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(Weak) Isotropic $\mathbb{R}^d$ $E\left[\{(x-\mu)\cdot w\}^2\right] = \sigma^2$ $\|w\| = 1$ $w \in R^d$

* The k-dim SVD subspace for F: $SVD(F) \supset span\{\mu_1, \dots, \mu_k\}$

$F = \sum_i w_i \underline{F_i}$

isotropic (μ_i)

isotropic proj: variance remains same.

$\sum w_i = 1$ $w_i \geq 0$.

Theorem: $F = \sum w_i F_i$ where F_i's general (non-isotropic)

σ^2_{iw} = max-variance of F_i along $W \equiv$ k-dim $SVD(F)$

the variances are different when you projected

Then, $$\sum_{i=1}^{k} w_i \, d(\mu_i, w)^2 \leq k \sum_{i=1}^{k} w_i \sigma_{iw}^2$$

Proof: $M = span\{\mu_1, \dots, \mu_k\}$ $\underbrace{\overset{SVD}{\Pi_m(x)}, \ \Pi_w(x)}_{\text{they don't need to be same.}}$

$$E\left[\|\Pi_m(x)\|^2\right] = \sum_i w_i \underset{i}{E}\left[\|\Pi_m(x)\|^2\right] = \sum_i w_i \left(\underset{i}{E}\left(\|\Pi_m(x) - \mu_i\|^2 + \|\mu_i\|^2\right)\right)$$

$$\geq \sum_i w_i \|\mu_i\|^2 = \underline{\sum_i w_i \|\Pi_W(\mu_i)\|^2} + \sum_i w_i \, d(\mu_i, w)^2$$

$W = SVD$ $v_1, \dots, v_k \equiv \{j\}$

$$E\left[\|\Pi_W(x)\|^2\right] = \sum_i w_i \left(\underset{i}{E}\left[\Pi_W(x-\mu_i)^2\right] + \|\Pi_W(\mu_i)^2\|\right)$$

$$= \sum_i w_i \sum_j \underset{i}{E}\left(\langle \Pi_W(x-\mu_i), v_j \rangle^2\right) + \sum_i w_i \|\Pi_W(\mu_i)\|^2$$

$$\leq k \sum_i w_i \sigma^2_{i,W} + \underline{\sum_i w_i \|\Pi_W(\mu_i)\|^2}$$

$$E\left[\|\Pi_W(x)\|^2\right] \geqslant E\left[\|\Pi_m(x)\|^2\right]$$

Log concave Function : $f: \forall x, y \quad \lambda \in [0,1]$

$$f(\lambda x + (1-\lambda)y) \geqslant f(x)^\lambda f(y)^{1-\lambda}$$

Theorem: Lovasz - Simonovitz

$\mathbb{R}^d \ni X \sim f(\cdot)$ which log-concave (μ, σ^2)

$$P\left[\|X-\mu\|^2 \geqslant t\sigma\right] \leqslant e^{-t+1} \quad \text{for all } t > 1$$

Sub-exponential

A r.v. $X\ (\mu, v)$ is sub-exponential (v, α) if

$$M_{X-\mu}(\lambda) \leq e^{\frac{v^2\lambda^2}{2}} \quad \forall |\lambda| \leq \frac{1}{\alpha}$$

Proposition: $P\left[X-\mu \geqslant t\right] \leqslant \begin{cases} e^{-t^2/2v^2} & 0 \leqslant t \leqslant v^2/\alpha \\ e^{-t/2\alpha} & t > v^2/\alpha \end{cases}$

Proof: Assume $\mu = 0$

$$P[X \geqslant t] \leq e^{-\lambda t} E\left[e^{\lambda X}\right] \leq e^{-\lambda t + \frac{\lambda^2 v^2}{2}} \quad \lambda \in [0, \alpha^{-1}]$$

$g(\lambda, t)$

$$g^*(t) = \inf_{\lambda \in \left[0, \frac{1}{\alpha}\right]} g(\lambda, t) \qquad \lambda^* = \frac{t}{v^2}$$

$$\lambda^* = \frac{t}{v^2} \leq \frac{1}{\alpha} \quad \text{— no problem}$$

$$\frac{t}{v^2} > \frac{1}{\alpha} \Rightarrow t > \frac{v^2}{\alpha}$$

$$\lambda^* = \frac{1}{\alpha}, \quad g^*(t) \leq -\frac{t}{2\alpha}$$

$$\{X_k\}_{k=1}^{n} \qquad X_k \sim (\mu_k, v_k^2) \qquad (v_k^2, \alpha_k)$$

independent $S_n \equiv \sum_{k=1}^{m} (X_k - \mu_k) \qquad M_{S_n}(\lambda)$

$$E\left[e^{\lambda \sum_k (X_k - \mu_k)}\right] = \prod_k E\left[e^{\lambda(X_k - \mu_k)}\right] \leq \prod_k e^{\frac{\lambda^2 v_k^2}{2}}$$

$$= e^{\lambda^2/2 \sum_k v_k^2}$$

$$X_k \longleftrightarrow |\lambda_k| \leq \frac{1}{\alpha_k}$$

→ Is the moment generating fn of sub-exp. sub-exp.? Yes,

$$\alpha^* = \max_k \alpha_k, \quad v^* = \sqrt{\sum_k v_k^2}$$

Result:

$$P\left[\frac{1}{n}\sum_k (X_k - \mu_k) \geq t\right] \leq e^{-\frac{n^2 t^2}{2(v^*)^2}} \qquad 0 \leq t \leq \frac{v_*^2}{n\alpha^*}$$

$$\leq e^{-\frac{nt}{2\alpha^*}} \qquad t > \frac{v_*^2}{n\alpha^*}$$

Exercise:

Defn: We say $X\ (\mu, \sigma^2)$ satisfies "Bernstein's condition" with parameter 'b' if $E[\|X-\mu\|^k] \leq \frac{k!}{2} \sigma^2 b^{k-2}$

If X satisfies Bernstein condition, X is sub-exponential with certain parameters.

→ What are these parameters?

(JLT works for subexpo.)

$$E[e^{\lambda(x-\mu)}] = E\left[\sum_{k=0}^{\infty} \frac{[\lambda(x-\mu)]^k}{k!}\right] \overset{\text{boundedness}}{=} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} E[(x-\mu)^k]$$

$$\leq \sum_{k=0}^{\infty} \underbrace{\frac{\lambda^k}{k!} \cdot \frac{k!}{2} \cdot \frac{\sigma^2}{b^2} \cdot b^k}_{\text{geometric series}}$$

For geometric series to converge $\underbrace{C}_{\text{constant}} \sum_{k=0}^{\infty} (\lambda b)^k$,

$(\lambda b) < 1 \Rightarrow |\lambda| \leq \frac{1}{b}$