IE 400 Principles of Engineering Management

The Simplex Algorithm-I:

Set 3

- So far, we have studied how to solve two-variable LP problems graphically.
- However, most real life problems have more than two variables!
- Therefore, we need to have another method to solve LPs with more than two variables.
- We are going to study **The Simplex Algorithm** which is quite useful in solving very large LP problems.
- Today, The Simplex Algorithm is used to solve LP problems in many industrial applications that involve thousands of variables and constraints.

- LP problems can have both equality and inequality constraints.
- LP problems can have nonnegative and urs (unrestricted in sign) variables.
- To use the Simplex Method, LP problems should be converted to ***Standard Form LP***.

Standard Form LP:

- All constraints are equalities, with constant nonnegative right-hand sides (RHS),
- All variables are nonnegative.

To Convert an LP into Standard Form:

- each inequality constraint is converted into an equality constraint by adding or subtracting nonnegative slack/excess variables,

- an inequality can be multiplied by -1 to get nonnegative RHS,

To Convert an LP into Standard Form:

- unrestricted variable can be represented as the difference of two new nonnegative variables.

  If x_i is urs, then let $x_i = x_i' - x_i''$ where $x_i', x_i'' \geq 0$.

  And then replace every occurrence of x_i with $x_i' - x_i''$ and add sign restrictions $x_i', x_i'' \geq 0$.

- P.S: If you come across a sign restriction $x_k \leq 0$, then let $x_k' = -x_k$ where $x_k' \geq 0$.

  And then replace every occurrence of x_k with $-x_k'$ and add the sign restriction $x_k' \geq 0$.

Converting LP into Standard Form:

Ex: Consider the following LP problem:

$$\max \quad x_1 + 3\,x_2$$

s.t.

$$2x_1 + 3x_2 + x_3 \leq 5 \qquad (1)$$

$$4x_1 + x_2 + 2x_3 \leq 11 \qquad (2)$$

$$3x_1 + 4x_2 + 2x_3 \geq 8 \qquad (3)$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$$

Converting LP into Standard Form:

a) First, we define a “slack” variable for each of the “≤” constraint to convert the inequality constraint into an equality constraint:

$$s_1 = 5 - 2x_1 - 3x_2 - x_3, \; s_1 \geq 0 \qquad (1)$$

$$s_2 = 11 - 4x_1 - x_2 - 2x_3, \; s_2 \geq 0 \qquad (2)$$

So that, the first two constraints become:

$$2x_1 + 3x_2 + x_3 + s_1 = 5 \qquad (1)$$

$$4x_1 + x_2 + 2x_3 + s_2 = 11 \qquad (2)$$

Converting LP into Standard Form:

b) Second, we define an "excess (surplus)" variable for each of the "≥" constraint to convert the inequality constraint into an equality constraint:

$$e_3 = 3x_1 + 4x_2 + 2x_3 - 8, \; e_3 \geq 0 \qquad (3)$$

So that, the third constraint becomes:

$$3x_1 + 4x_2 + 2x_3 - e_3 = 8 \qquad (3)$$

Converting LP into Standard Form:

For reasons that will become clear later, if the i^{th} constraint is:

- "≤" constraint, then the slack variable is represented as s_i ,
- "≥" constraint, then the excess variable is represented as e_i.

Converting LP into Standard Form:

$$\begin{aligned} \max \quad & x_1 + 3x_2 \\ \text{s.t.} \quad & \\ & 2x_1 + 3x_2 + x_3 \leq 5 \quad \color{red}{(1)} \\ & 4x_1 + x_2 + 2x_3 \leq 11 \quad \color{red}{(2)} \\ & 3x_1 + 4x_2 + 2x_3 \geq 8 \quad \color{red}{(3)} \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & z = x_1 + 3x_2 \\ \text{s.t.} \quad & \\ & 2x_1 + 3x_2 + x_3 + \mathbf{s_1} = 5 \quad \color{red}{(1)} \\ & 4x_1 + x_2 + 2x_3 + \mathbf{s_2} = 11 \quad \color{red}{(2)} \\ & 3x_1 + 4x_2 + 2x_3 - \mathbf{e_3} = 8 \quad \color{red}{(3)} \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0, \\ & s_1 \geq 0,\ s_2 \geq 0,\ e_3 \geq 0. \end{aligned}$$

Standard Form LP:

Suppose that we converted an LP with *m* constraints into a standard form. Also assume that after the conversion, we have *n* variables as x_1, x_2, x_3,…, x_n.

Standard Form LP:

Suppose that we converted an LP with *m* constraints into a standard form. Also assume that after the conversion, we have *n* variables as $x_1, x_2, x_3, \ldots, x_n$.

$$\max\ (\min)\quad c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

$$\text{s.t.}$$

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2\\
\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots &\\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

$$x_i \geq 0 \quad (i = 1,2,\ldots.,n)$$

Standard Form LP:

If we define:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

and

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

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and

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

LP can be written as the system of equations :

$$Ax=b$$

Standard Form LP:

- For Ax=b to have a solution, rank (A|b)=rank (A).
- We also assume that all redundant constraints are removed, so rank(A)=m.

- i.e.

$$x_1 + 2x_2 + x_3 = 4 \quad (1)$$

$$x_2 - x_3 = 1 \quad (2)$$

$$2x_1 + 6x_2 = 10 \quad (3)$$

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Constraint (3) can be written as a linear combination of (1) and (2):

2 [(1)+(2)] = (3)

Remove the redundant constraint

Standard Form LP:

- Before proceeding any further with the discussion of the simplex algorithm, we should define the concept of a basic solution to a linear system.
- **Basic Solution:** A solution to Ax=b is called a basic solution if it is obtained by setting ***n-m*** variables equal to 0 and solving for the remaining ***m*** variables whose columns are linearly independent.

Standard Form LP:

- The ***n-m*** variables whose values are set to 0 are called **nonbasic variables**.
- The remaining ***m*** variables are called **basic variables**.

Standard Form LP:

For $Ax=b$, where A is a $m \times n$ matrix, $\text{rank}(A)=m$, $b \geq 0$,

- pick m linearly independent columns from A,
- rearrange A such that these chosen columns are the first m columns in A.

Standard Form LP:

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- pick m linearly independent columns from A,
- rearrange A such that these chosen columns are the first m columns in A.

Since elementary column and row operations do not change the system of linear equations, matrix A can be brought into a form $A=[B_{m \times m} \;\; N_{m \times (n-m)}]$, where $B_{m \times m}$ is invertible.

Standard Form LP:

Let $x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}$ be the corresponding partition in x.

$$Ax = b \equiv \begin{bmatrix} B & N \end{bmatrix} \begin{pmatrix} x_B \\ x_N \end{pmatrix} = b \Rightarrow Bx_B + Nx_N = b$$

Standard Form LP:

Let $x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}$ be the corresponding partition in x.

$$(Ax = b) \equiv \left(\begin{bmatrix} B & N \end{bmatrix} \begin{pmatrix} x_B \\ x_N \end{pmatrix} = b \right) \Rightarrow Bx_B + Nx_N = b$$

If we set $x_N = 0$, then $x_B = B^{-1}b$ will be a unique solution.

For every such B choice, $\exists$ a unique solution $\begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}$.

Standard Form LP:

If a basic solution $x = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix} \geq 0$,

then x is called a basic feasible solution (bfs).

Basic Solutions:

- Consider the system of equations:

$$\begin{aligned} x_1 + x_2 + x_3 \quad &= 6 \\ x_2 \quad + x_4 &= 3 \end{aligned}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Basic Solutions:

- Consider the system of equations:

$$
\begin{aligned}
x_1 + x_2 + x_3 \quad &= 6 \\
x_2 \quad + x_4 &= 3 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}
$$

$$\Rightarrow$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \qquad b = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

Basic Solutions:

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

1. Let us chose:

$$B = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\substack{\text{The columns are} \\ \text{linearly independent}}} \qquad x_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad x_N = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

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$$\text{Setting } x_N = 0 \text{ and solving for } x_B : \; Bx_B = b \Rightarrow x_B = B^{-1}b$$

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The columns are linearly independent

$$\text{Setting } x_N = 0 \text{ and solving for } x_B: \; Bx_B = b \Rightarrow x_B = B^{-1}b$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1}\begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

Basic Solutions:

$$\text{Since } x_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \geq 0, \quad x = \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 0 \\ 0 \end{pmatrix} \text{ is a bfs.}$$

Basic Solutions:

Let us consider different ways of forming B:

2.

$$\underbrace{B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{\text{The columns are Linearly dependent!}} \qquad x_B = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \qquad x_N = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix}$$

Basic Solutions:

Let us consider different ways of forming B:

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$$B = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{\substack{\text{The columns are} \\ \text{Linearly dependent!}}} \quad x_B = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \quad x_N = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix}$$

Hence the choice of $x_B = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$ cannot be a basic solution.

Basic Solutions:

3. Let us chose:

$$B = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{The columns are linearly independent}} \qquad x_B = \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} \qquad x_N = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$$

Basic Solutions:

3. Let us chose:

$$B = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\substack{\text{The columns are} \\ \text{linearly independent}}} \qquad x_B = \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} \qquad x_N = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$$

Setting $x_N = 0$ and solving for x_B :

$$x_B = B^{-1}b = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} \Rightarrow x = \begin{pmatrix} 6 \\ 0 \\ 0 \\ 3 \end{pmatrix} \text{is a bfs.}$$

Basic Solutions:

4. Let us chose:

$$B = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{\text{The columns are linearly independent}} \qquad x_B = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \qquad x_N = \begin{pmatrix} x_1 \\ x_4 \end{pmatrix}$$

Basic Solutions:

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$$B = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{\text{The columns are linearly independent}} \qquad x_B = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \qquad x_N = \begin{pmatrix} x_1 \\ x_4 \end{pmatrix}$$

Setting $x_N = 0$ and solving for x_B :

$$x_B = B^{-1}b = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \Rightarrow x = \begin{pmatrix} 0 \\ 3 \\ 3 \\ 0 \end{pmatrix} \text{ is a bfs.}$$

Basic Solutions:

5. Let us chose:

$$B = \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{\text{The columns are linearly independent}} \quad x_B = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} \quad x_N = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$$

Basic Solutions:

5. Let us chose:

$$B = \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{\text{The columns are linearly independent}} \qquad x_B = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} \qquad x_N = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$$

$$\text{Setting } x_N = 0 \text{ and solving for } x_B:$$

$$x_B = B^{-1}b = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix} \Rightarrow x = \begin{pmatrix} 0 \\ 6 \\ 0 \\ -3 \end{pmatrix},$$

Basic Solutions:

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Setting $x_N = 0$ and solving for x_B :

$$x_B = B^{-1}b = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix} \Rightarrow x = \begin{pmatrix} 0 \\ 6 \\ 0 \\ -3 \end{pmatrix},$$

This basic solution is not feasible!

Basic Solutions:

6. Let us chose:

$$\underbrace{B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{The columns are linearly independent}} \quad x_B = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \quad x_N = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Basic Solutions:

6. Let us chose:

$$B = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{The columns are linearly independent}} \qquad x_B = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \qquad x_N = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Setting $x_N = 0$ and solving for x_B :

$$x_B = B^{-1}b = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} \Rightarrow x = \begin{pmatrix} 0 \\ 0 \\ 6 \\ 3 \end{pmatrix} \text{ is a bfs.}$$

Basic Feasible Solutions:

So, this system of equations has 4 basic feasible solutions:

$$\begin{pmatrix}3\\3\\0\\0\end{pmatrix}, \begin{pmatrix}6\\0\\0\\3\end{pmatrix}, \begin{pmatrix}0\\3\\3\\0\end{pmatrix}, \begin{pmatrix}0\\0\\6\\3\end{pmatrix}$$

Basic Solutions:

- Consider the system of equations:

$$x_1 + x_2 \leq 6$$

$$x_2 \leq 3$$

$$x_1, x_2 \geq 0$$

(a)

(b)

Basic Solutions:

- Consider the system of equations:

$$x_1 + x_2 \leq 6$$

$$x_2 \leq 3$$

$$x_1, x_2 \geq 0$$

(a)

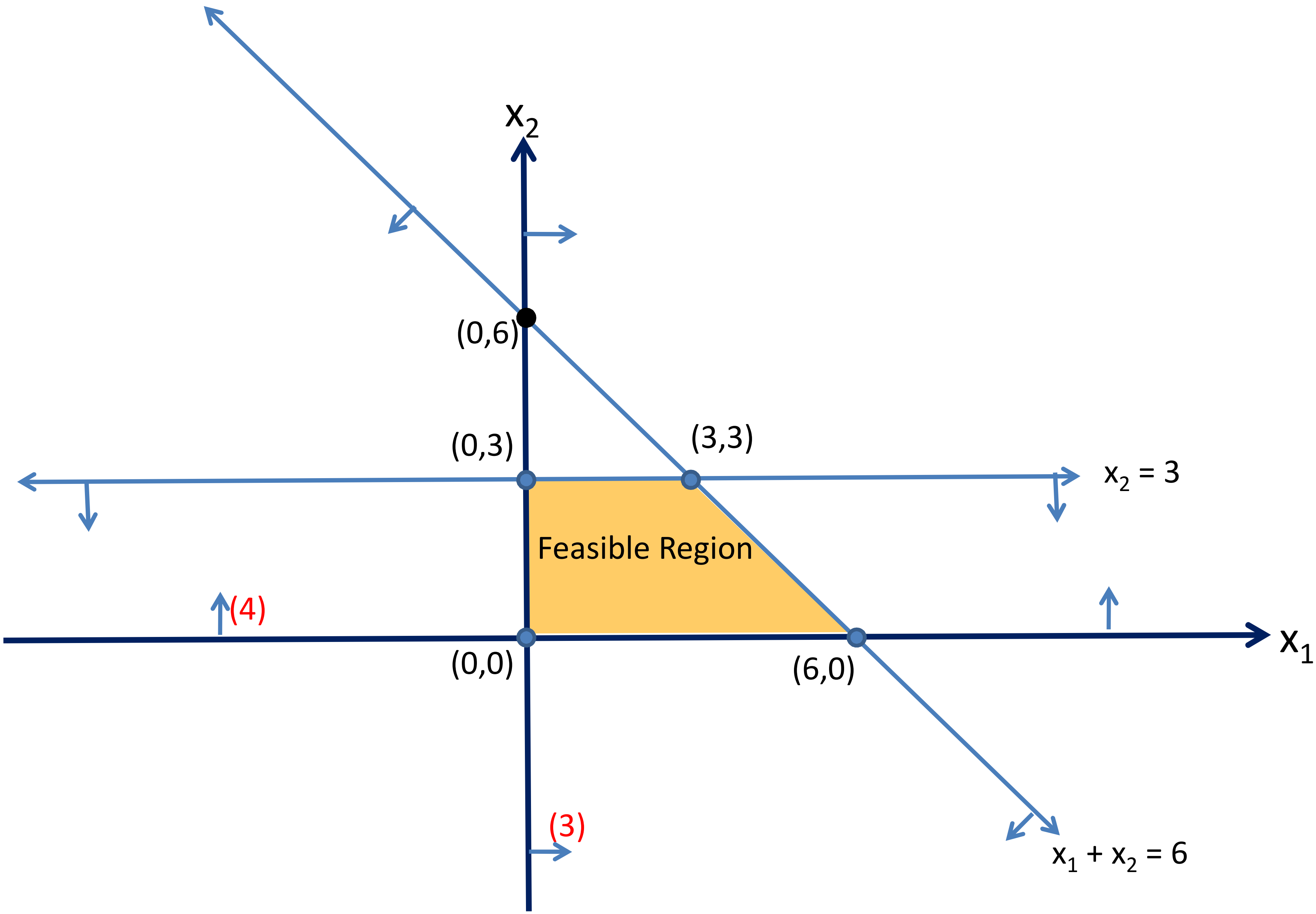
- Converting into a standard form LP:

$$x_1 + x_2 + x_3 = 6$$

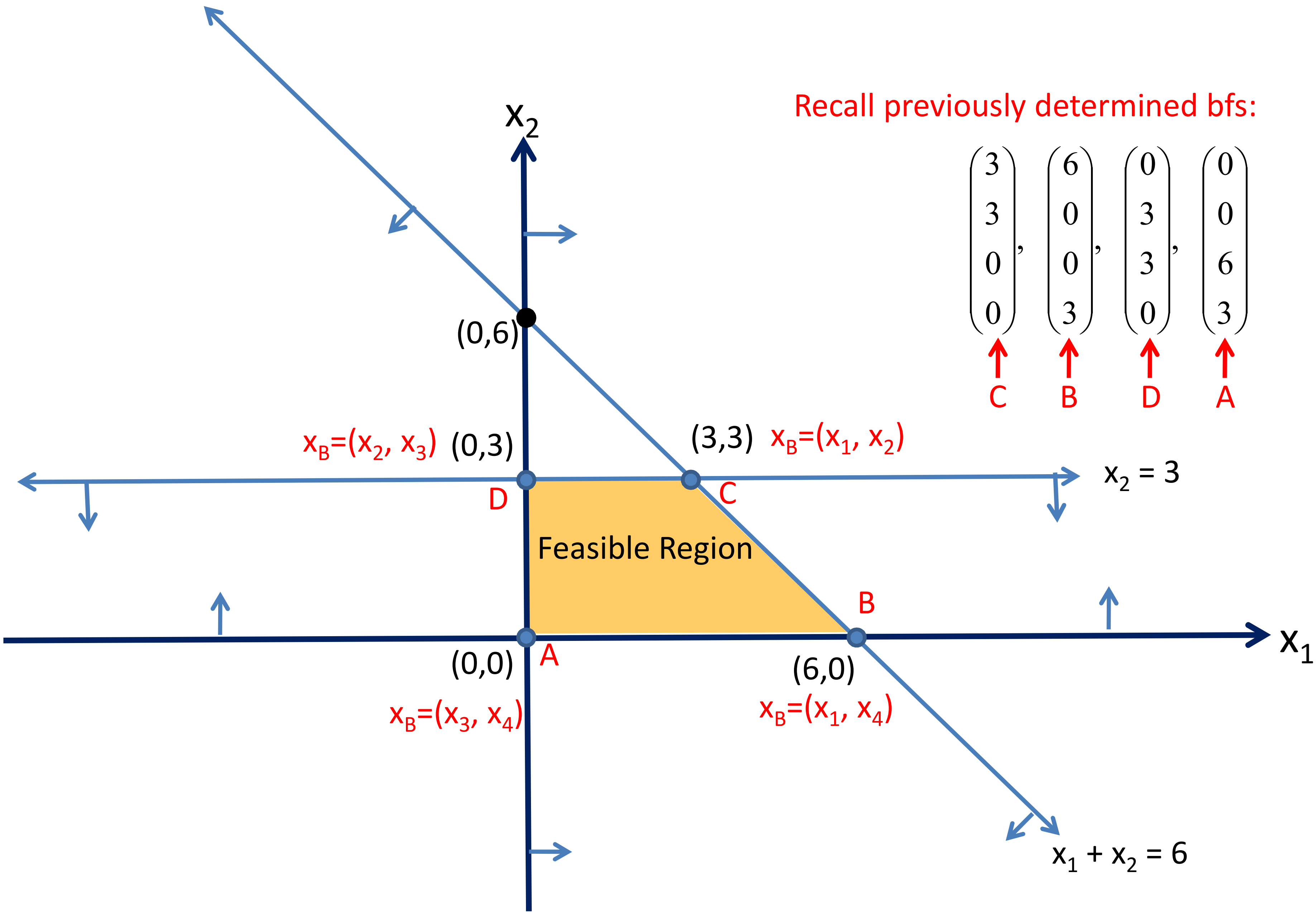
$$x_2 + x_4 = 3$$

$$x_1, x_2, x_3, x_4 \geq 0$$

(b)



x_2
(0,6)
(0,3)
(3,3)
$x_2 = 3$
Feasible Region
(4)
x_1
(0,0)
(6,0)
(3)
$x_1 + x_2 = 6$


Recall previously determined bfs:
$\begin{pmatrix}3\\3\\0\\0\end{pmatrix}, \begin{pmatrix}6\\0\\0\\3\end{pmatrix}, \begin{pmatrix}0\\3\\3\\0\end{pmatrix}, \begin{pmatrix}0\\0\\6\\3\end{pmatrix}$
C
B
D
A
x_2
x_1
(0,6)
(0,3)
$x_B=(x_2, x_3)$
D
(3,3)
$x_B=(x_1, x_2)$
C
$x_2 = 3$
Feasible Region
B
(0,0)
A
$x_B=(x_3, x_4)$
(6,0)
$x_B=(x_1, x_4)$
$x_1 + x_2 = 6$

Basic Feasible Solutions:

$$\text{The number of basic feasible solutions } \leq \binom{n}{m}$$

Theorem:

A point in the feasible region of an LP is an extreme point if and only if it is a basic feasible solution to the LP.

Fundamental Theorem of LP:

- If feasible set of an LP is non-empty, then there is at least one bfs.
- If an LP has an optimal solution, then there is a bfs which is optimal.

This Theorem Implies:

- Finding an optimal solution to an LP problem is equivalent to finding the best bfs!

$$\text{The number of bfs} \leq \binom{n}{m}$$

- Therefore, we should search for the best bfs.

- One way is to enumerate all bfs and choose the one that gives he best objective function value.

- One way is to enumerate all bfs and choose the one that gives he best objective function value.

- However, enumerating all bfs can be very expensive!

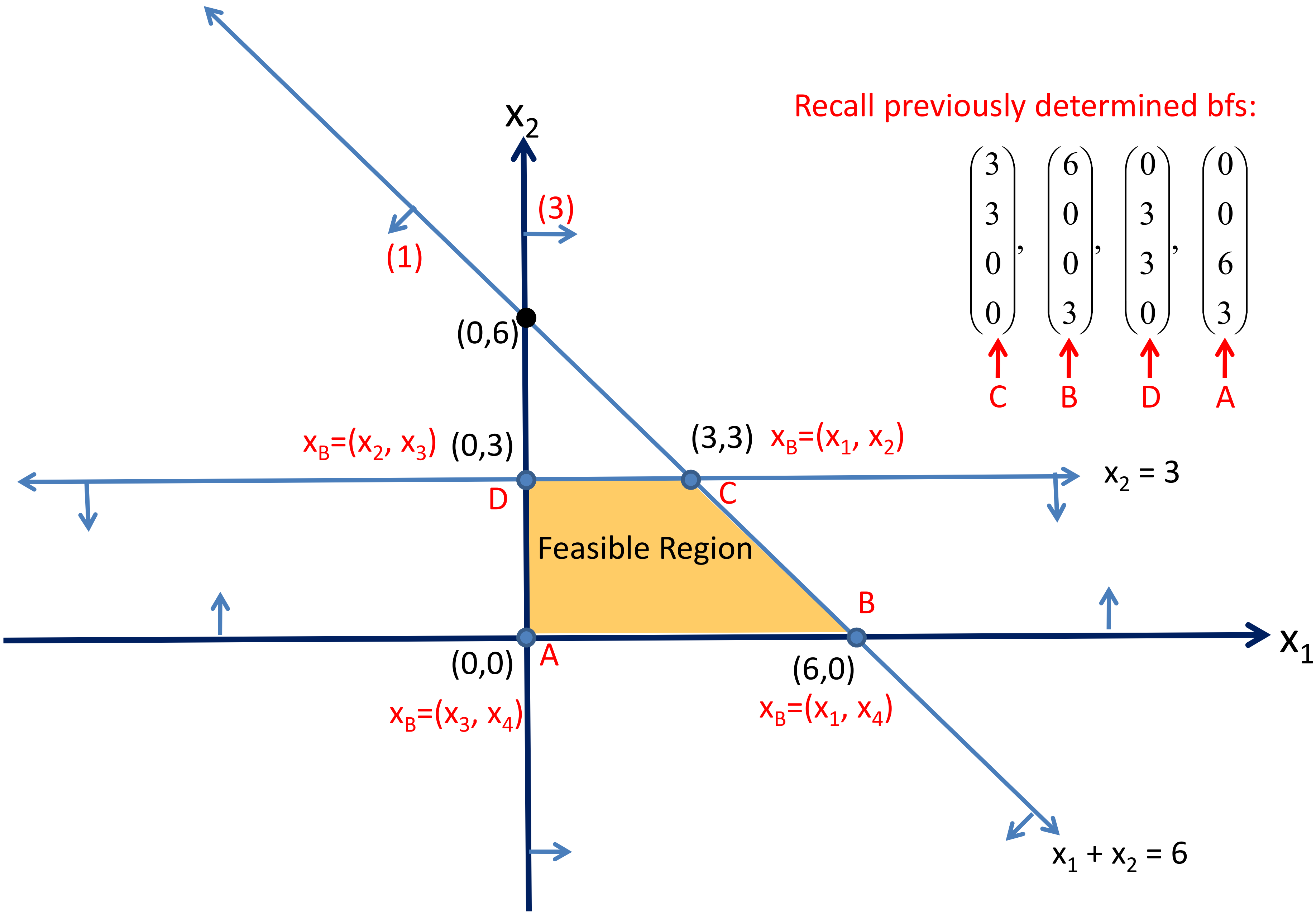
$$\text{For example, for } n = 20 \text{ and } m = 10, \binom{n}{m} \text{ is } 184756!$$

- Simplex Algorithm does this in a clever way. Usually it finds an optimal solution within *3m* enumeration.

- For any LP with *m* constraints, two bfs are said to be **adjacent** if their set of basic variables have *m-1* basic variables in common.

  (Intuitively, two bfs are adjacent if they both lie on the same edge of the boundary of the feasible region)

- Simplex Algorithm goes from an extreme point to an adjacent extreme point with a better objective value.



Recall previously determined bfs:
(3, 3, 0, 0), (6, 0, 0, 3), (0, 3, 3, 0), (0, 0, 6, 3)
C B D A
x2
x1
(3)
(1)
(0,6)
(0,3)
(3,3)
(0,0)
(6,0)
xB=(x2, x3)
xB=(x1, x2)
xB=(x3, x4)
xB=(x1, x4)
D
C
B
A
x2 = 3
Feasible Region
x1 + x2 = 6

General Description of the Simplex Algorithm:

1. Convert the LP problem into a standard form LP.

2. Obtain a bfs to the LP. This bfs is called the **initial bfs**. In general, the most recent bfs is called the **current bfs**. Therefore, at the beginning the initial bfs is the current bfs.

General Description of the Simplex Algorithm:

3. Determine if the current bfs is an optimal solution or not.
4. If the current bfs is not optimal, then find an adjacent bfs with a better objective function value (one nonbasic variable becomes basic and one basic variable becomes nonbasic).
5. Go to Step 3.