Stochastic Inventory Systems - Extensions

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On-Line Course Notes

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Newsvendor model and periodic review extensions

Newsvendor Model
Periodic Review Extensions

- Starting inventory
- Multiple periods
  - Infinite horizon, backorder
  - Infinite horizon, lost sales
  - Service level models
  - Setup costs

Newsvendor model – Starting inventory

- Earlier, we showed that the cost function $C(y)$ is convex in y.
- Let y^* be the optimal value of the cost function $C(y)$. Since $C(y)$ is convex,
  - For $y<y^*$, $C(y)$ is decreasing in y
  - For $y>y^*$, $C(y)$ is increasing in y
- Therefore, ordering policy should be
  - if starting inventory is less than y^*, order up to y^*
  - If starting inventory is more than or equal to y^*, do not order

Periodic Review Inventory Systems

- The solution of newsvendor model cannot be used repeatedly in a multi-period setting
  - Remaining inventory in one period can be used to satisfy the demand in next period
  - Any unsatisfied demand in one period can be satisfied in next period (in backorder case)
  - Replenishment of multiple periods can be combined to save from fixed costs (in setup case)

Periodic Review Inventory Systems

- A basic model
  - **Infinite horizon** (infinitely many periods)
  - Unsatisfied demand is **backordered**
  - No setup costs for ordering
  - Find the order-up-to-level to minimize inventory holding and shortage costs
- Note: finite horizon models result in order-up-to levels smaller than order-up-to levels that are computed with infinite horizon assumption

Periodic Review: Infinite Horizon, Backorders

- Let D_1, D_2,… be infinite sequence of demands (independent and identical).
- The policy is to order up to Q in each period
- All excess demand is backordered
- <u>Lead time is zero</u>.
- S is the sales price of the item, c is the cost, p is the backlogging cost (loss of goodwill) and h is the inventory holding cost per period (charged per unit of any inventory left at the end of a period)

Periodic Review: Infinite Horizon, Backorders

- Number of units sold
  - In period 1 = $min\{Q,D_1\}$
  - In period 2 = $max\{D_1\text{-}Q,0\}+min\{Q,D_2\}$
  - In period 3 = $max\{D_2\text{-}Q,0\}+min\{Q,D_3\}$
  - ..
- But, $min\{Q, D_i\}+max\{D_i\text{-}Q, 0\} = D_i$. Therefore, the demand in every period is eventually completely purchased and sold (except the last period)

Periodic Review: Infinite Horizon, Backorders

Cost for n periods

$$cQ + (c - S)E(D_1 + D_2 + \ldots + D_{n-1}) - SE[\min\{Q, D_n\}] + nC(Q)$$

where

$$C(Q) = hE[\max\{Q - D, 0\} + pE[\max\{D - Q, 0\}]$$

$$= h\int_0^Q (Q - x)f(x)dx + p\int_Q^\infty (x - Q)f(x)dx$$

Divide the total cost by n and take the limit as $n \to \infty$

to find the long term average costs per period

$$(c - s)\mu + C(Q)$$

To find the optimal Q, take the derivative wrt Q and set to 0

$$C'(Q) = 0$$

Periodic Review: Infinite Horizon, Backorders

The one period cost function

$$C(Q) = h\int_{0}^{Q}(Q-x)f(x)dx + p\int_{Q}^{\infty}(x-Q)f(x)dx$$

is nothing but the same cost function that we have for the one period newsvendor problem, where h can be interpreted as the overage cost and p as the underage cost

To find the derivative of C(Q), use Leibniz's rule to get

$$C'(Q) = h\int_{0}^{Q}f(x)dx - p\int_{Q}^{\infty}f(x)dx = hF(Q) - p[1-F(Q)]$$

Then, $C'(Q) = (h+p)F(Q) - p$

Thus, $C'(Q^*) = 0 \Rightarrow F(Q^*) = \dfrac{p}{p+h}$

will give Q^* that minimizes average costs

Periodic Review: Infinite Horizon, Lost Sales

- Let D_1, D_2,… be infinite sequence of demands (independent and identical).
- The policy is to order up to Q in each period
- All excess demand is lost
- <u>Lead time is zero.</u>
- S is the sales price of the item, c is the cost, p is the backlogging cost (loss of goodwill) and h is the inventory holding cost per period (charged per unit of any inventory left at the end of a period)

Periodic Review: Infinite Horizon, Lost Sales

- Number of units sold
  - In period 1 = $min\{Q,D_1\}$
  - In period 2 = $min\{Q,D_2\}$
  - In period 3 = $min\{Q,D_3\}$
  - ..
- The number of units purchased in one period is exactly equal to the number of units sold in the previous period (and nothing else since there is no backordering), since the policy is to order up to Q

Periodic Review: Infinite Horizon, Lost Sales

$$E[\min\{Q,D\}] = E[D] - E[\max\{D-Q,0\}] = \mu - \int_Q^\infty (x-Q)\, f(x)dx$$

Cost for n periods

$$cQ + [(n-1)c - nS][\mu - \int_Q^\infty (x-Q)\, f(x)dx] + nC(Q)$$

where

$$C(Q) = h\int_0^Q (Q-x)\, f(x)dx + p\int_Q^\infty (x-Q)\, f(x)dx$$

Divide the total cost by n and take the limit as $n \to \infty$

to find the long term average costs per period

$$(c-s)[\mu - \int_Q^\infty (x-Q)\, f(x)dx] + C(Q)$$

To find the optimal Q, take the derivative wrt Q and set to 0

Periodic Review: Infinite Horizon, Lost Sales

The derivative of $\mu - \int_{Q}^{\infty} (x-Q) f(x) dx$ wrt Q is $(1-F(Q))$

Thus, the derivative of the cost function with respect to Q

$(c\text{-}S)(1-F(Q)) + hF(Q) - p(1-F(Q))$

Equating this to zero will give the optimality condition

$$\mathrm{F(Q^*)} = \frac{p + S\text{-}c}{p + S + h\text{-}c}$$

This is similar to the earlier condition, except that we have an additional S-c on both sides to incorporate lost profit, in case of lost sales

Periodic Review, Service Level Constraints

- Service level constraints can be incorporated to periodic review systems
- For the case of no set-up costs and no leadtime
- Type I service level (α)
  - *Pr{D<=Q)=F(Q)=* α
- Type II service level *(*β)

$$\beta = 1 - \frac{n(Q)}{\mu}$$

$n(Q)$: average demand not satisfied μ : average demand

$$n(Q) = E[\max\{D - Q, 0\}] = \int_Q^\infty (x - Q) f(x) dx$$

$n(Q) = \sigma L(z)$ for Normal demand

Periodic Review, Set-up costs

- Analysis of periodic review systems so far assumes that there is no set-up cost associated with ordering
- In a continuous review setting with set-up costs, we derive an *(R,Q)* system: when the inventory level drops to *R*, create an order of size *Q*
- The problem with this approach in the periodic review setting is that since the inventory is reviewed periodically, it is impossible to place an order at the instant that the inventory reaches R
- To overcome this problem, a new operating policy is used in periodic systems
  - If the inventory level is less than or equal to s, order enough to take the inventory level to S
  - If the inventory level is above s, do not order

Periodic Review, Set-up costs

- These ordering policies are called *(s,S)* policies or re-order point, order-up-to-level policies.
- These policies are optimal in periodic systems with convex (or linear) inventory holding and backlog cost structures and when ordering cost has a fixed component
- However it is quite difficult to come up with the optimal values of *(s,S)*
- Many practical systems use approximations. One approximation is to compute a *(Q,R)* policy and set $s=R$ and $S=R+Q$.

Practical Issues (Final)

- A items – Continuous Review – cannot afford to stock more or less – precision is required. Hence, planner may devote more time.
- C items – Less concern. Their presence are necessary, but decisions should not take much effort. – Periodic Review with relatively long review periods. Less concern with forecasting, etc. No harm in keeping large safety stocks.
- B items – In between

Practical Issues

- C items + B items – Decide on the SS by using a service level

  SS = k*σ(over: lead-time + review period), where k is determined by a service level (you can have the same k for most of the items to be consistent – check total safety stock investment to make a final decision on k)