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$$*\begin{cases} E\left[\pi_i(\alpha)^{2q}\right] \le \frac{q!}{2}(4v)^q \\ \Rightarrow P\left[\left|\sum_{i=1}^{k}(\pi_i(\alpha)^2-1)\right| \ge 4v\sqrt{2kt}+4vt\right] \le 2e^{-t} \quad (1) \end{cases}$$

$Y \sim t_y(2) = -$

Gamma Distribution.

Consider a r.v. Y with gamma dist ε, $a, b \ge 0$

$EY = ab$
$\mathrm{Var}\, Y = ab^2$
$\& 2$

$$E[e^{tY}] = \frac{1}{\Gamma(a)\, b^a}\int_0^\infty 2^{a-1} e^{-2\left(\frac{1}{b}-t\right)}$$

$\ln \downarrow$

$$\psi_Y(t) = -a \ln(1-bt)$$

$X \equiv Y - EY$ $\quad X$ is centered

$$\psi_X(t) = \ln E e^{t(Y-EY)} = -abt - a\ln(1-bt) \to 0 < t < \frac{1}{b}$$

$0 < bt < 1$

If $t \in (0,1)$ $\quad -t - \ln(1-t) \le \frac{t^2}{2(1-t)}$

$$\psi_X(t) \le \frac{ab^2t^2}{2(1-bt)}$$

- Gamma dists have fractals. (fractal distribution)

Defn: A r.v. Z is said to be <u>sub-gamma</u> on the right tail with $\mathrm{Var}(Z) = v$,
$\mathrm{Scale}(Z) = c$

if $\psi_Z(t) \leq \dfrac{v t^2}{2(1-ct)}$ $\quad t \in \left(0, \frac{1}{c}\right)$

$Z \in \Gamma_+(v, c)$

$Z \in \Gamma_-(v, c)$

Addition of normal = normal

" of $(\text{normal})^2$ = chi- square

Compute Cramer's Transform

$$\psi_Z^*(t) \triangleq \sup_{\lambda \geq 0} \left(\lambda t - \psi_Z(\lambda)\right)$$

$\psi_Z(\lambda)$ is defined $\lambda \in (0, \frac{1}{2})$, $\lambda = \frac{1}{c}\left(1 - \dfrac{1}{\sqrt{1+\frac{2ct}{v}}}\right)$

$$\psi_Z^*(t) \triangleq \frac{v}{c^2}\left(1 + \frac{ct}{v} - \sqrt{1+\frac{2ct}{v}}\right) = \frac{v}{c^2} h\left(\frac{ct}{v}\right)$$

$h(u) = 1+u - \sqrt{1+2u}$, $h^{-1}(u) = u + \sqrt{2u}$

If $Z \in \Gamma_+(v, c)$ $\quad P[Z > t] \leq e^{-\frac{v}{c^2} h\left(\frac{ct}{v}\right)}$

$f^{-1} \circ f(x) = x$ and $f(x) = y$.

$f(x)$ is increasing.

$\dfrac{df^{-1}}{dy} \cdot f'(x) = 1 \Rightarrow \dfrac{df^{-1}}{dy} = \dfrac{1}{f'(x)} > 0$. $f'(x) > 0$; so f^{-1} is also increasing.

Decide $h(i)$ is increasing / decreasing ! $[1+2u]^{1/2}$.

X $h' = 1 - \frac{1}{\cancel{2}}(1+2u)^{-1/2} \cdot \cancel{2} = 1 - \frac{1}{(1+2u)^{1/2}}$

$a = \frac{ct}{v}$.

$$P\left[Z > \frac{va}{c}\right] \leq e^{-v/c^2 \, h(a)}$$

$$P\left[Z > \frac{v}{c}\left(u + \sqrt{2u}\right)\right] \leq e^{-\frac{v}{c} \cdot u}$$

$Z \in \Gamma_+(v, c)$

$t = \frac{vu}{c^2}$

$$P\left[Z > ct + 2\sqrt{vt}\right] \leq e^{-t}$$

* the bounds of sub-gaussians are valid sub-gamma.

Bernstein's Inequality - if you have these conditions, they are sub-gam $\frac{v}{c}$

Let $X_1, X_2, \ldots X_k$ be an independent real-valued rv's.

Assume $\exists v, c > 0$ s.t $\sum_{i=1}^{k} E[X_i]^2 < v$ and

(*) looks similar to this

$\sum_{i=1}^{k} E[(X_t)^q] \leq \frac{1}{2} q! \, v \, c^{q-2}$ for $\forall$ integers $q \geq 3$. Then,

if I define $S = \sum_{i=1}^{k}(X_i - EX_i)$ and for $x \in \left(0, \frac{1}{c}\right)$ and $t > 0$.

$\psi_S(\lambda) \leq \frac{v x^2}{2(1-c)}$ and $\psi_S^*(t) \geq \frac{v}{c^2} h\left(\frac{ct}{v}\right)$ and

$$P\left[S \geq \underbrace{\sqrt{2ct}}_{?} + ct\right) \leq e^{-t}$$

Proof: Part 1. RTP: $\psi_S(\lambda) \leq \frac{v\lambda^2}{2(1-c)}$

Let's define $\phi(t) = e^t - t - 1$, $\lambda > 0$

$$\phi(\lambda X_i) = e^{\lambda X_i} - (\lambda X_i + 1) = \frac{\lambda^2 X_i^2}{2} + \sum_{q \geq 3} \frac{\lambda^q X_i^q}{q!}$$

$$\leq \frac{\lambda^2 X_i^2}{2} + \sum_{q \geq 3} \frac{\lambda^q (X_{i+})^q}{q!}$$

monotonic convergence function.

$$\sum_{i=1}^{k} E[\phi(\lambda X_i)] \leq \sum_{i=1}^{k}\left(\frac{\lambda^2 E[X_i^2]}{2} + \sum_{q \geq 3} \frac{\lambda^q E[(X_{i+})]^q}{q!}\right) \leq \frac{v}{2} \sum_{q \geq 0} \lambda^q c^{q-2}$$

$0 < \lambda < \frac{1}{c}$ $\qquad \leq \frac{v\lambda^2}{2(1-c\lambda)}$

$$\psi_S(\lambda) = \ln E\left[e^{\sum_i \lambda (X_i - E(X_i))}\right] \qquad = -\sum_i \phi(\lambda X_i)$$

$$\leq \sum (Ee^{\lambda X_i} - 1 - \lambda X_i)$$

$$= \ln E\left[\prod_i e^{\lambda (X_i - \underbrace{EX_i}_{\text{constant}})}\right] = \sum_i \left(\ln E\, e^{\lambda X_i} - \lambda E X_i\right)$$

$$= \sum_i \left(\ln E\, e^{\lambda X_i} - \lambda E X_i\right) \qquad \ln \cancel{E}[\underbrace{EX_i}_{\text{constant}}]$$

$x > 0$, $\ln x \leq x - 1$.

$$P\left[\sup_{v \in T} \left| \sum_{i=1}^{k} (\Pi_i(v)^2 - 1)\right| \geq 4v(\sqrt{2kt} + t)\right] \leq 2N^2 e^{-t}$$

t has to be non-negative.

$t = \ln\left(\frac{N^2}{8}\right)$

$$P\left[\sup_{v\in T}\left|\,\|\pi(v)\|^2-1\right| \geq \frac{8v\ln\left(N/\sqrt{\delta}\right)}{k} + \sqrt{\frac{8v\ln\left(N/\sqrt{\delta}\right)}{k}}\right] \leq \delta$$

$$k \geq 100\cdot\varepsilon^{-2}N^2\ln\left(\frac{N}{\sqrt{\delta}}\right)$$

$$\varepsilon \geq \frac{2\varepsilon^2}{25v} \qquad \leq \frac{4\varepsilon}{5}$$

$$P\left[\sup_{v\in T}\left|\,\|\pi(v)\|^2-1\right| \leq \varepsilon\right] \geq 1-\delta$$

norm of projection = −1, +1

basis vectors are gaussian → the summation are χ^2

special case of subgaussian