5.5.2020

Frequency-based approach

Fictitious play.

$N = 3$

Prob of Mutation = 0.3

$$A \xrightarrow{0.5} B, \quad A \xrightarrow{0.5} C$$

– equal probability

$N = 5$

$$A \text{ --- } B \qquad = \frac{(1-\alpha)}{N-1}$$

initial — all other

(it's equally probable

$$= \frac{1}{N} + \left(1 - \frac{1}{N}\right)\left(1 - \frac{\alpha N}{(N-1)}\right)^n$$

Spectral Theory

Operator: takes a function and gives another vector. Ex, derivative, inverse, fourier transform, matrix

two element set: a straight line $(x_1, x_2;\ x_1 + x_2 = 1)$

three " set: triangle. $(x_1, x_2, x_3:\ x_1 + x_2 + x_3 = 1)$

$x_1, x_2, x_3 \geq 0$

For n probability, we have $(n-1)$ simplex

$$x_i \geq 0, \quad x_1 + \dots + x_i + \dots + x_n = 1$$

- I take a point in simplex, apply P and get another point. (which should be in simplex)
- P should guarantee that point should be in simplex & it's coming from spectral theory.

$$P = \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix}. \qquad \begin{bmatrix} 0.7-\lambda & 0.3 \\ 0.6 & 0.4-\lambda \end{bmatrix}$$

- if you take any transition matrix, largest eigen value will be 1.

$$(0.7-\lambda)(0.4-\lambda) - 0.18 = 0.$$
$$0.28 - 1.1\lambda + \lambda^2 - 0.18 = 0.$$
$$0.1 - 1.1\lambda + \lambda^2 = 0$$
$$0.1 \quad -\lambda \qquad \lambda = 0.1$$
$$1 \quad -\lambda \qquad \lambda = 1$$

- The norm of the vector cannot increase. (if it is a probability vector) and it cannot decrease as well (should be 1).
- Since Sum of the rows is 1, I'm operating vector to its convex comb.
- Summation of convex comb. is 1.
- $\underbrace{\text{Point} \xrightarrow{P} \text{Point} \xrightarrow{P} \text{Point} \cdots \longrightarrow \text{Point}}$

n-step Probability

sequence of points in Simplex

P has at most eigenvalue 1. (can't be increased)

det(P) < 1, meaning you decrease the distance with the operator

So, it's not possible to move out of the simplex.

Moreover, if you keep applying P, you reached a point. (if exists)

- Unique invariant distribution.

$pi * P = pi$ / $(1\ 0) \xrightarrow{P} pi$

Last part

$Pv = av \qquad mod(a) \leq 1$

$P^2 v = a^2 v \qquad v = a_1 v_1 + \cdots a_k v_k$

$Pv = a_1^2 v_1 + \cdots + a_k^2 v_k \qquad P^t$ can be done t-th power of the eigenvalue of P as above

$P^t v = a_1^t v_1 + \cdots + a_k^t v_k$

- 2nd eigenvalue very much less than 1, it will depend on eigengap.
- Eigen structure, gives how fast it converges (if it converges)
- You can reach pi as fast as eigen-gap.
- To compute eigen values: all of them is different

  repeated / eigen values are non-real / imaginary. (comes in conjugate pairs)

n - #repetition times