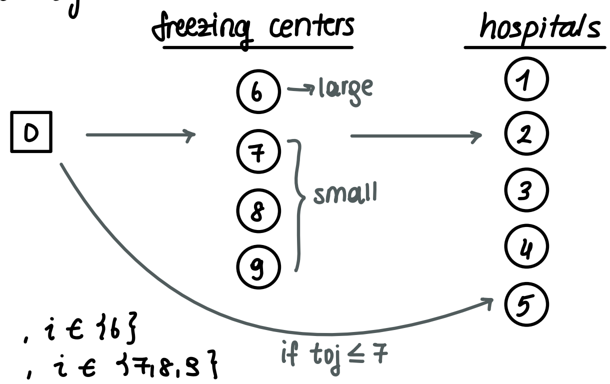
Question 1

- $G = (N, A)$
- max 7 days package
- node $D \rightarrow$ production plant
- nodes $1, 2, 3, 4, 5 \rightarrow$ hospitals (demand points) $\Rightarrow$ Let's denote them as set H, $H = \{1, 2, 3, 4, 5\}$
- node $6 \rightarrow$ large center $\Rightarrow$ + 1 day to re-package + 7 more days
- nodes $7, 8, 9 \rightarrow$ small centers $\Rightarrow$ + 3 more days



d_j = demand of hospital $j \in H$
dl_j = deadline of hospital $j \in H$

t_{ij} = travel time from i to j, $(ij) \in A$
c_{ij} = unit cost of traversing arc $(ij) \in A$

l_i = unit cost of freezing a package at the large center, $i \in \{6\}$
s_i = unit cost of freezing a package at a small center, $i \in \{7, 8, 9\}$

! If $t_{0j} \le 7$, directly go to the hospital ; if $t_{0j} > 7$ use a freezing center

a) Decision variable : $X_{ij} = \begin{cases} 1, & \text{if vaccine is transported over the arc } (ij) \in A \\ 0, & \text{o/w} \end{cases}$

Model :

$$\min \underbrace{\sum_{j \in H} d_j C_{0j} X_{0j}}_{\text{cost of direct transportations}} + \underbrace{\sum_{i \in \{6,7,8,9\}} \sum_{j \in H} d_j (C_{0i} + C_{ij}) X_{ij}}_{\text{cost of transportantions using centers}} + \underbrace{\sum_{j \in H} d_j l_6 X_{6j}}_{\text{cost of freezing at the large center}} + \underbrace{\sum_{\substack{i \in \{7,8,9\} \\ j \in H}} \sum d_j s_i X_{ij}}_{\text{cost of freezing at small centers}}$$

s.t.

$$X_{0j} + \sum_{i \in \{6,7,8,9\}} X_{ij} = 1 \qquad \forall j \in H$$

→ Each vaccine either goes to a hospital directly or should make a stop at a freezin center, this constraint also ensures freezing at most once

$$X_{0j}\, t_{0j} \le 7 \qquad \forall j \in H$$
$$X_{0j}\, t_{0j} \le dl_j \qquad \forall j \in H$$

} If vaccine directly goes to the hospital, travel time should not exceed 7 days and also the deadline

$$X_{ij}\, t_{0i} \le 7 \qquad \forall i \in \{6,7,8,9\} \; j \in H$$

→ If a freezing center is used, time between the production plant and center should not exceed 7 days

$$X_{ij}\, t_{ij} \le 3 \qquad \forall i \in \{7,8,9\} \; j \in H$$

→ small centers give us 3 more days

$$X_{6j}\, t_{6j} \le 7 \qquad \forall j \in H$$

→ large center gives us 7 more days

$$X_{ij}\,(t_{0i} + t_{ij}) \le dl_j \qquad \forall i \in \{7,8,9\},\; j \in H$$
$$X_{6j}\,(t_{06} + t_{0j} + 1) \le dl_j \qquad \forall j \in H$$

} deadline constraints

$$X_{ij} \in \{0, 1\} \qquad \forall (ij) \in A$$

b) • m alternative locations for large freezing centers , $M = \{6, \dots, 6+m\}$
• n alternative locations for small freezing centers , $S = \{7+m, \dots, 7+m+n\}$
• hospitals , $H = \{1, 2, \dots, 5\}$
• node 0 is the production plant

+ additional parameters : fl_j = fixed cost of locating a large center at $j \in M$
fs_j = fixed cost of locating a small center at $j \in S$

Decision variables :

$$x_{ij} = \begin{cases} 1, & \text{if vaccine is transported over the arc } (ij) \in A \\ 0, & \text{o/w} \end{cases}$$

$$y_i = \begin{cases} 1, & \text{if a large center is opened at } i \in M \\ 0, & \text{o/w} \end{cases}$$

$$z_i = \begin{cases} 1, & \text{if a small center is opened at } i \in S \\ 0, & \text{o/w} \end{cases}$$

Model :

$$\min \underbrace{\sum_{j \in H} d_j c_{0j} x_{0j}}_{\text{direct transportation cost}} + \underbrace{\sum_{i \in M \cup S} \sum_{j \in H} d_j (c_{0i} + c_{ij}) x_{ij}}_{\text{transportation cost when using a center}} + \underbrace{\sum_{i \in M} \sum_{j \in H} d_j l_i x_{ij}}_{\text{freezing cost for large centers}} + \underbrace{\sum_{i \in S} \sum_{j \in H} d_j s_i x_{ij}}_{\text{freezing cost for small centers}} + \underbrace{\sum_{i \in M} fl_i y_i}_{\text{large center opening cost}} + \underbrace{\sum_{i \in S} fs_i z_i}_{\text{small center opening cost}}$$

s.t.

$$x_{0j} + \sum_{i \in M \cup S} x_{ij} = 1 \quad \forall j \in H$$

$$\left.\begin{array}{ll} x_{ij} \le y_i & \forall i \in M, j \in H \\ x_{ij} \le z_i & \forall i \in S, j \in H \end{array}\right\} \text{No allocation to centers if there is no center}$$

$$\left.\begin{array}{ll} x_{0j} t_{0j} \le 7 & \forall j \in H \\ x_{0j} t_{0j} \le dl_j & \forall j \in H \end{array}\right\} \text{direct transportation time and deadline constraint}$$

$$x_{ij} t_{0i} \le 7 \quad \forall i \in M \cup S, j \in H \rightarrow \text{vaccine should arrive to centers within 7 days}$$

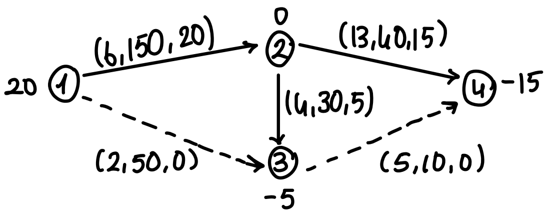
$$\left.\begin{array}{ll} x_{ij} t_{ij} \le 7 & \forall i \in M, j \in H \\ x_{ij} t_{ij} \le 3 & \forall i \in S, j \in H \end{array}\right\} \text{extra days gained from centers}$$

$$\left.\begin{array}{ll} x_{ij} (t_{0i} + t_{ij}) \le dl_j & \forall i \in S, j \in H \\ x_{ij} (t_{0i} + t_{ij} + 1) \le dl_j & \forall i \in M, j \in H \end{array}\right\} \text{deadline constrains when using a center}$$

$$\begin{array}{ll} x_{ij} \in \{0,1\} & \forall (ij) \in A \\ y_i \in \{0,1\} & \forall i \in M \\ z_i \in \{0,1\} & \forall i \in S \end{array}$$

Question 2

(1) Construct a feasible tree :



$(C_{ij}\ ,\ u_{ij}\ ,\ x_{ij})$
(cost, capacity, flow)

current cost $= (6 \times 20) + (13 \times 15) + (4 \times 5) = 335$

(2) For arcs in the tree : (using $\pi_i - \pi_j = C_{ij}$, find π_i's)

$$\left.\begin{aligned}(12) &\rightarrow \pi_1 - \pi_2 = C_{12} = 6\\ (23) &\rightarrow \pi_2 - \pi_3 = C_{23} = 4\\ (24) &\rightarrow \pi_2 - \pi_4 = C_{24} = 13\end{aligned}\right\}$$

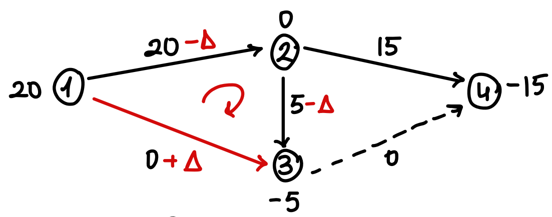
3 quations, 4 unkowns
So, let $\pi_4 = 0$ and solve:

$\pi_1 = 19$, $\pi_2 = 13$, $\pi_3 = 9$

(3) For co-tree arcs (if $x_{ij} = 0$, check if $\pi_i - \pi_j \leq C_{ij}$
if $x_{ij} = u_{ij}$, check if $\pi_i - \pi_j \geq C_{ij}$)

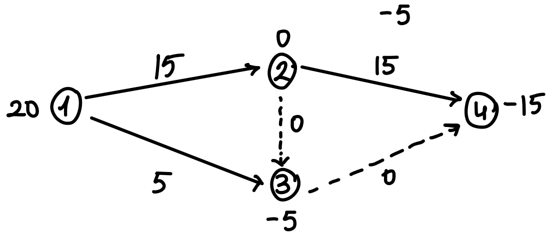
$(13) \rightarrow$ since $x_{13} = 0$ check for $\pi_1 - \pi_3 \leq C_{13} \Rightarrow 19 - 9 \overset{?}{\not\leq} 2$ X

Enter arc (13) :



$\Delta = 5$

New tree :



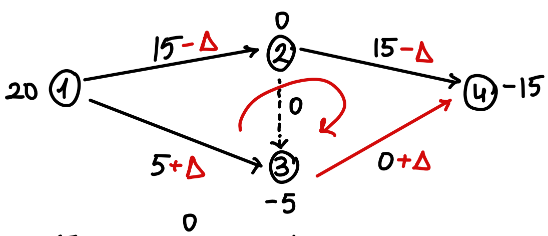
(1) For arcs in the tree :

$$\left.\begin{aligned}(12) &\rightarrow \pi_1 - \pi_2 = C_{12} = 6\\ (13) &\rightarrow \pi_1 - \pi_3 = C_{13} = 2\\ (24) &\rightarrow \pi_2 - \pi_4 = C_{24} = 13\end{aligned}\right\}$$

$\pi_4 = 0$; $\pi_1 = 19$, $\pi_2 = 13$, $\pi_3 = 17$

(2) For co-tree arcs : $(23) \rightarrow$ since $x_{23} = 0$ check for $\pi_2 - \pi_3 \leq C_{23} \Rightarrow 13 - 17 \leq 4$ ✓
$(34) \rightarrow$ since $x_{23} = 0$ check for $\pi_3 - \pi_4 \leq C_{34} \Rightarrow 17 - 0 \not\leq 5$ X

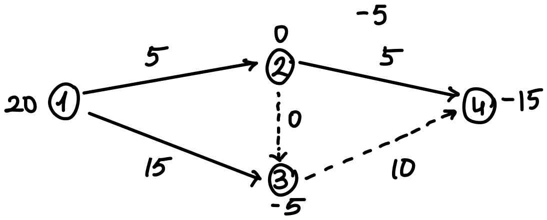
Enter arc (34) :



$\Delta = \min\{15, 15, \underbrace{45, 10}\} = 10$

max extra allowable flows for arcs (13) and (34)

New tree :



! arc (34) is not in the tree since $x_{34} = u_{34}$

(1) For arcs in the tree :

$$\left.\begin{aligned}(12) &\rightarrow \pi_1 - \pi_2 = C_{12} = 6\\ (13) &\rightarrow \pi_1 - \pi_3 = C_{13} = 2\\ (24) &\rightarrow \pi_2 - \pi_4 = C_{24} = 13\end{aligned}\right\}$$

$\pi_4 = 0$; $\pi_1 = 19$, $\pi_2 = 13$, $\pi_3 = 17$

(2) For co-tree arcs : $(23) \rightarrow$ since $x_{23} = 0$ check for $\pi_2 - \pi_3 \leq C_{23} \Rightarrow 13 - 17 \leq 4$ ✓
$(34) \rightarrow$ since $x_{34} = u_{34}$ check for $\pi_3 - \pi_4 \geq C_{34} \Rightarrow 17 - 0 \geq 5$ ✓

⇝ all satisfied ⇒ solution is optimal

cost : $(5 \times 6) + (5 \times 13) + (15 \times 2) + (10 \times 5) = 175$ //

Question 3

Distance (km)		Customer (j) 1	2	3	4	5
Location (i)	1	65	55	40	80	30
	2	70	70	55	80	20
	3	80	30	60	75	80
	4	30	40	80	40	70
	5	60	75	80	10	25
	6	20	60	75	30	80

6 possible locations with opening costs $f = [5\ 6\ 4\ 4\ 2\ 3]$
5 customers with penalty costs $P_j = 3 \quad \forall j$
(if not covered)

covering distance : $T = 35$ km
t_{ij} : distance between $i \to j$

Construct the coverage matrix $a_{ij} = \begin{cases} 1, & \text{if } t_{ij} \le T \\ 0, & \text{o/w} \end{cases}$

L \ C	1	2	3	4	5	f_i			
1	0	1	1	0	1	~~5~~	~~4~~	~~1~~	0
2	0	0	1	0	1	~~6~~	~~3~~	2	
3	0	1	0	0	0	~~4~~	3		
4	1	1	0	1	0	~~4~~	~~1~~	0	
5	0	0	0	1	1	~~2~~	1		
6	1	0	0	1	0	~~3~~	0		

leftovers : $f_i - \sum_j a_{ij} w_j$

Dual in open form :

$$
\begin{aligned}
\max\quad & w_1 + w_2 + w_3 + w_4 + w_5 \\
\text{s.t.}\quad & w_2 + w_3 + w_5 \le f_1 = 5 \\
& w_3 + w_5 \le f_2 = 6 \\
& w_2 \le f_3 = 4 \\
& w_1 + w_2 + w_4 \le f_4 = 4 \\
& w_4 + w_5 \le f_5 = 2 \\
& w_1 + w_4 \le f_6 = 3 \\
& 0 \le w_1, w_2, \dots, w_7 \le 3
\end{aligned}
$$

$\Rightarrow w^* = [3 \quad 1 \quad 3 \quad 0 \quad 1]$

Now, find y^* and z^* using CS conditions :

(1) $(f_i - \sum_j a_{ij} w_j)\, y_i = 0 \quad \forall j$

(2) $w_j \left(\sum_i a_{ij} y_i + z_j - 1 \right) = 0 \quad \forall j$

(3) $(w_j - P_j)\, z_j = 0 \quad \forall j$

For $i = 2,3,5$, $f_i - \sum_j a_{ij} w_j \neq 0 \Rightarrow y_2 = y_3 = y_5 = 0$

For $j = 2,4,5$, $w_j \neq P_j = 3 \Rightarrow z_2 = z_4 = z_5 = 0$

For $j = 1,2,3,5$, $w_j \neq 0 \Rightarrow \sum_i a_{ij} y_i + z_j = 1$

$j=1 \Rightarrow \cancel{y_4} + y_6 + z_1 = 1$

$j=2 \Rightarrow y_1^{1} + \cancel{y_3}^{0} + \cancel{y_4}^{0} + \cancel{z_2}^{0} = 1 \rightarrow y_4 = 0$

$j=3 \Rightarrow y_1^{1} + \cancel{y_2}^{0} + \cancel{z_3}^{0} = 1 \rightarrow z_3 = 0$

$j=5 \Rightarrow y_1 + \cancel{y_2}^{0} + \cancel{y_5}^{0} + \cancel{z_5}^{0} = 1 \rightarrow y_1 = 1$

We have 2 cases :

(1) $y_6 = 1,\ z_1 = 0$ in this case, facilities 1 and 6 are opened, there is no uncovered customer
$\Rightarrow$ cost $= 5 + 3 = 8$

(2) $y_6 = 0,\ z_1 = 1$ in this case, only 1 is opened and customer 1 is uncovered
$\Rightarrow$ cost $= 5 + 3 = 8$

Check feasibility : If we only open 1, 4 will also be uncovered, but $z_4 = 0 \Rightarrow$ infeasible
So, we choose case 1 : $y_6 = 1,\ z_1 = 0$

Final solution :

$y^* = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]$
$z^* = [0 \quad 0 \quad 0 \quad 0 \quad 0]$
$w^* = [3 \quad 1 \quad 3 \quad 0 \quad 1]$ total cost $= y_1 \cdot f_1 + y_6 \cdot f_6 = 5 + 3 = 8$

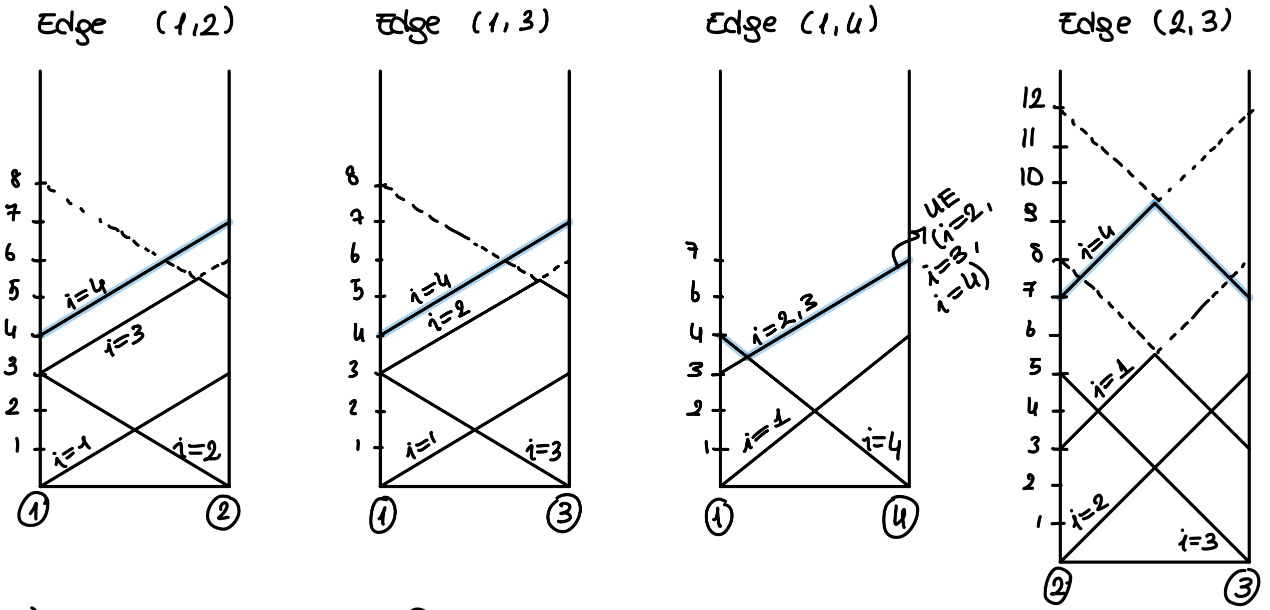
Question 4

Open 1 DC, minimize longest distance to any customer → minimize max distance

a) if we choose node 1 → max {0, 3, 3, 4} = 4 * ⇒ locate a DC at node 1
if we choose node 2 → max {3, 0, 5, 7} = 7
if we choose node 3 → max {3, 5, 0, 7} = 7
if we choose node 4 → max {4, 7, 7, 0} = 7

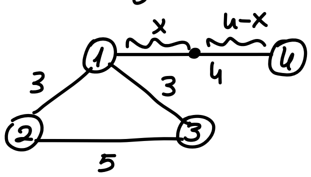
b) Now, DC can be located at the edges too → (absolute p-center problem)

Hakimi Algorithm



- (1,2) → 1-center is at 1, worst distance is 4 to node 4 (cost = 4)
- (1,3) → 1-center is at 1, worst distance is 4 to node 4 (cost = 4)

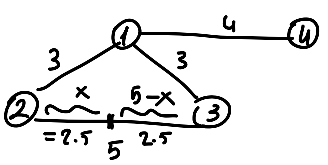
- upper envelop for edge (1,4) is determined by nodes 2, 3 and 4.



distance to 2 or 3 → 3 + x
distance to 4 → 4 − x

$3 + x = 4 - x$
$x = 0.5$

1-center is 0.5 away from node 1, worst distance = 3.5 (cost = 3.5)



distance to 4 → 7 + x OR 12 − x
distance to 1 → 3 + x OR 8 − x

$7 + x = 12 - x$
$2x = 5$
$x = 2.5$

worst distance = 9.5 to node 4 if locate between 2 and 3

But if we locate either on 2 or 3 → worst distance = 7 to node 4

⇒ min {max distance} = 3.5 when opening a center 0.5 away from node 1, cost = 3.5

Question 5

- set of available sites for base stations → $M = \{1, \dots, m\}$
- set of districts → $N = \{1, \dots, n\}$

f = fixed cost of locating a base station

C^R : cost of buying a red-type ambulance

C^B : cost of buying a blue-type ambulance

P_i : population of district $i \in N$

t_{ij} : travel time from $j \in M$ to $i \in N$

! Red-type coverage : $t_{ij} \leq 5$

Blue-type coverage : $t_{ij} \leq 9$

⇒ Let's define parameters for coverage : $a^R_{ij} = \begin{cases} 1, & \text{if } t_{ij} \leq 5, \ j \in M, \ i \in N \\ 0, & \text{o/w} \end{cases}$

$$a^B_{ij} = \begin{cases} 1, & \text{if } t_{ij} \leq 9, \ j \in M, \ i \in N \\ 0, & \text{o/w} \end{cases}$$

a) Minimize cost while covering everybody, both red-type and blue-type coverage is necessary

Decision variables :

$x_j = \begin{cases} 1, & \text{if base } j \text{ is opened}, \ j \in M \\ 0, & \text{o/w} \end{cases}$

$y^R_j = \begin{cases} 1, & \text{if there is at least one red-type ambulance at base } j \in M \\ 0, & \text{o/w} \end{cases}$

$y^B_j = \begin{cases} 1, & \text{if there is at least one blue-type ambulance at base } j \in M \\ 0, & \text{o/w} \end{cases}$

Model :

$$\min \sum_{j \in M} \left(f \cdot x_j + C^R y^R_j + C^B y^B_j \right)$$

s.t.

$\sum_{j \in M} a^R_{ij} \, y^R_j \geq 1 \quad \forall i \in N$ → Every district is covered by at least one red-type ambulance

$\sum_{j \in M} a^B_{ij} \, y^B_j \geq 1 \quad \forall i \in N$ → Every district is covered by at least one blue-type ambulance

$\left.\begin{array}{ll} y_j^R \leq x_j & \forall j \in M \\ y_j^B \leq x_j & \forall j \in M \end{array}\right\}$ an ambulance can be located only if there is a base

$x_j, \ y_j^R, \ y_j^B \in \{0,1\} \quad \forall j \in M$

b) C^* → optimal value found via the model in part (a)

Budget = $C < C^*$

+ additional decision variable : $w_i = \begin{cases} 1, & \text{if district } i \text{ is both served by red and blue type ambulances}, \ i \in N \\ 0, & \text{o/w} \end{cases}$

$$\max \sum_{i \in N} w_i$$

s.t. $\sum_{j \in M} (f \cdot x_j + C^R y_j^R + C^B y_j^B) \leq C$

$\sum_{j \in M} a_{ij}^R \, y_j^R \geq w_i \quad \forall i \in N$

$\sum_{j \in M} a_{ij}^B \, y_j^B \geq w_i \quad \forall i \in N$

$y_j^B \leq x_j \quad \forall j \in M$

$y_j^R \leq x_j \quad \forall j \in M$

$y_j^B, \ y_j^R, \ x_j, \ w_i \in \{0,1\} \quad \forall j \in M, \ i \in N$

Solve this model for different budget levels : C, $C + \Delta$, $C + 2\Delta$, ... and record how many districts ($w_i = 1$) are served. The increase in covered districts shows how many additional areas would receive both services if the budget slightly increases.

c) maximize total number of people covered within a budget, C

\+ additional decision variable : $w_i = \begin{cases} 1, & \text{if district } i \text{ is both served by red and blue type ambulances,} \\ 0, & \text{o/w} \end{cases}$ $i \in N$

Model:

$$\max \sum_{i \in N} P_i w_i$$

$$\text{s.t.} \quad \sum_{j \in M} (f \cdot x_j + C^R y_j^R + C^B y_j^B) \leq C \quad \rightarrow \text{Budget constraint}$$

$$\sum_{j \in M} a_{ij}^R y_j^R \geq w_i \quad \forall i \in N$$

$$\sum_{j \in M} a_{ij}^B y_j^B \geq w_i \quad \forall i \in N$$

} ensure that district is considered covered ($w_i = 1$) only if it is within reach of at least one red-type and at least one blue-type ambulance

$$y_j^R \leq x_j \quad \forall j \in M$$

$$y_j^B \leq x_j \quad \forall j \in M$$

} an ambulance can be located only if there is a base

d) basic coverage (blue-type coverage) is a must
maximize red-type coverage

\+ additional decision variables : $z_i^R = \begin{cases} 1, & \text{if district } i \text{ is served by a red-type ambulance}, \ i \in N \\ 0, & \text{o/w} \end{cases}$

$z_i^B = \begin{cases} 1, & \text{if district } i \text{ is served by a blue-type ambulance}, \ i \in N \\ 0, & \text{o/w} \end{cases}$

Model:

$$\max \sum_{i \in N} P_i z_i^R$$

$$\text{s.t.} \quad \sum_{j \in M} (f \cdot x_j + C^R y_j^R + C^B y_j^B) \leq C \quad \rightarrow \text{Budget constraint}$$

$$\sum_{j \in M} a_{ij}^B y_j^B \geq 1 \quad \forall i \in N \quad \rightarrow \text{Every district should be served by a blue-type ambulance}$$

$$\sum_{j \in M} a_{ij}^R y_j^R \geq z_i^R \quad \forall i \in N$$

$$y_j^R \leq x_j \quad \forall j \in M$$

$$y_j^B \leq x_j \quad \forall j \in M$$

$$x_j, y_j^R, y_j^B \in \{0,1\} \quad \forall j \in M$$

$$z_i^R, z_i^B \in \{0,1\} \quad \forall i \in N$$

e) Only one type of coverage is enough, maximize total appreciation

\+ additional parameters : q = level of appreciation per person if served by a blue-type ambulance
$3q$ = level of appreciation per person if served by a red-type ambulance
Q = minimum number of people receiving service required by the government (lower bound)

! use the same decision variables : $x_j, y_j^R, y_j^B, z_i^R, z_i^B$

\+ additional decision variable :

$u_i = \begin{cases} 1, & \text{if district } i \text{ is served by at least one type of ambulance} \\ 0, & \text{o/w} \end{cases}$

If a district is served by both red and blue type ambulances, the total appreciation would be :
$3q + q = 4q$ (double counting)

! To avoid double counting, we will formulate the appreciation level per district as follows:

$9 \cdot u_i + 29 \cdot z_i^R \rightarrow$ if only covered by blue-type $\Rightarrow u_i = 1,\ z_i^R = 0 \Rightarrow 9$
if only covered by red-type $\Rightarrow u_i = 1,\ z_i^R = 1 \Rightarrow 39$
if covered by both blue and red $\Rightarrow u_i = 1,\ z_i^R = 1 \Rightarrow 39$
↳ avoided double counting

Model: $\max \sum_{i \in N} p_i (9 \cdot u_i + 29 \cdot z_i^R)$

s.t. $\sum_{j \in M} (f \cdot x_j + C^R y_j^R + C^B y_j^B) \le C$ → Budget constraint

$\sum_{j \in M} a_{ij}^R y_j^R \ge z_i^R \quad \forall i \in N$

$\sum_{j \in M} a_{ij}^B y_j^B \ge z_i^B \quad \forall i \in N$

$u_i \le z_i^R + z_i^B \quad \forall i \in N$

if covered at least by one type ($u_i = 1$), at least one of z_i^B or z_i^R should be 1.
→ If both z_i^B and z_i^R are zero, u_i can not be 1.

$\sum_{i \in N} p_i \cdot u_i \ge Q$

→ total # of people receiving service should be at least Q

f) hospital k located at Ankara
total journey: base j ⟶ district i ⟶ hospital k
$t_{ij} + (2\text{ min}) + d_{ik}$

+ d_{ik} = travel time between district i to hospital k
+ p = number of bases to open

+ new decision variables ⇒ $b_{ij}^R = \begin{cases} 1, & \text{if district } i \text{ is served by base } j \text{ with a red-type ambulance} \\ 0, & \text{otherwise} \end{cases}$

$b_{ij}^B = \begin{cases} 1, & \text{if district } i \text{ is served by base } j \text{ with a blue-type ambulance} \\ 0, & \text{otherwise} \end{cases}$

Model: $\max \sum_{i \in N} p_i z_i^R$

$\sum_{j \in M} x_j = p$ → total number of bases opened should be p

$\sum_{j \in M} b_{ij}^B \ge 1 \quad \forall i \in N$ → blue-type coverage is necessary

$\sum_{j \in M} b_{ij}^R \ge z_i^R \quad \forall i \in N$

$b_{ij}^R \le a_{ij}^R y_j^R \quad \forall i \in N,\ j \in M$
$b_{ij}^B \le a_{ij}^B y_j^B \quad \forall j \in N,\ j \in M$

} assignment (base-district) is only made if coverage is satisfied and there is ambulance at the base (according to each type)

$y_j^R \le x_j \quad \forall j \in M$
$y_j^B \le x_j \quad \forall j \in M$

$(t_{ij} + 2 + d_{ik})\, b_{ij}^R \le 12 \quad \forall i \in N,\ j \in M$
$(t_{ij} + 2 + d_{ik})\, b_{ij}^B \le 14 \quad \forall i \in N,\ j \in M$

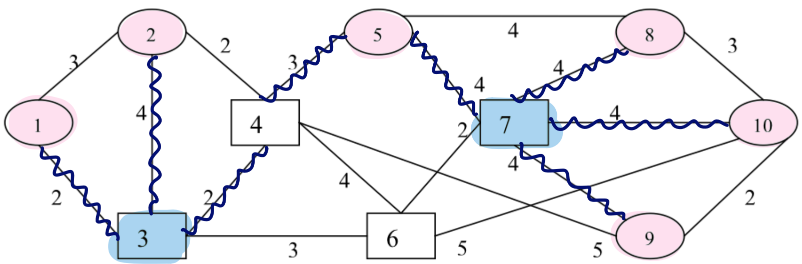
} time constraints

$x_j,\ y_j^R,\ y_j^B \in \{0,1\} \quad \forall j \in M$

$z_i^R,\ z_i^B \in \{0,1\} \quad \forall i \in N$

$b_{ij}^R,\ b_{ij}^B \in \{0,1\} \quad \forall i,j$

Question 6



$1, 2, 5, 8, 9, 10 \rightarrow$ customers

$3, 4, 6, 7 \rightarrow$ alt. sites

a) Coverage matrix $A = [a_{ij}]$ for $T = 5$, $a_{ij} = \begin{cases} 1 & \text{if } d_{ij} \le T \\ 0 & \text{o/w} \end{cases}$

$$\begin{array}{c|c} L \backslash C & \begin{matrix} 1 & 2 & 5 & 8 & 9 & 10 \end{matrix} \\ \begin{matrix} 3 \\ 4 \\ 6 \\ 7 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

b) Dual greedy approach $(p_j = 3, \ \forall j)$

$$\begin{array}{c|c|l} L \backslash C & \begin{matrix} 1 & 2 & 5 & 8 & 9 & 10 \end{matrix} & f_i \\ \begin{matrix} 3 \\ 4 \\ 6 \\ 7 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} & \begin{matrix} \cancel{5} \ \cancel{2} \ 0 \\ \cancel{9} \ \cancel{6} \ \cancel{4} \ 1 \\ \cancel{6} \ \cancel{3} \ 1 \\ \cancel{8} \ \cancel{5} \ \cancel{2} \ 0 \end{matrix} \\ w^* & \begin{matrix} 3 & 2 & 0 & 3 & 3 & 2 \end{matrix} & \end{array}$$

$$\begin{array}{cc} & \begin{matrix} 1 & 2 & 5 & 8 & 9 & 10 \end{matrix} \\ w^* = & [\begin{matrix} 3 & 2 & 0 & 3 & 3 & 2 \end{matrix}] \\ z = & [\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}] \end{array}$$

$$\begin{array}{cc} & \begin{matrix} 3 & 4 & 6 & 7 \end{matrix} \\ y = & [\begin{matrix} 1 & 0 & 0 & 1 \end{matrix}] \end{array}$$

CS conditions

$(f_i - \sum_j a_{ij} w_j)\, y_i = 0 \quad \forall i \rightarrow$ for $i = 4, 6$ $\ f_i - \sum_j a_{ij} w_j \neq 0 \Rightarrow y_4 = y_6 = 0$

$(w_j - p_j)\, z_j = 0 \quad \forall j \rightarrow$ for $j = 2, 5, 10$ $\ w_j \neq p_j \Rightarrow z_2 = z_5 = z_{10} = 0$

$(\sum_i a_{ij} y_i + z_j - 1)\, w_j = 0 \quad \forall j \rightarrow$ for $j = 1, 2, 8, 9, 10$ $\ w_j \neq 0$

$$\left.\begin{array}{lll} j=1 & \Rightarrow & y_3 + \cancel{y_4} + \cancel{y_6} + \cancel{z_1} = 1 \\ j=2 & \Rightarrow & y_3 + \cancel{y_4} + \cancel{z_2} = 1 \\ j=8 & \Rightarrow & y_7 + \cancel{z_8} = 1 \\ j=9 & \Rightarrow & \cancel{y_4} + y_7 + \cancel{z_9} = 1 \\ j=10 & \Rightarrow & \cancel{y_6} + y_7 + \cancel{z_{10}} = 1 \end{array}\right\} \quad \begin{array}{l} y_7 = 1 \\ z_9 = 0 \\ z_8 = 0 \\ y_3 = 1 \\ z_1 = 0 \end{array}$$

$\rightsquigarrow$ optimal solution is to open at 3 an 7, 3 covers 1, 2, 5
7 covers 5, 8, 9, 10

no uncovered customers

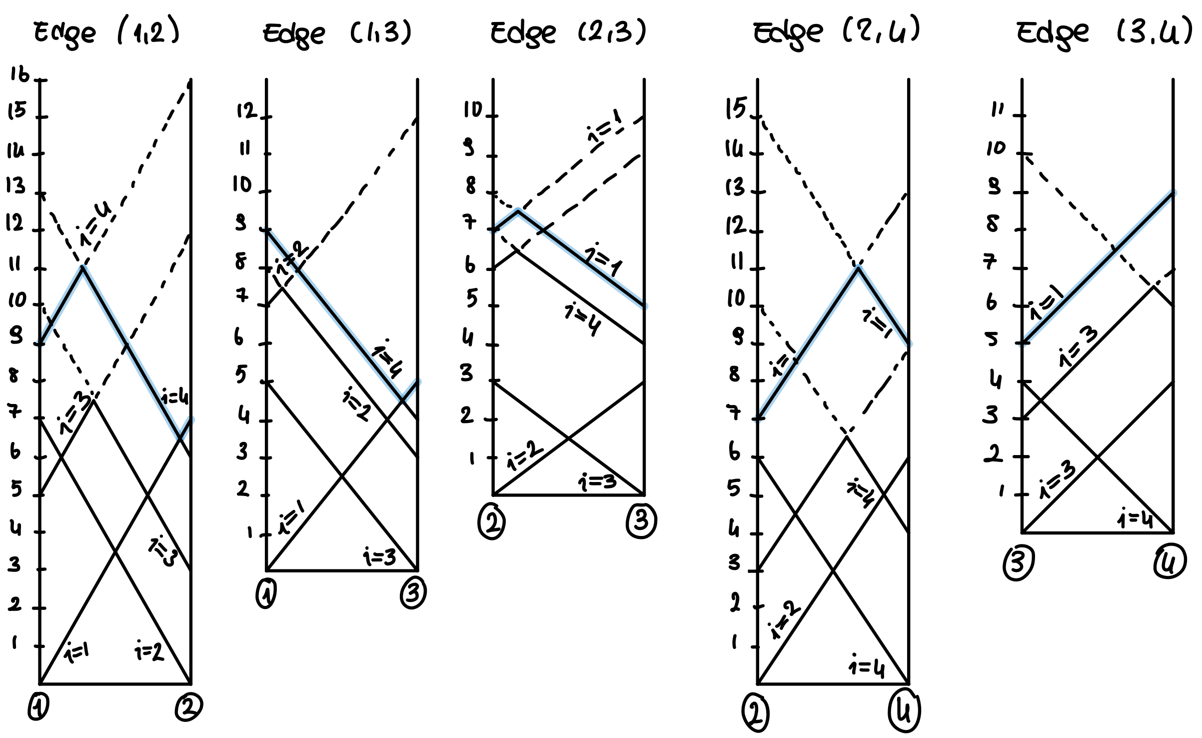
total cost $= 5 + 8 = 13$

Question 7

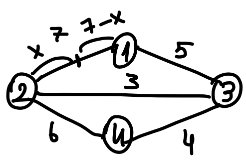
a) 1 - center problem (vertex restricted)

if locate at 1 → $\max\{0,7,5,9\} = 9$
" " " 2 → $\max\{7,0,3,6\} = 7$
" " " 3 → $\max\{5,3,0,4\} = 5$ * ⇒ 1 - center at 3
" " " 4 → $\max\{9,6,4,0\} = 9$ cost = 5

b) Absolute 1 - center problem : Hakimi Algorithm
Edges → (1,2) , (1,3), (2,3) , (2,4) , (3,4)



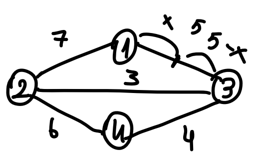
(1,2) → upper envelop determined by 1 and 4



$7 - x = 6 + x$
$2x = 1$
$x = 0.5$

1-center at 0.5 away from 2
worst distance = 6.5

(1,3) → upper envelop determined by 1 and 4



$x = 9 - x$
$2x = 9$
$x = 4.5$

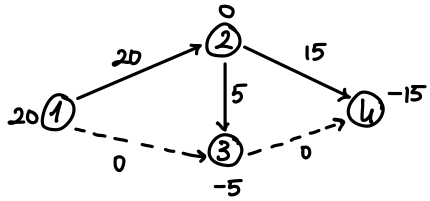
1 - center at 0.5 away from 3
worst distance = 4.5 * ⇒ optimal solution is to open 0.5 away from 3 on edge (1,3)

(2,3) → 1 - center at 3 , worst distance = 5
(2,4) → 1 - center at 2 , worst distance = 7
(3,4) → 1 - center at 3 , worst distance = 5

cost = 4.5 //

Question 8

(1) Find a feasible tree :



cost = $20 \times 4 + 5 \times 4 + 15 \times 2 = 130$

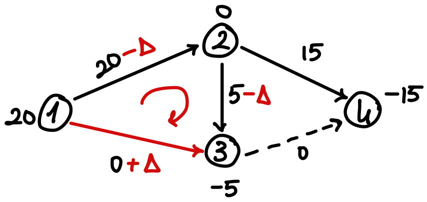
(2) Arcs on the tree :

$(12) \rightarrow \pi_1 - \pi_2 = C_{12} = 4$
$(23) \rightarrow \pi_2 - \pi_3 = C_{23} = 4$
$(24) \rightarrow \pi_2 - \pi_4 = C_{24} = 2$

$\pi_4 = 0$; $\pi_1 = 6$, $\pi_2 = 2$, $\pi_3 = -2$

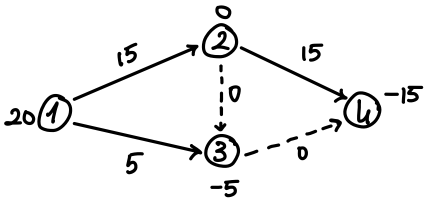
(3) Co-tree arcs : $(13) \rightarrow$ since $X_{13} = 0$ check for $\pi_1 - \pi_3 \leq C_{13} \Rightarrow 6-(-2) \not\leq 2$ X

Enter arc (13) :



$\Delta = \min\{20, 5, 50\} = 5$

New tree :



(1) Arcs on the tree :

$(12) \rightarrow \pi_1 - \pi_2 = C_{12} = 4$
$(13) \rightarrow \pi_1 - \pi_3 = C_{13} = 2$
$(24) \rightarrow \pi_2 - \pi_4 = C_{24} = 2$

$\pi_4 = 0$; $\pi_1 = 6$, $\pi_2 = 2$, $\pi_3 = 4$

(2) Co-tree arcs :

$(23) \rightarrow$ since $X_{23} = 0$ check for $\pi_2 - \pi_3 \leq C_{23} \Rightarrow 2 - 4 \leq 4$ ✓
$(34) \rightarrow$ since $X_{34} = 0$ check for $\pi_3 - \pi_4 \leq C_{34} \Rightarrow 4 - 0 \leq 5$ ✓

→ all satisfied , solution is optimal

cost = $(15 \times 4) + (5 \times 2) + (15 \times 2) = 100$ //