sim} (0, \sigma^2)$

Let $0 \leq a \leq \sqrt{2}\, N\sigma^2$

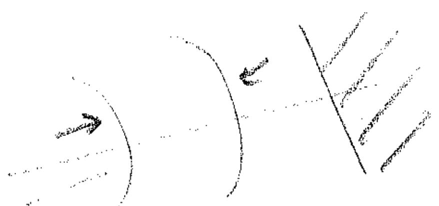
Assume that

$|E(X_i^s)| \leq \sigma^2 s!$

for $s = 3, 4, \dots \left\lfloor \frac{a^2}{4n\sigma^2} \right\rfloor$

$P[|X| \geq a] \leq 3e^{-a^2/12n\sigma^2}$

most of the points lie in



$E[e^{tX}]$ $X \sim N(\mu, \sigma^2)$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{tX} e^{-\frac{(X-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{t\sigma y} e^{-y^2/2} \qquad \frac{x-\mu}{\sigma} = y$$

$$= \frac{e^{\mu t + \frac{1}{2}t^2\sigma^2}}{2\pi} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(y-\sigma t)^2} dy$$

$$X^{\gamma} = (X_1 + X_2 + \dots + X_N)^{\gamma} = \sum_{(\gamma_1 \dots \gamma_N) : \sum \gamma_i = \gamma} \frac{\gamma!}{\gamma_1! \dots \gamma_N!} X_1^{r_1} X_2^{r_2} \dots X_N^{r_N}$$

$$E[X^{\gamma}] = \sum — E[X_1^{\gamma_1}]\, E[X_2^{\gamma_2}] \dots E[X_N^{\gamma_N}]$$

→ because of independency

$\sum \gamma_i \geq 2$. — why?

$$E[X^\gamma] \leq \gamma! \overbrace{\sum_{(\gamma_1, \ldots, \gamma_N)} \sigma^{2\,(\#\text{ of non-zero } \gamma_i \text{ in the set})}}^{*}$$

't' sub - nonzero $r_i$ — $\binom{n}{t}$

$$Y_1 + Y_2 + \cdots + Y_t = r - 2t$$

$$Y_i \geq 2$$

coefficient $(X^2 + X^3 + \cdots + X^\gamma)^t$, $X^r$ in this set.

$$\Rightarrow \frac{X^{2t}(1 - X^{r-1})^t}{(1-X)^t} = X^{2t}(1 - X^{\gamma - 1})^t (1 - X)^{-t}$$

$$= X^{r-2t}$$

$$\binom{r-t-1}{t-1} \binom{r-2t+t-1}{t-1}$$

$$E[X^r] \leq r! \underbrace{\sum_{t=1}^{r/2} f(t)}_{(**)} \qquad f(t) = \binom{n}{t}\binom{\gamma - t - 1}{t-1} \sigma^{2t}$$

ub. $\qquad \frac{n^t}{t!} \quad 2^{\gamma - t - 1} \quad \sigma^{2t}$

(*) and (**) are same

$$\leq \gamma! \sum_{t=1}^{r/2} h(t) \qquad = \frac{(n\sigma^2)^t}{t!} 2^{\gamma - t - 1}$$

where $h(t) = \frac{(n\sigma^2)^t}{t!} 2^{\gamma - t - 1}$ Moreover,

control - no need to write

$$\frac{h(t)}{h(t-1)} = \frac{n\sigma^2}{2t} = \frac{(n\sigma^2)\ \cancel{2^{\gamma - t - 1}}}{\cancel{t!}} \cdot \frac{\cancel{(t-1)!}}{\cancel{(n\sigma^2)^{t-1}}\ \cancel{2^{\gamma - t}}} = \frac{n\sigma^2}{2t}$$

$\frac{n\sigma^2}{2t} \geq 2$ . where $t \leq \frac{r}{2} \leq \frac{n\sigma^2}{2}$ and $S \leq \frac{n\sigma^2}{2}$

$r = 3, 4, \ldots S$

So,

$$\leq r! \sum_{t=1}^{r/2} h(t) \leq r!\, h\left(\frac{r}{2}\right)\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right)$$

$$= \frac{r!}{(r/2)!} 2^{r/2} (n\sigma^2)^{r/2}$$

To sum up,

$$E[X^r] \leq \frac{r!}{(r/2)!} 2^{r/2} (n\sigma^2)^{r/2}$$

$$P[|X| > a] = P[|X|^r > a^r] \leq \frac{r!\,(n\sigma^2)^{r/2}\, 2^{r/2}}{(r/2)!\, a^r} \equiv g(r)$$

$$\leq \left(\frac{2rn\sigma^2}{a^2}\right)^{r/2}$$

$n/N$ – nb. of points

Therefore, for even 'r' $\quad \frac{g(r)}{g(r-2)} = \frac{\overbrace{4(r-1)n\sigma^2}^{\text{decreases}}}{a^2}$

$$(r-1) \leq \frac{a^2}{4n\sigma^2} \Rightarrow \frac{4(r-1)n\sigma^2}{a^2} \leq 1$$ which means $g(r)$ is decreasing fn.

it will decrease till r satisfies this

$r$ to be largest even integer $\leq \frac{a^2}{6n\sigma^2}$

? not sure

Tail probability $\leq e^{-\gamma/2}$

$e^1 \cdot e^{-\frac{a^2}{12n\sigma^2}} \leq 3 \cdot e^{-\frac{a^2}{12n\sigma^2}}$

Full projection lies somewhere in the annulus