IE 400 Principles of Engineering Management

Graphical Solution of 2-variable LP Problems

Graphical Solution of 2-variable LP Problems

Ex 1.a)

$$\max \quad x_1 + 3\,x_2$$

s.t.

$$x_1 + x_2 \leq 6 \qquad (1)$$

$$-x_1 + 2x_2 \leq 8 \qquad (2)$$

$$x_1, x_2 \geq 0 \qquad (3), (4)$$

Graphical Solution of 2-variable LP Problems

max $\quad x_1 + 3\,x_2$

s.t.

$x_1 + x_2 \le 6$ (1)

$-x_1 + 2x_2 \le 8$ (2)

$x_1, x_2 \ge 0$ (3), (4)

General Form:

max $\quad cx$

s.t.

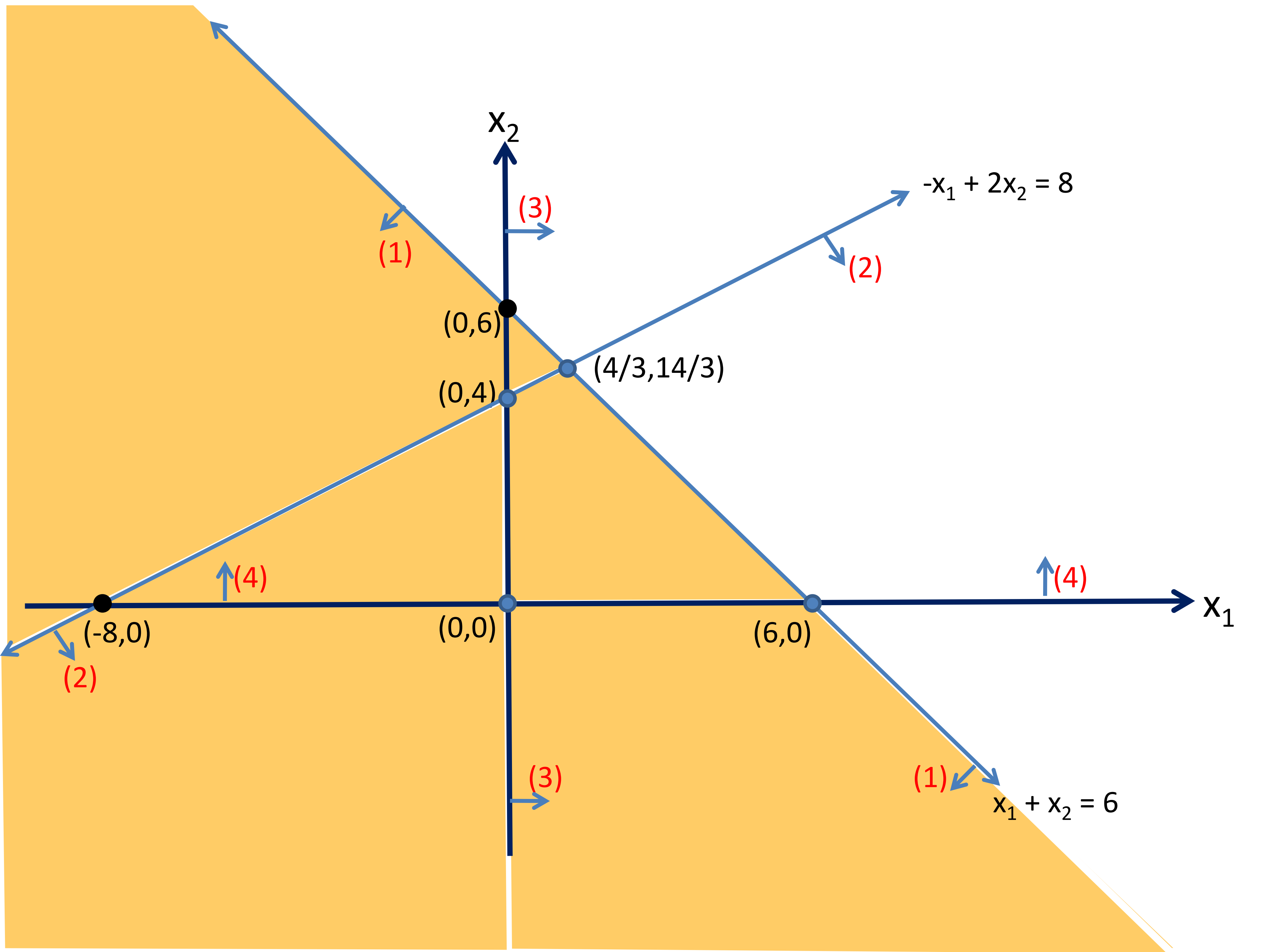
$$Ax \le b$$

$$x \ge 0$$

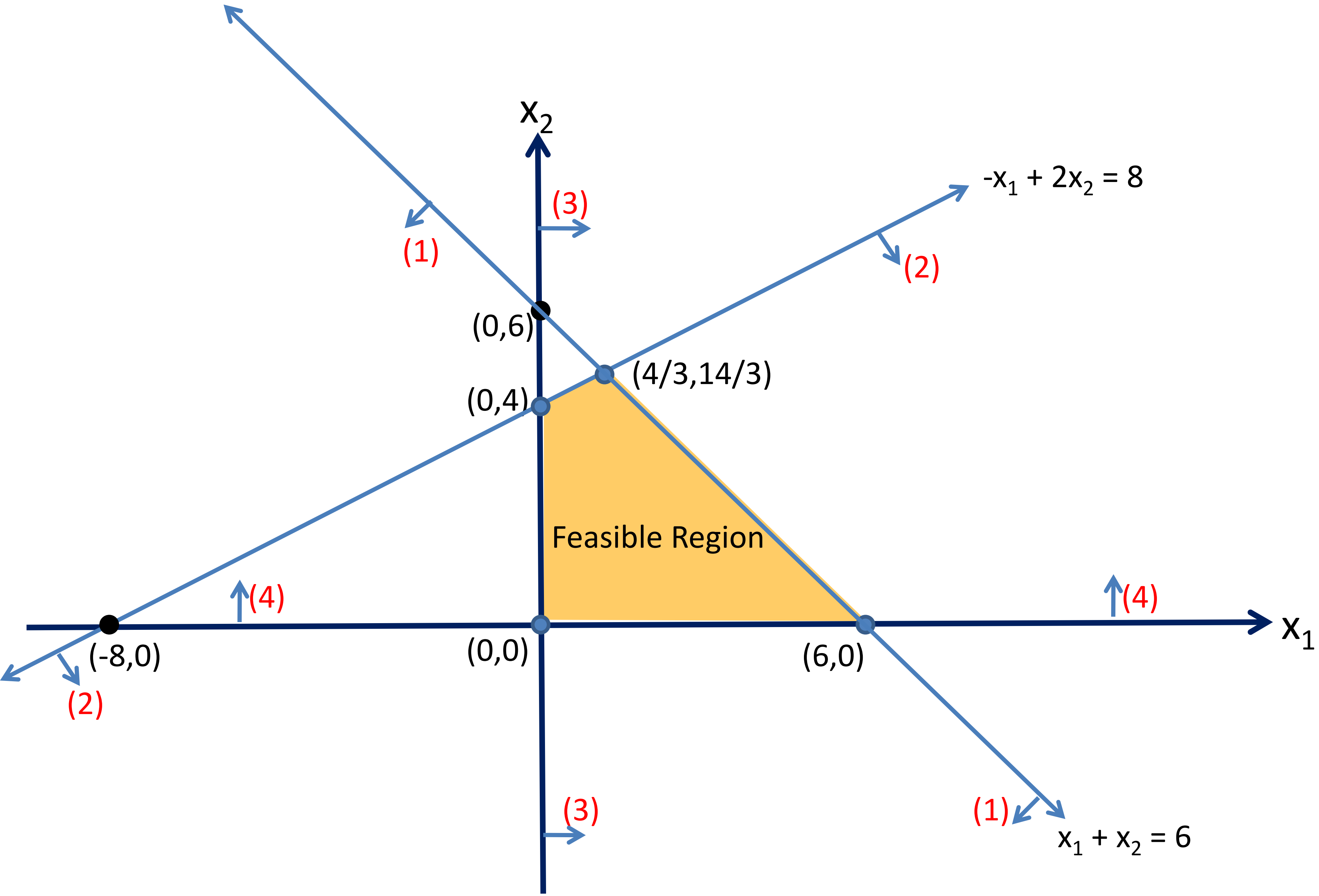
where,

$$c = \begin{bmatrix} 1 & 3 \end{bmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

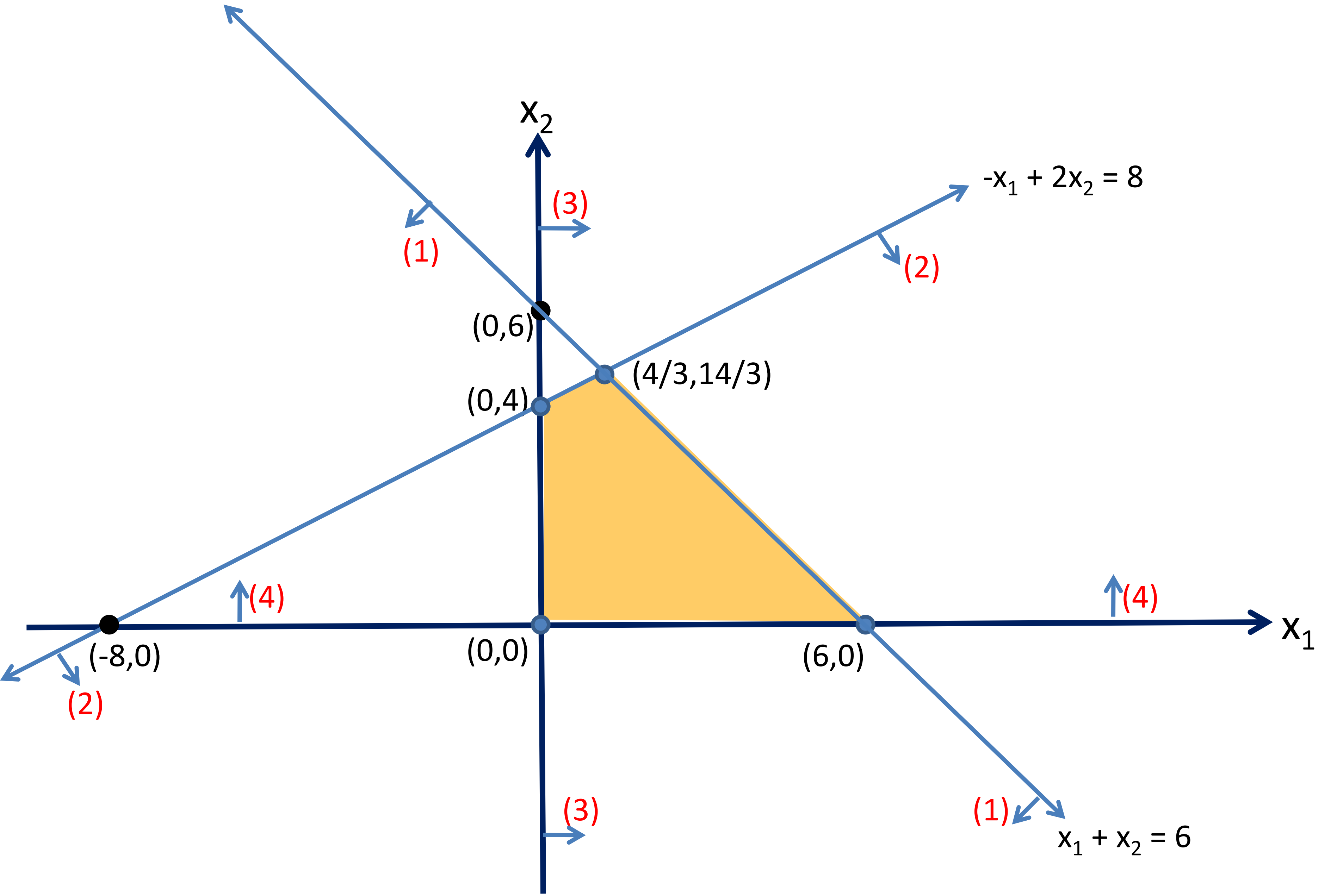
$$A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$



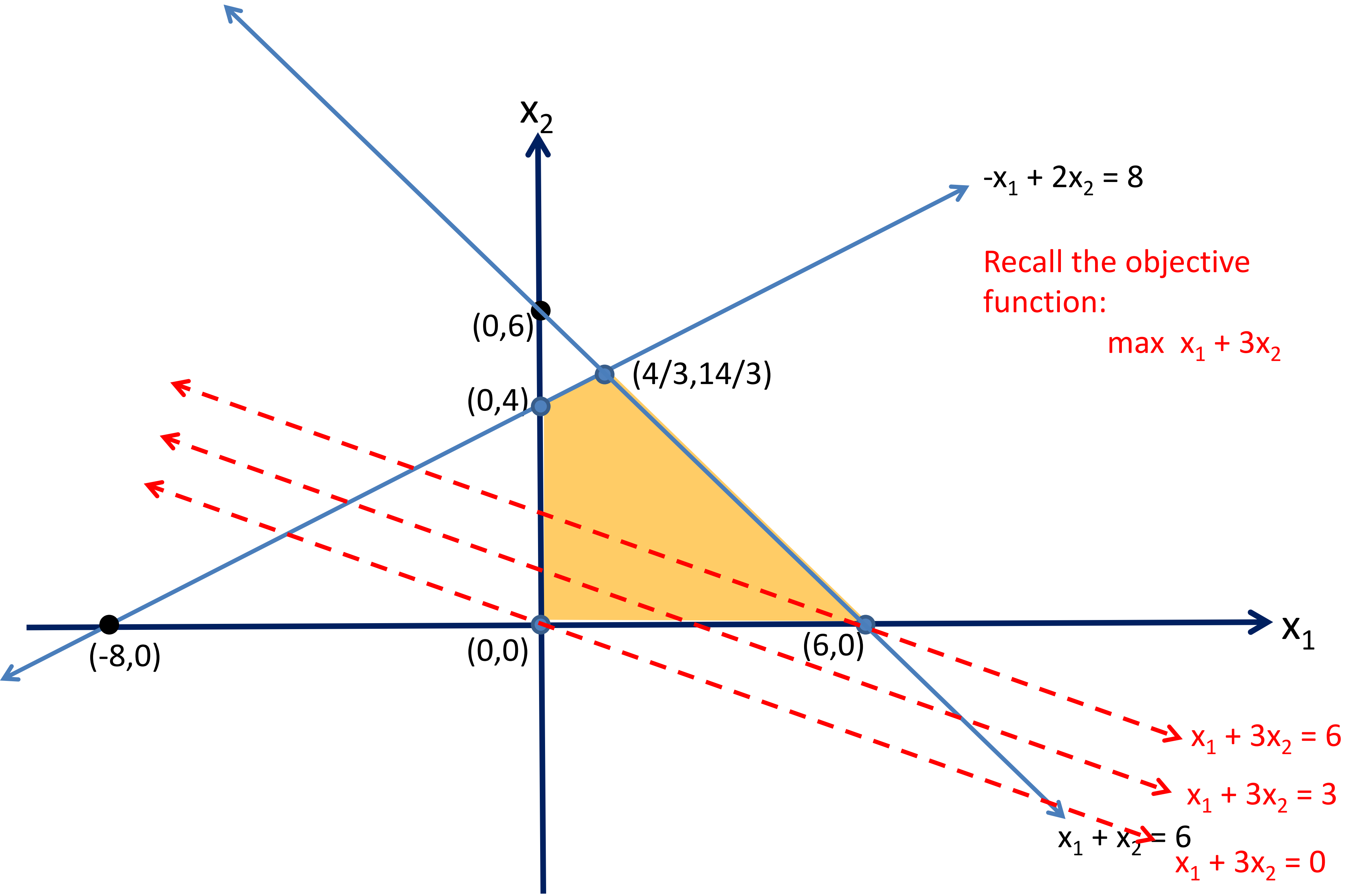
x_2
x_1
$-x_1 + 2x_2 = 8$
$x_1 + x_2 = 6$
(0,6)
(4/3,14/3)
(0,4)
(-8,0)
(0,0)
(6,0)
(1)
(2)
(3)
(4)
(4)
(2)
(3)
(1)



x_2
$-x_1 + 2x_2 = 8$
(3)
(1)
(2)
(0,6)
(4/3,14/3)
(0,4)
Feasible Region
(4)
(4)
x_1
(-8,0)
(0,0)
(6,0)
(2)
(3)
(1)
$x_1 + x_2 = 6$



x_2
$-x_1 + 2x_2 = 8$
(3)
(1)
(2)
(0,6)
(4/3,14/3)
(0,4)
(4)
(4)
x_1
(-8,0)
(0,0)
(6,0)
(2)
(3)
(1)
$x_1 + x_2 = 6$



x_2
$-x_1 + 2x_2 = 8$
Recall the objective function:
max $x_1 + 3x_2$
(0,6)
(4/3,14/3)
(0,4)
(-8,0)
(0,0)
(6,0)
x_1
$x_1 + 3x_2 = 6$
$x_1 + 3x_2 = 3$
$x_1 + x_2 = 6$
$x_1 + 3x_2 = 0$

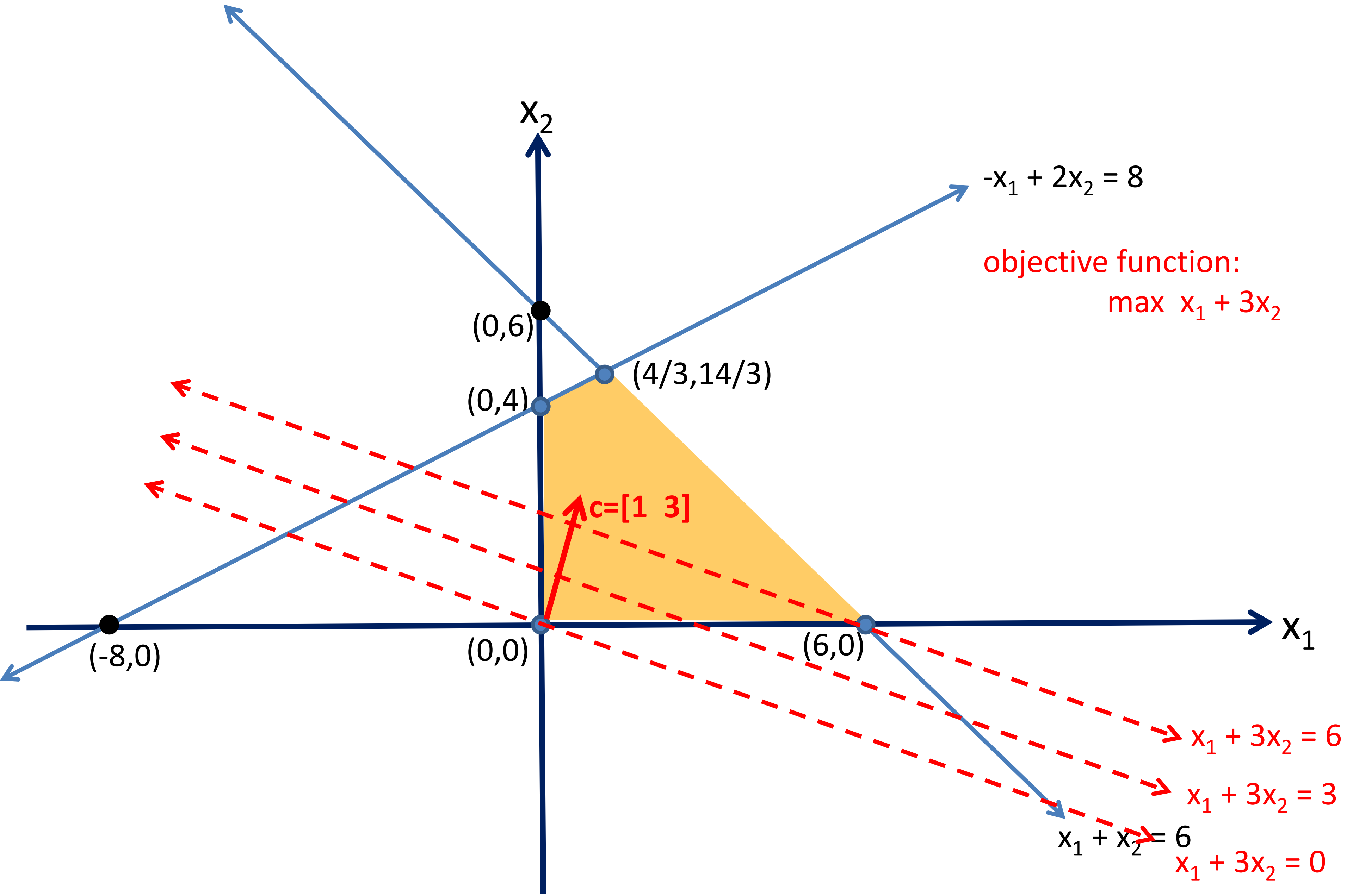
Graphical Solution of 2-variable LP Problems

A line on which all points have the same z-value ($z = x_1 + 3x_2$) is called :

- **Isoprofit line** for maximization problems,
- **Isocost line** for minimization problems.

Graphical Solution of 2-variable LP Problems

Consider the coefficients of x_1 and x_2 in the objective function and let $\mathbf{c}$=[1 3].


x2
-x1 + 2x2 = 8
objective function:
max x1 + 3x2
(0,6)
(4/3,14/3)
(0,4)
c=[1 3]
(-8,0)
(0,0)
(6,0)
x1
x1 + 3x2 = 6
x1 + 3x2 = 3
x1 + x2 = 6
x1 + 3x2 = 0

Graphical Solution of 2-variable LP Problems

Note that as we move the isoprofit lines in the direction of **c**, the total profit increases!

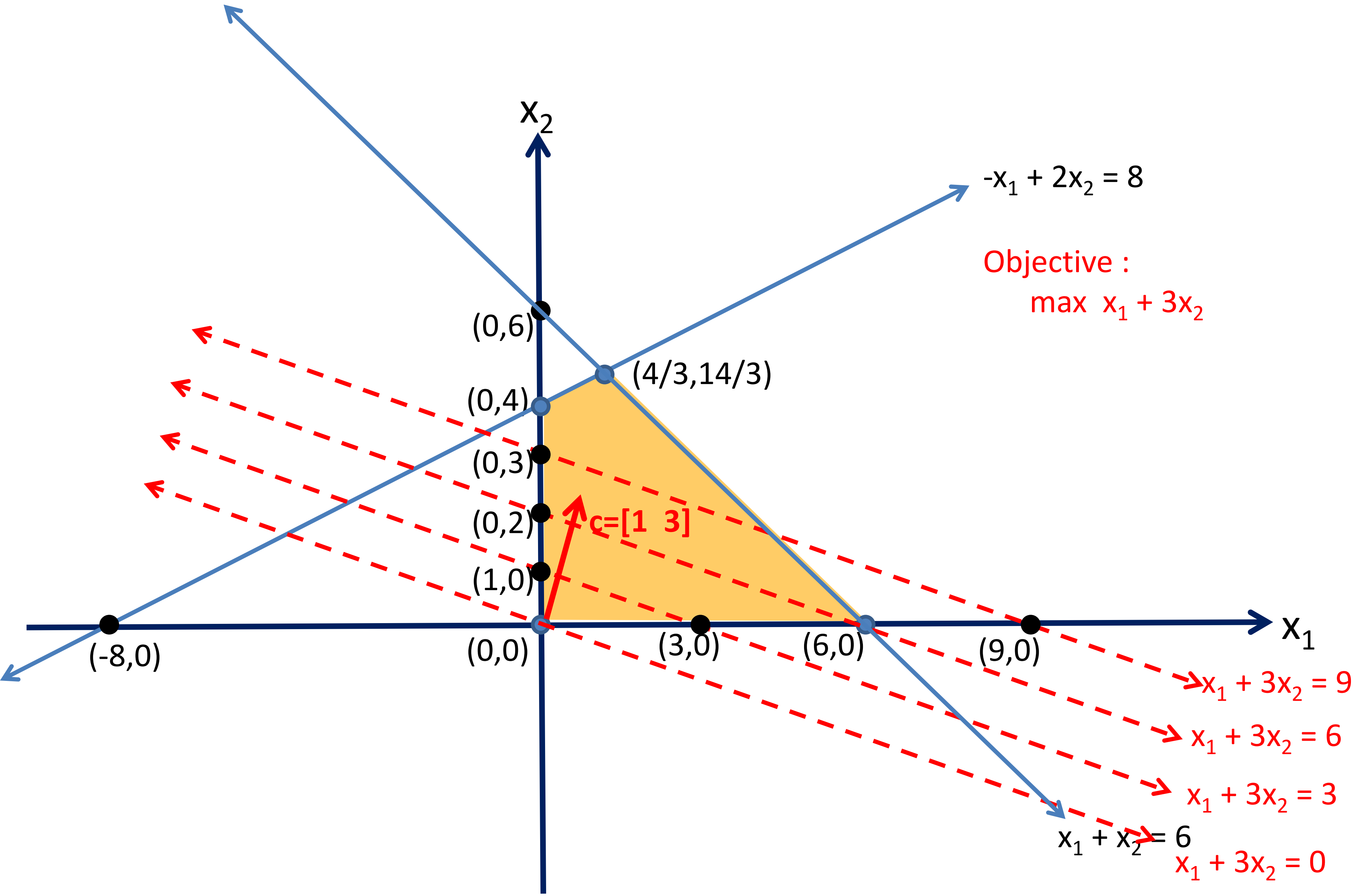
Graphical Solution of 2-variable LP Problems

For a **maximization problem**, the vector **c**, given by the coefficients of x_1 and x_2 in the objective function, is called the **improving direction**.

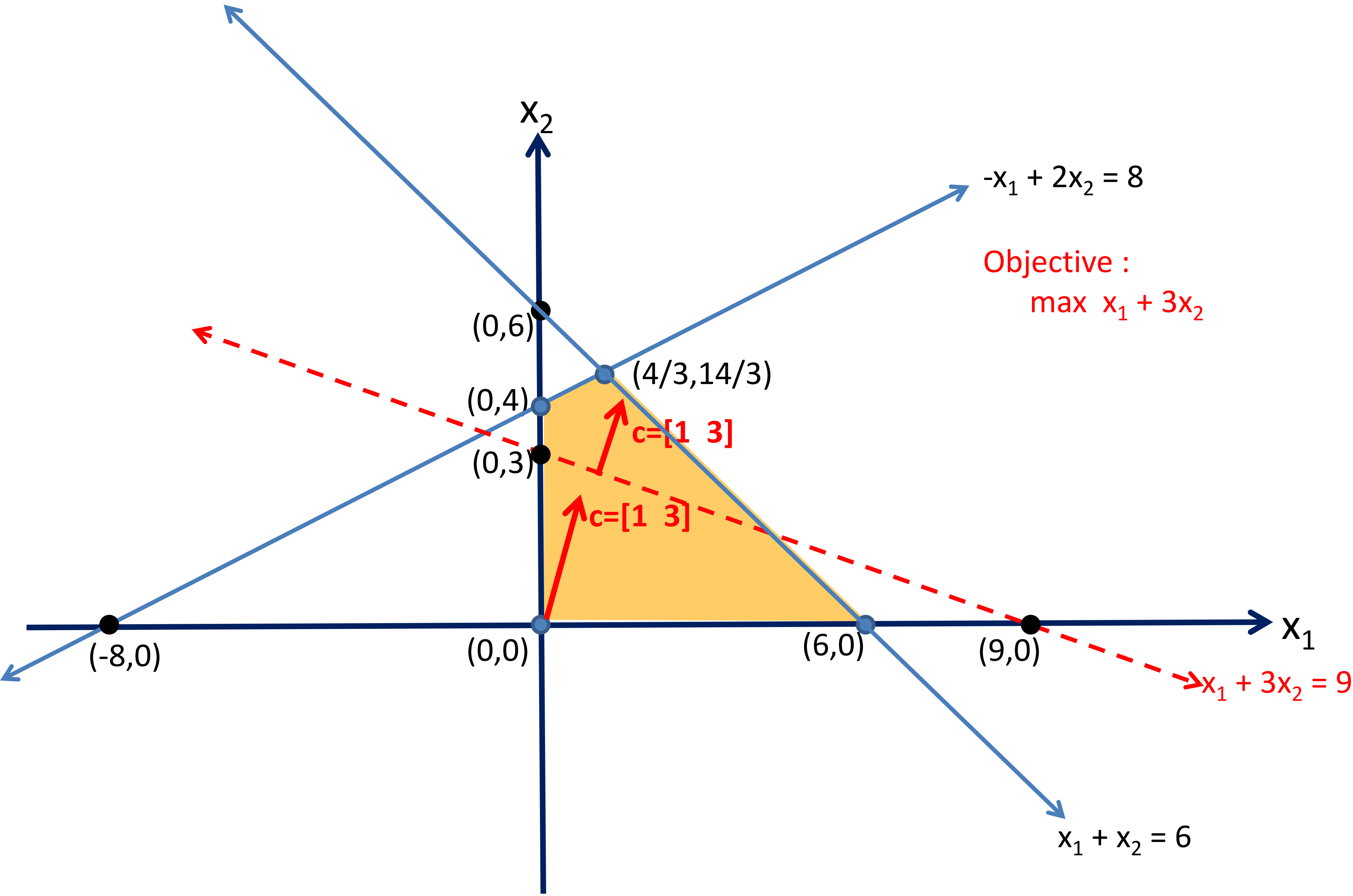
On the other hand, **-c is the improving direction for minimization problems**!

Graphical Solution of 2-variable LP Problems

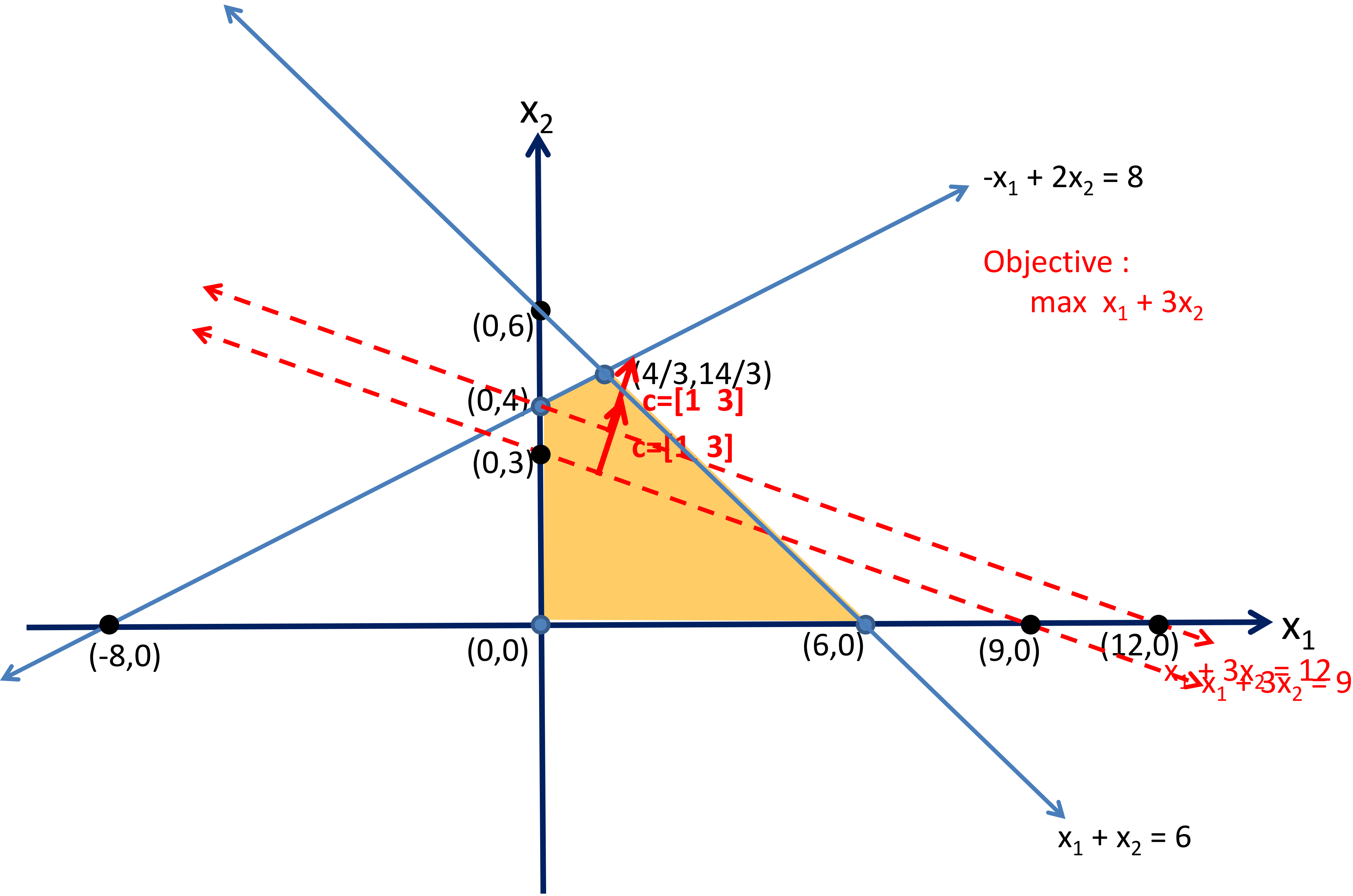
To solve an LP problem in two variables, we should try to push isoprofit (isocost) lines in the improving direction as much as possible while still staying in the feasible region.



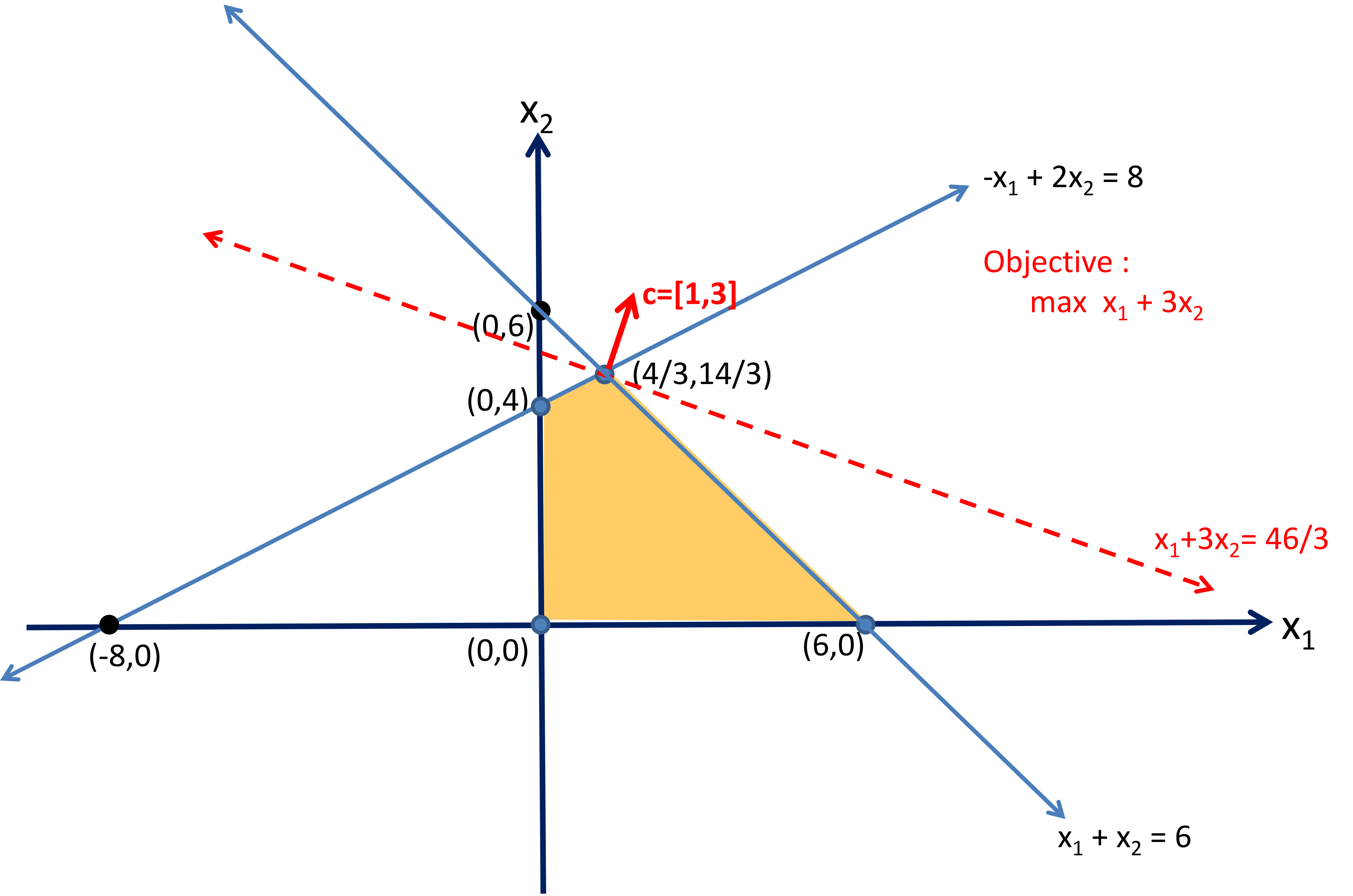
x2
x1
-x1 + 2x2 = 8
Objective :
max x1 + 3x2
(0,6)
(4/3,14/3)
(0,4)
(0,3)
(0,2)
(1,0)
c=[1 3]
(-8,0)
(0,0)
(3,0)
(6,0)
(9,0)
x1 + 3x2 = 9
x1 + 3x2 = 6
x1 + 3x2 = 3
x1 + x2 = 6
x1 + 3x2 = 0



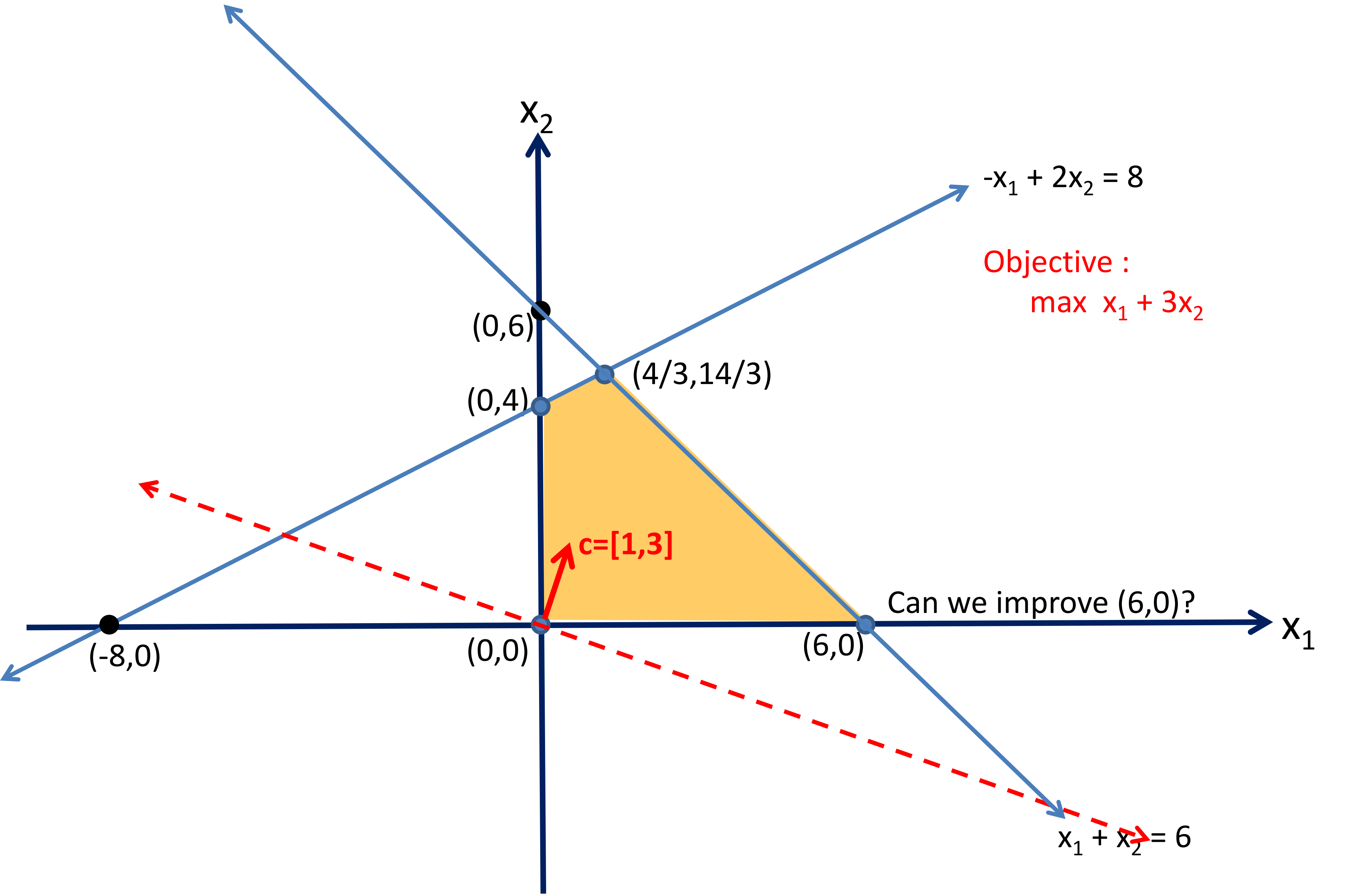
x_2
$-x_1 + 2x_2 = 8$
Objective :
max $x_1 + 3x_2$
(0,6)
(4/3,14/3)
(0,4)
c=[1 3]
(0,3)
c=[1 3]
x_1
(-8,0)
(0,0)
(6,0)
(9,0)
$x_1 + 3x_2 = 9$
$x_1 + x_2 = 6$

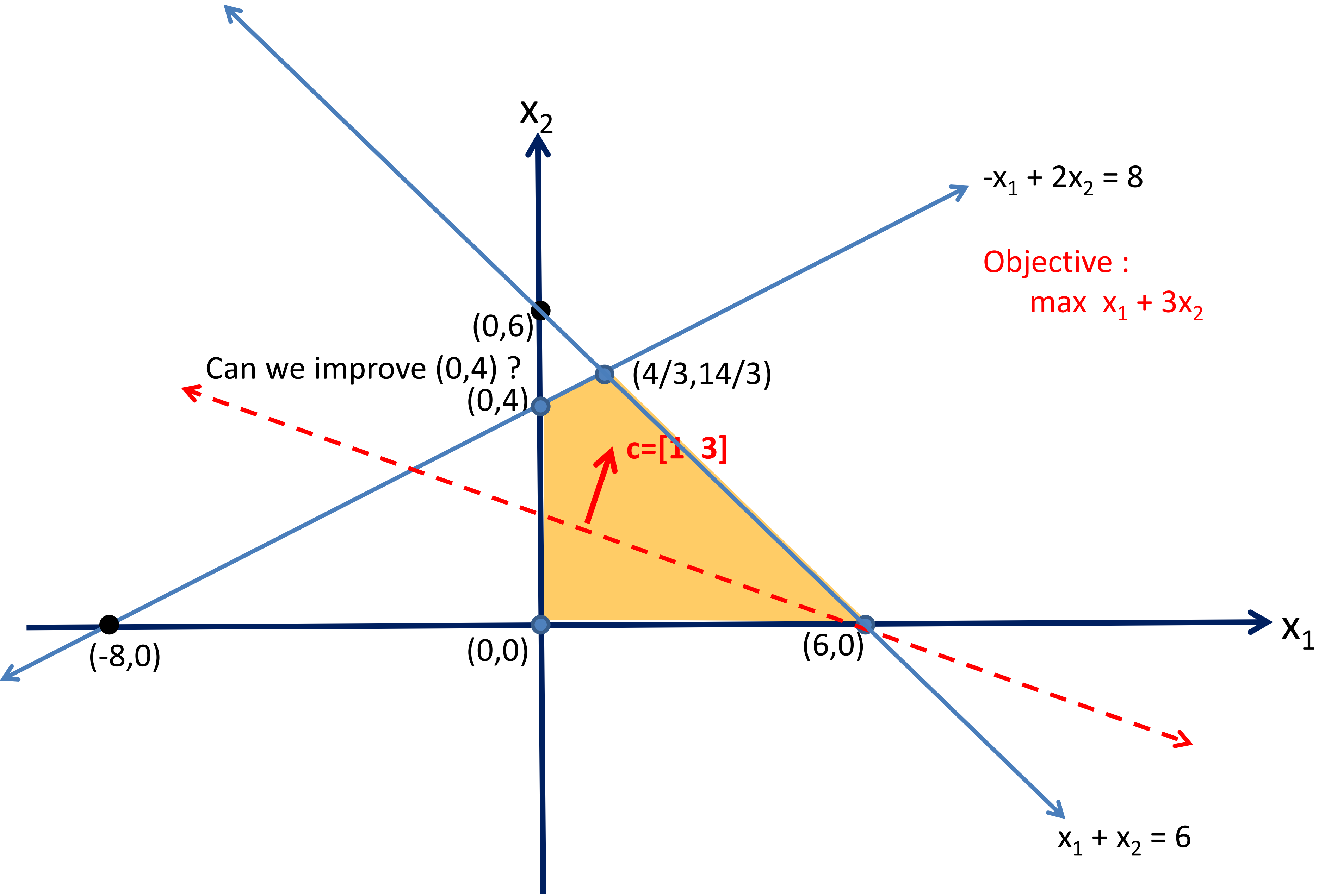


x2
x1
-x1 + 2x2 = 8
Objective :
max x1 + 3x2
(0,6)
(4/3,14/3)
(0,4)
c=[1 3]
c=[1 3]
(0,3)
(-8,0)
(0,0)
(6,0)
(9,0)
(12,0)
x1 + 3x2 = 12
x1 + 3x2 = 9
x1 + x2 = 6

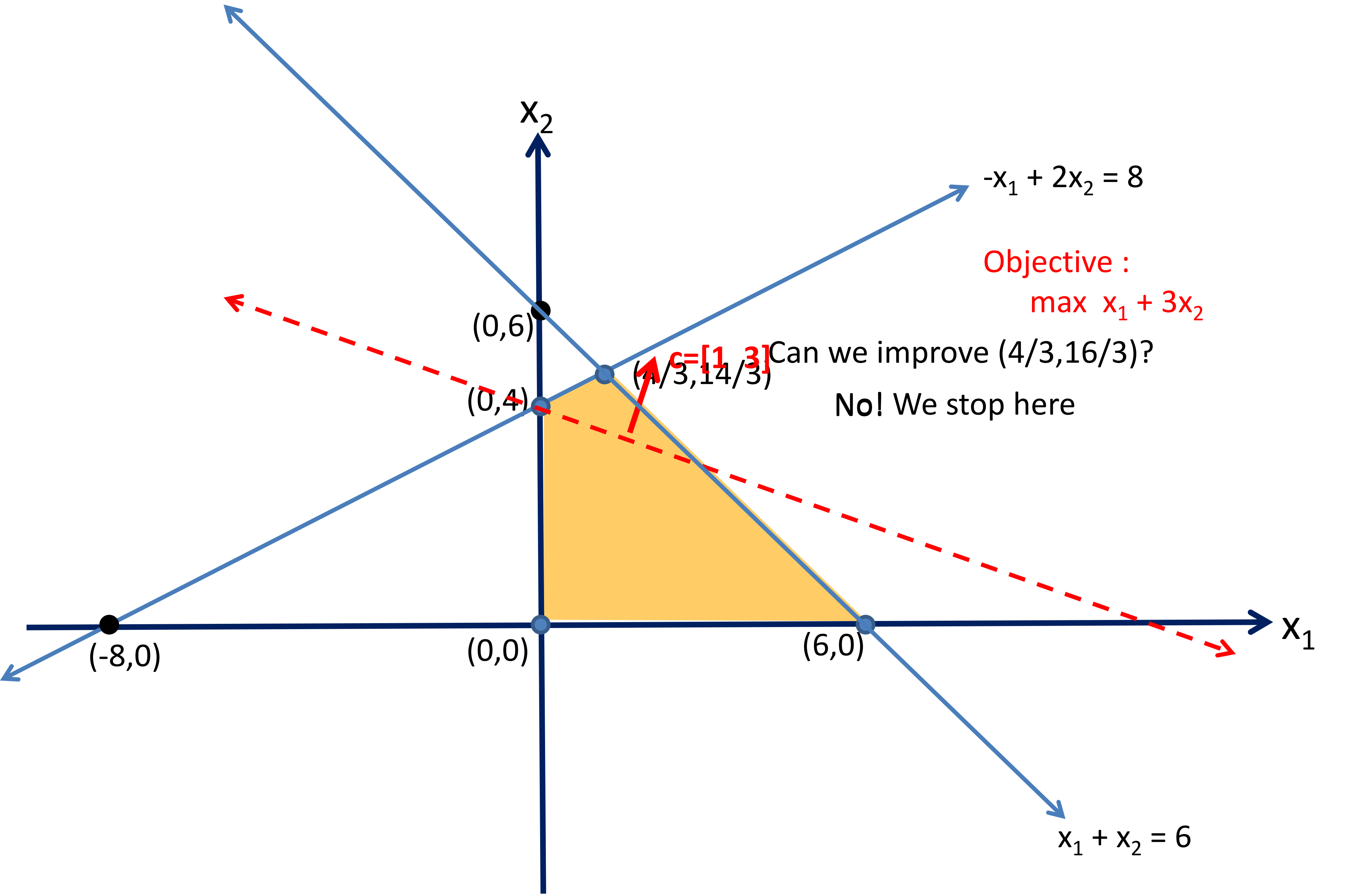


x_2
$-x_1 + 2x_2 = 8$
Objective :
max $x_1 + 3x_2$
c=[1,3]
(0,6)
(4/3,14/3)
(0,4)
$x_1+3x_2= 46/3$
x_1
(-8,0)
(0,0)
(6,0)
$x_1 + x_2 = 6$

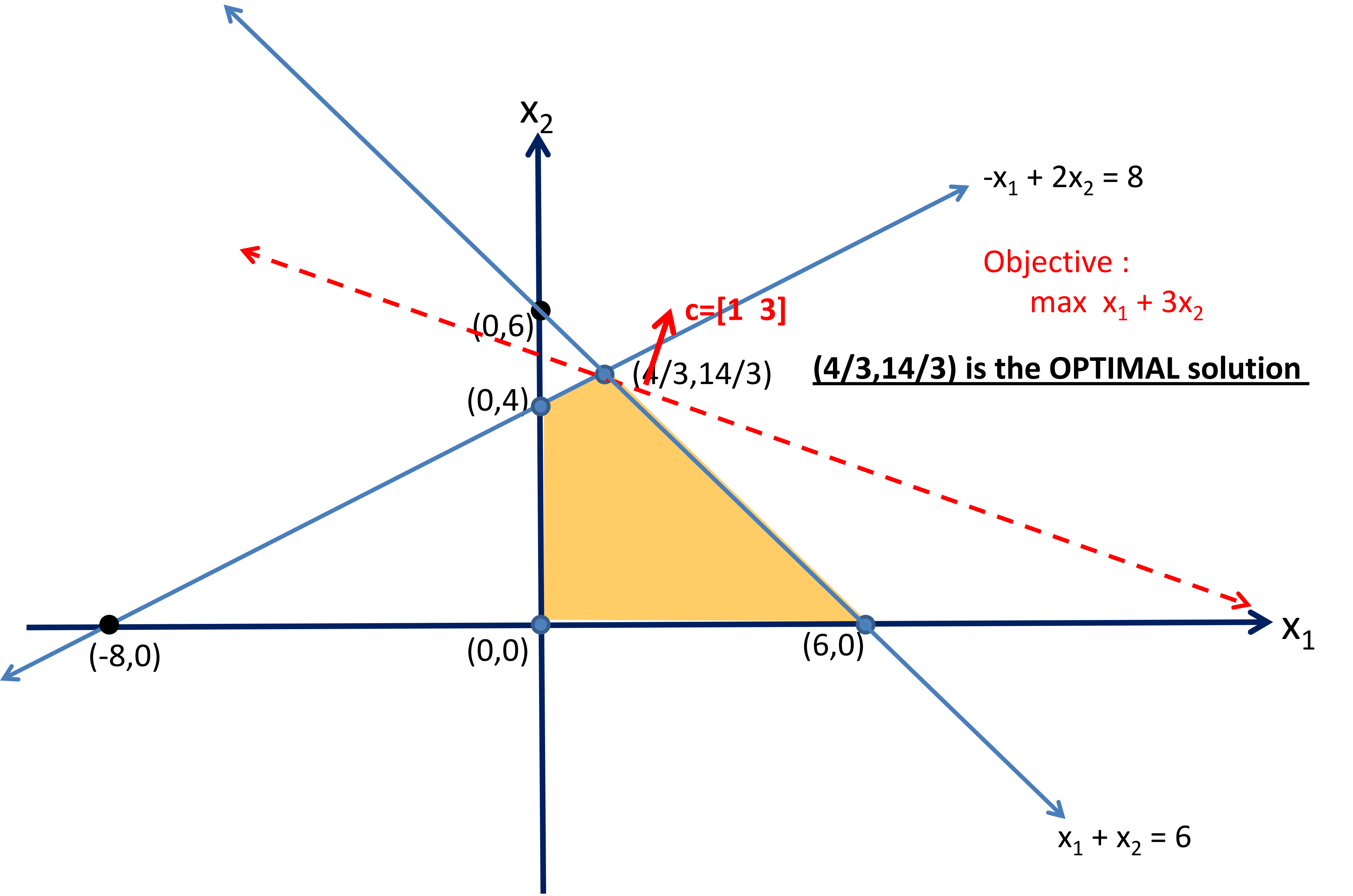

x_2
$-x_1 + 2x_2 = 8$
Objective :
max $x_1 + 3x_2$
(0,6)
(4/3,14/3)
(0,4)
c=[1,3]
Can we improve (6,0)?
x_1
(-8,0)
(0,0)
(6,0)
$x_1 + x_2 = 6$



x_2
x_1
$-x_1 + 2x_2 = 8$
Objective :
max $x_1 + 3x_2$
(0,6)
Can we improve (0,4) ?
(4/3,14/3)
(0,4)
c=[1 3]
(-8,0)
(0,0)
(6,0)
$x_1 + x_2 = 6$



x2
x1
-x1 + 2x2 = 8
Objective :
max x1 + 3x2
c=[1, 3]
Can we improve (4/3,16/3)?
No! We stop here
(0,6)
(0,4)
(4/3,14/3)
(-8,0)
(0,0)
(6,0)
x1 + x2 = 6



x_2
$-x_1 + 2x_2 = 8$
Objective :
max $x_1 + 3x_2$
c=[1 3]
(0,6)
(4/3,14/3)
(4/3,14/3) is the OPTIMAL solution
(0,4)
(-8,0)
(0,0)
(6,0)
x_1
$x_1 + x_2 = 6$

Graphical Solution of 2-variable LP Problems

Point (4/3, 14/3) is the last point in the feasible region when we move in the improving direction. So, if we move in the same direction any further, there is no feasible point.

Therefore, point z*= (4/3, 14/3) is the maximizing point. The **optimal value** of the LP problem is 46/3.

Graphical Solution of 2-variable LP Problems

Let us modify the objective function while keeping the constraints unchanged:

Ex 1.a)

max $\quad x_1 + 3\,x_2$

s.t.

$x_1 + x_2 \le 6 \qquad (1)$

$-x_1 + 2x_2 \le 8 \qquad (2)$

$x_1, x_2 \ge 0 \qquad (3), (4)$

Ex 1.b)

max $\quad x_1 - 2\,x_2$

s.t.

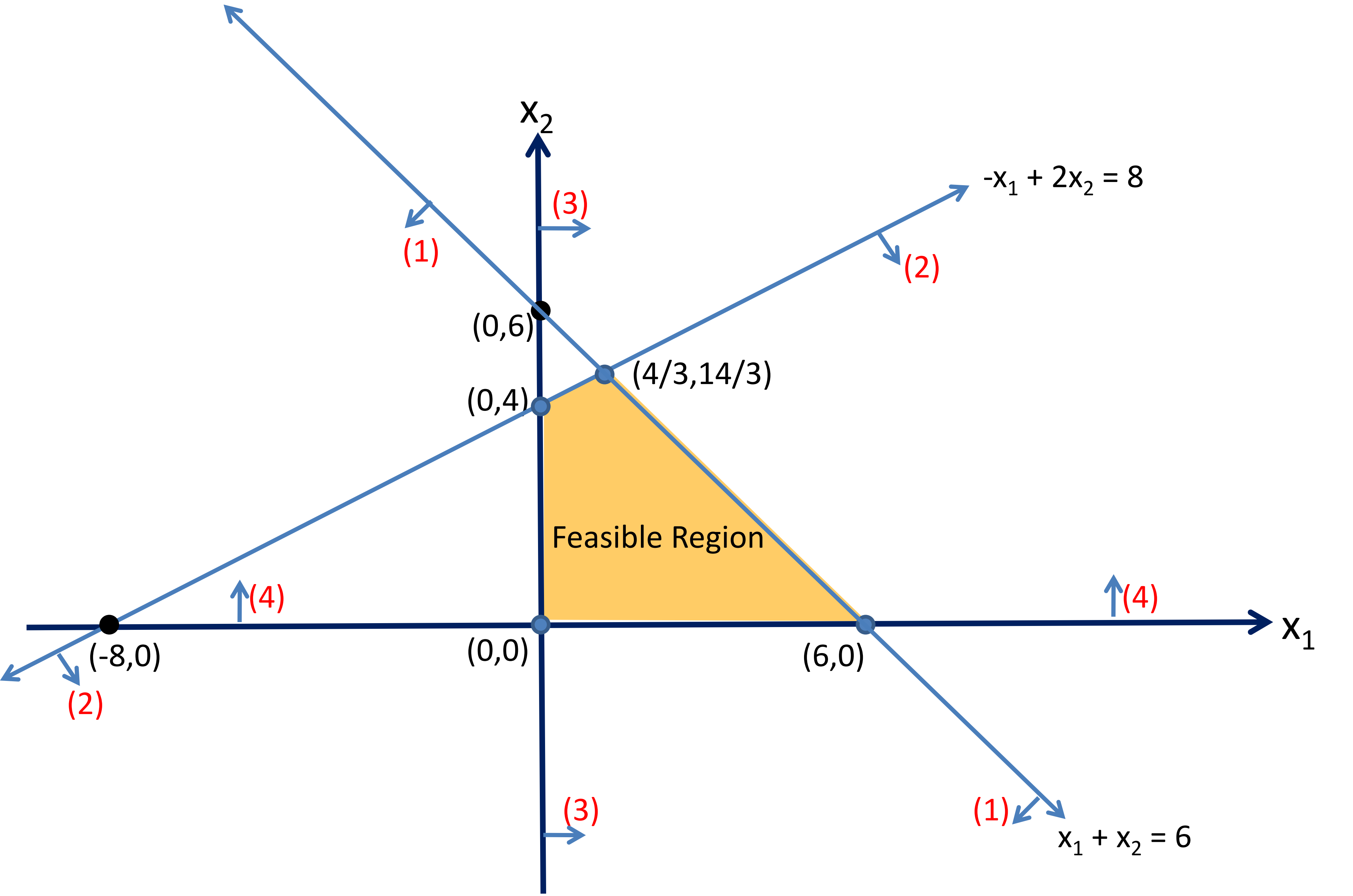
$x_1 + x_2 \le 6 \qquad (1)$

$-x_1 + 2x_2 \le 8 \qquad (2)$

$x_1, x_2 \ge 0 \qquad (3), (4)$

Graphical Solution of 2-variable LP Problems

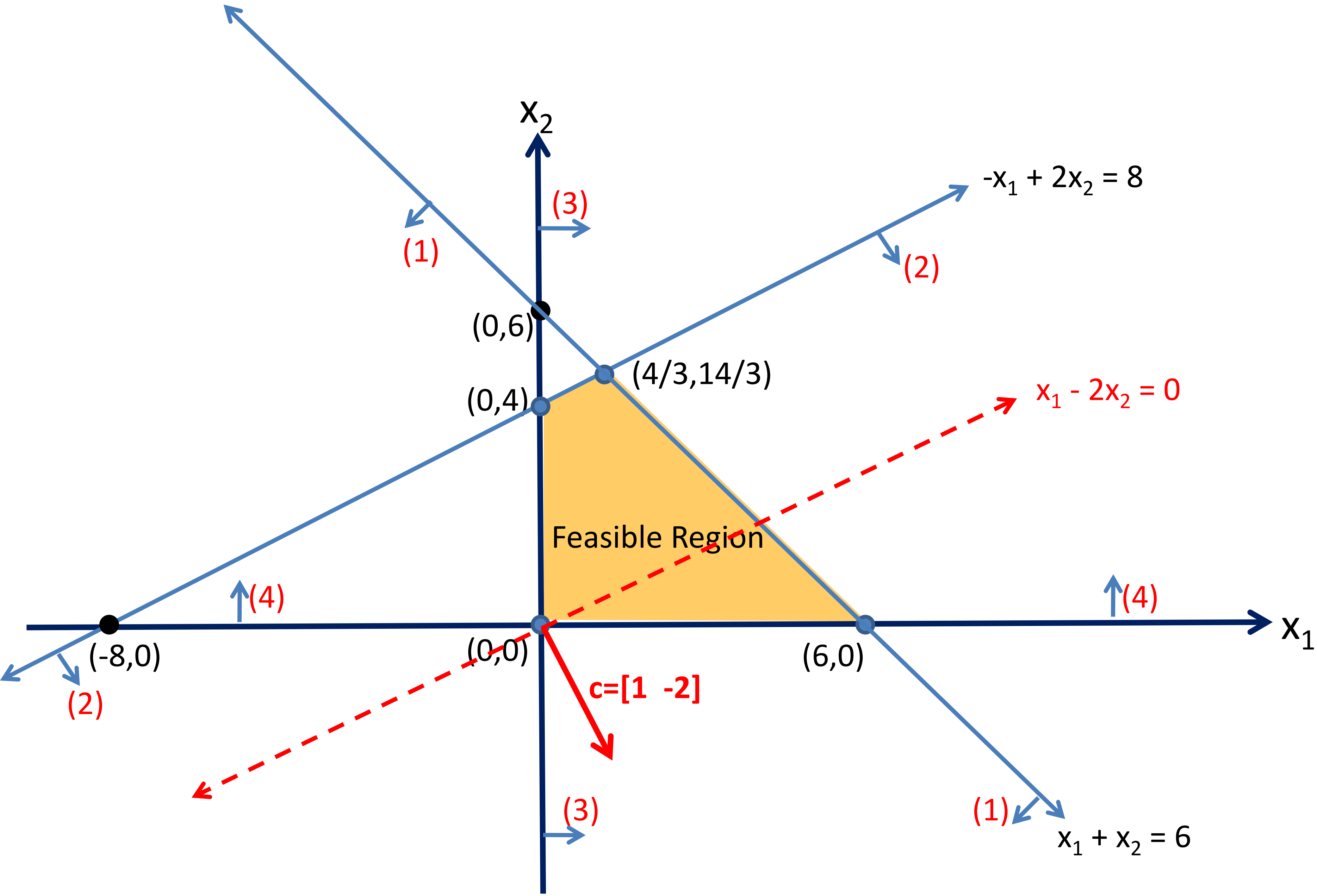
Does feasible region change?



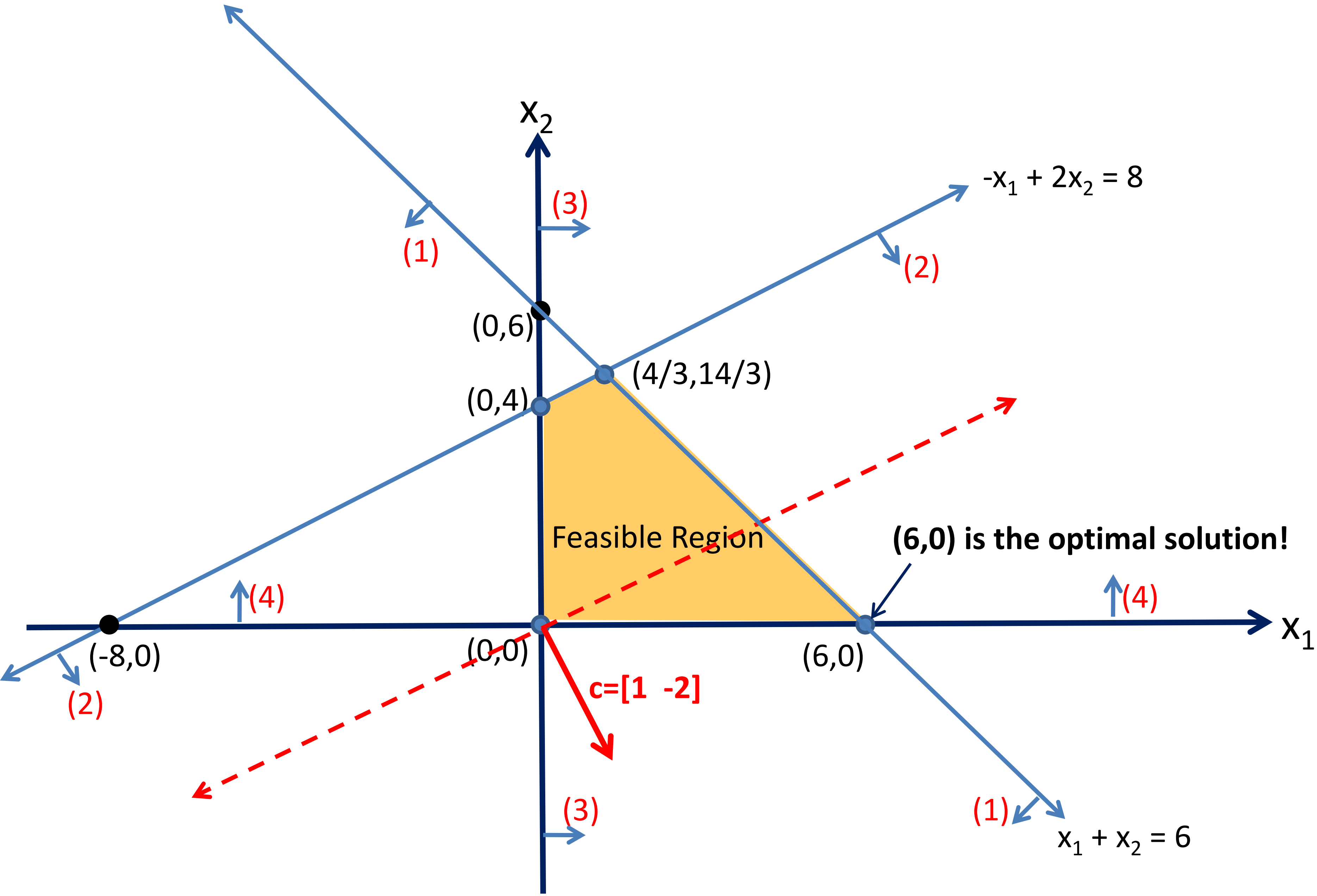
x2
-x1 + 2x2 = 8
(3)
(1)
(2)
(0,6)
(4/3,14/3)
(0,4)
Feasible Region
(4)
(4)
x1
(-8,0)
(0,0)
(6,0)
(2)
(3)
(1)
x1 + x2 = 6

Graphical Solution of 2-variable LP Problems

What about the improving direction?



x_2
x_1
$-x_1 + 2x_2 = 8$
$x_1 - 2x_2 = 0$
$x_1 + x_2 = 6$
(1)
(2)
(3)
(4)
(0,6)
(4/3,14/3)
(0,4)
Feasible Region
(-8,0)
(0,0)
(6,0)
c=[1 -2]



x_2
$-x_1 + 2x_2 = 8$
(3)
(1)
(2)
(0,6)
(4/3,14/3)
(0,4)
Feasible Region
(6,0) is the optimal solution!
(4)
(4)
x_1
(-8,0)
(0,0)
(6,0)
(2)
c=[1 -2]
(3)
(1)
$x_1 + x_2 = 6$

Graphical Solution of 2-variable LP Problems

Now let us consider:

Ex 1.c)

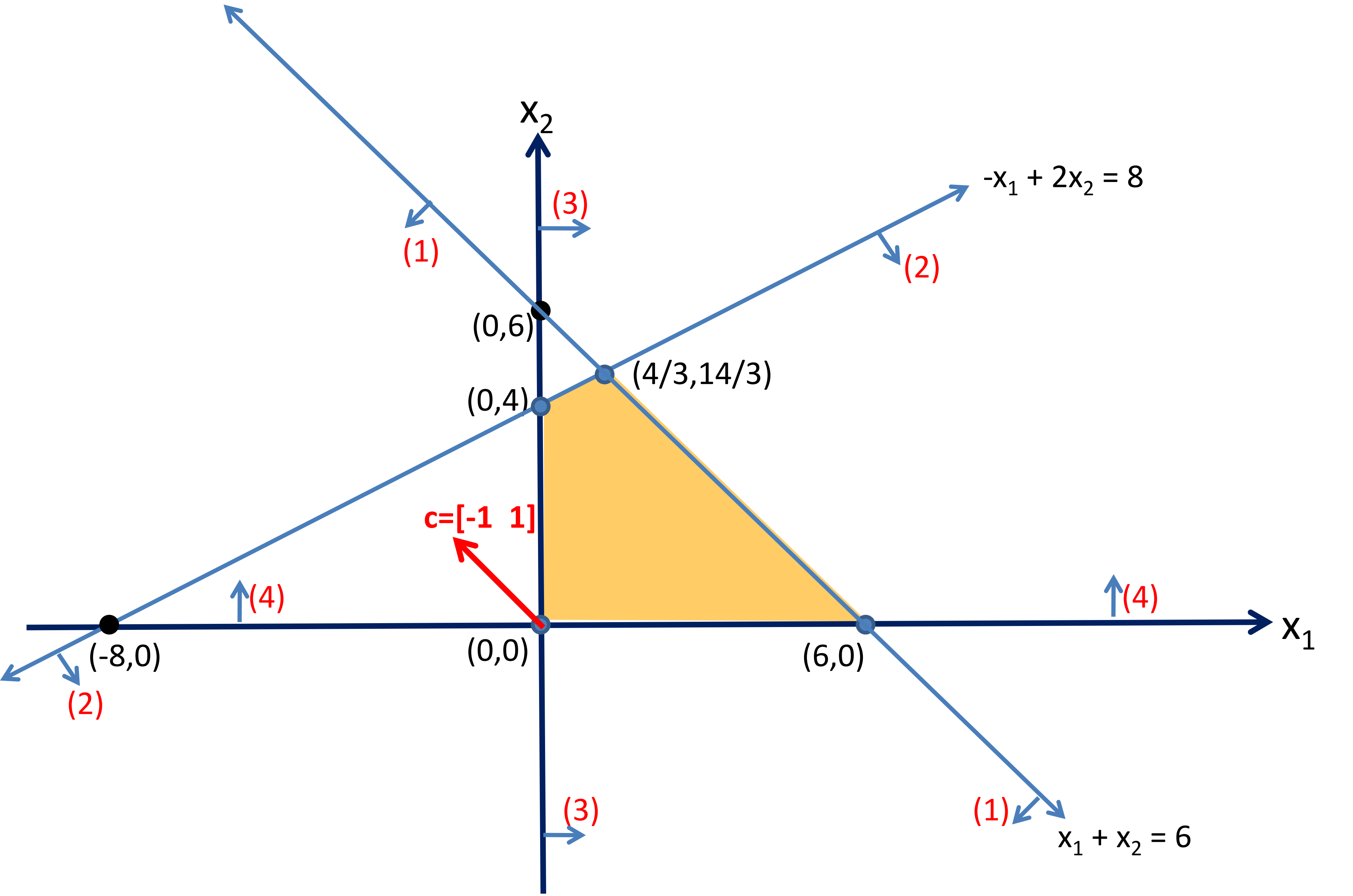
max $-x_1 + x_2$

s.t.

$x_1 + x_2 \leq 6$ (1)

$-x_1 + 2x_2 \leq 8$ (2)

$x_1, x_2 \geq 0$ (3), (4)


x_2
x_1
$-x_1 + 2x_2 = 8$
$x_1 + x_2 = 6$
(1)
(2)
(3)
(4)
(0,6)
(4/3,14/3)
(0,4)
c=[-1 1]
(-8,0)
(0,0)
(6,0)

Graphical Solution of 2-variable LP Problems

Now, let us consider:

Ex 1.c)

$$\max \quad -x_1 + x_2$$

s.t.

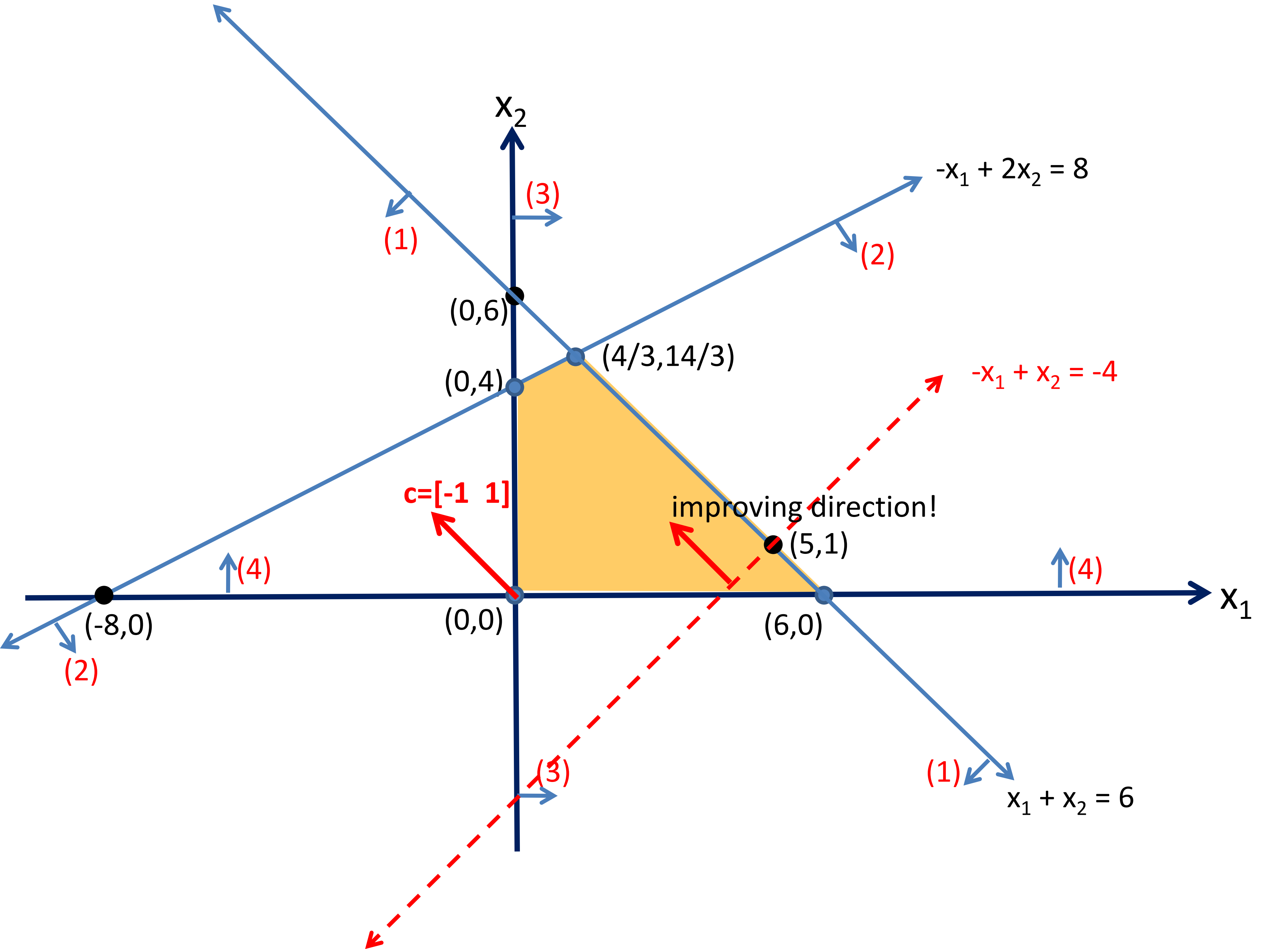
$$x_1 + x_2 \leq 6 \qquad (1)$$

$$-x_1 + 2x_2 \leq 8 \qquad (2)$$

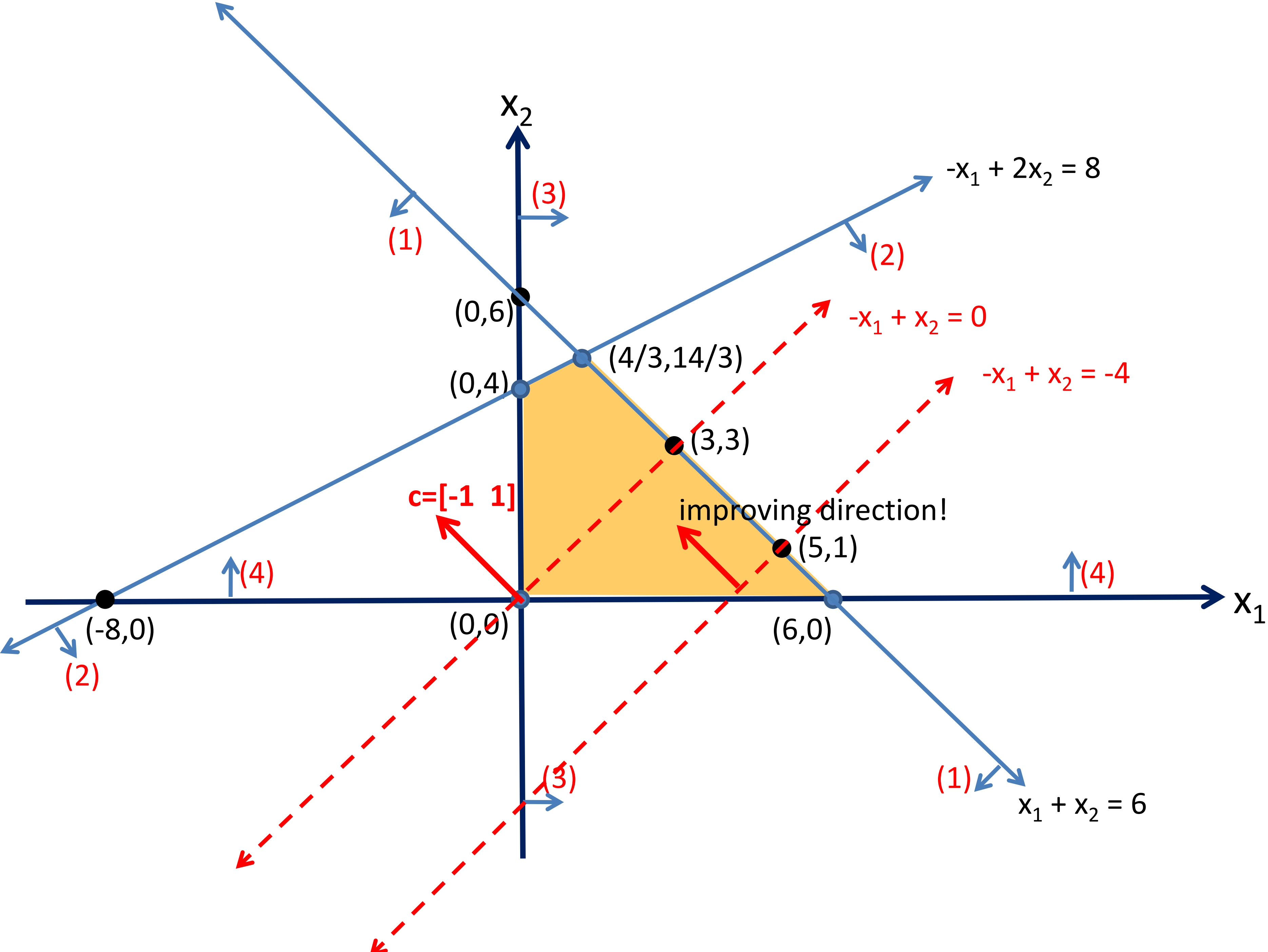
$$x_1, x_2 \geq 0 \qquad (3), (4)$$

Question:

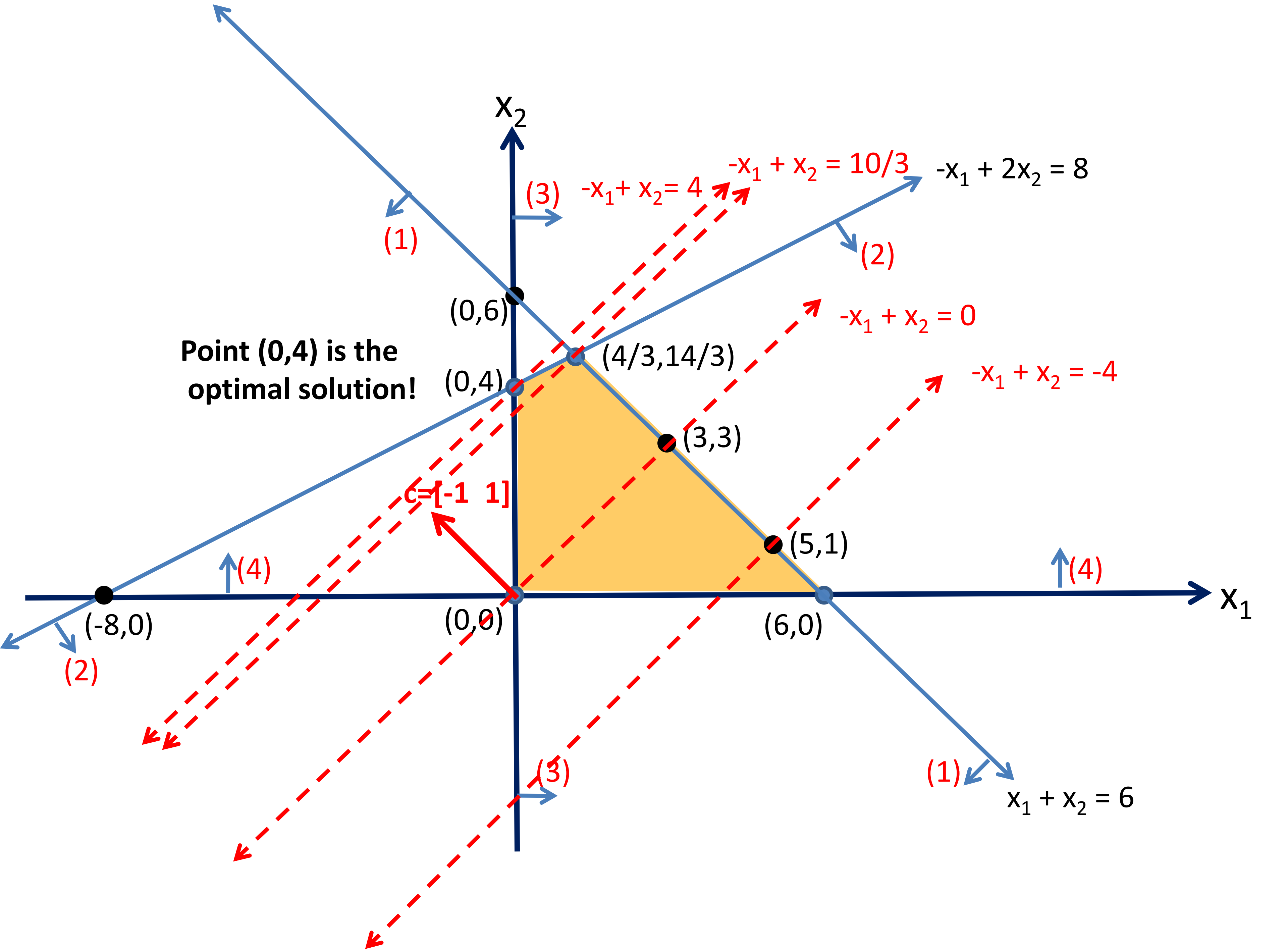
Is it possible for (5,1) be the optimal?



x_2
x_1
$-x_1 + 2x_2 = 8$
$x_1 + x_2 = 6$
$-x_1 + x_2 = -4$
(1)
(2)
(3)
(4)
(0,6)
(4/3,14/3)
(0,4)
c=[-1 1]
improving direction!
(5,1)
(-8,0)
(0,0)
(6,0)



x_2
x_1
$-x_1 + 2x_2 = 8$
$x_1 + x_2 = 6$
$-x_1 + x_2 = 0$
$-x_1 + x_2 = -4$
(1)
(2)
(3)
(4)
(0,6)
(0,4)
(4/3,14/3)
(3,3)
(5,1)
(-8,0)
(0,0)
(6,0)
c=[-1 1]
improving direction!



x_2
x_1
$-x_1 + x_2 = 10/3$
$-x_1 + 2x_2 = 8$
$-x_1 + x_2 = 4$
(3)
(1)
(2)
(0,6)
$-x_1 + x_2 = 0$
Point (0,4) is the optimal solution!
(4/3,14/3)
(0,4)
$-x_1 + x_2 = -4$
(3,3)
c=[-1 1]
(5,1)
(4)
(4)
(-8,0)
(0,0)
(6,0)
(2)
(3)
(1)
$x_1 + x_2 = 6$

Graphical Solution of 2-variable LP Problems

Now, let us consider:

Ex 1.d)

$\max \quad -x_1 - x_2$

s.t.

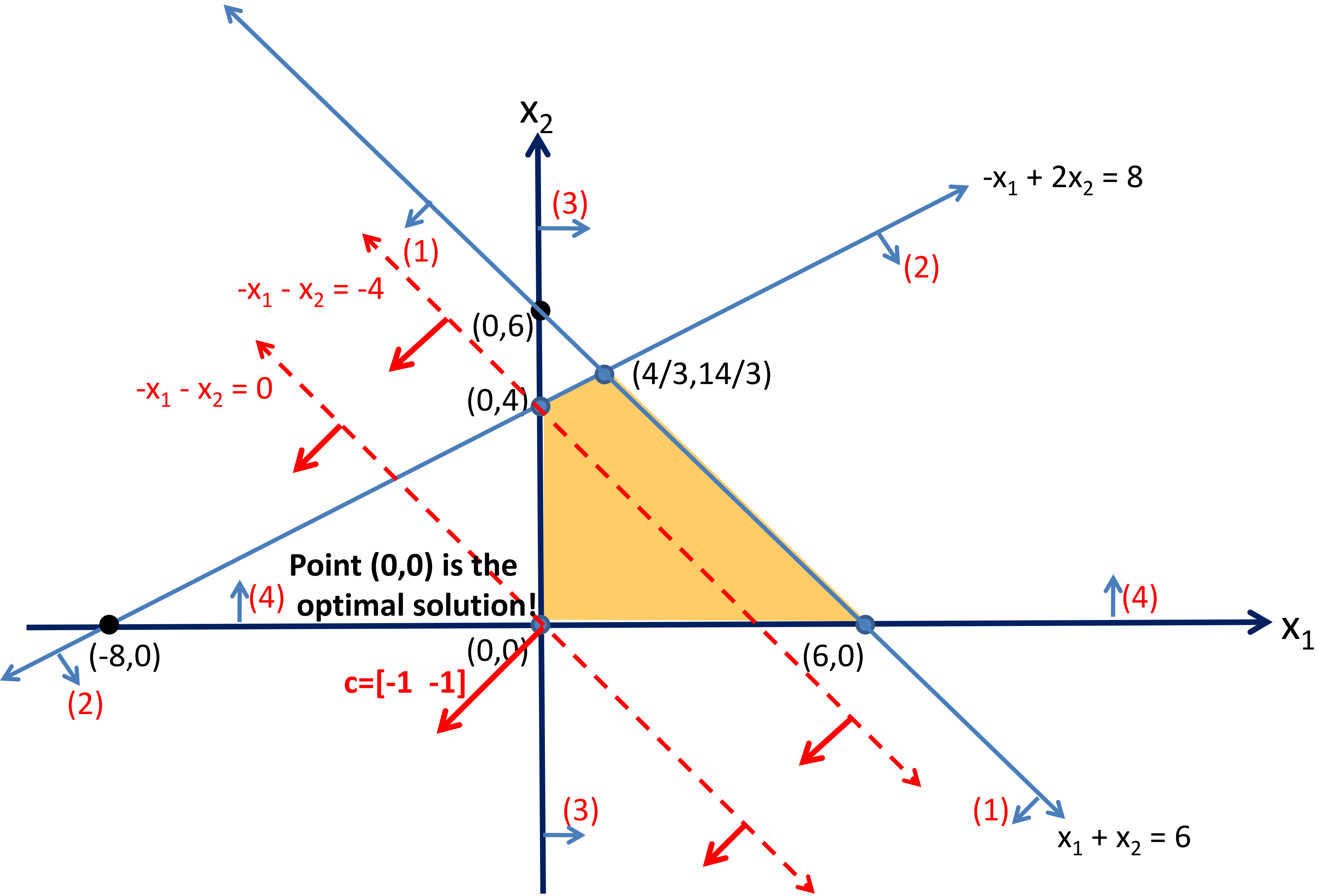
$x_1 + x_2 \leq 6$ (1)

$-x_1 + 2x_2 \leq 8$ (2)

$x_1, x_2 \geq 0$ (3), (4)

Question:

Is it possible for (0,4) be the optimal?



x_2
x_1
$-x_1 + 2x_2 = 8$
$x_1 + x_2 = 6$
$-x_1 - x_2 = -4$
$-x_1 - x_2 = 0$
(1)
(2)
(3)
(4)
(0,6)
(0,4)
(4/3,14/3)
(-8,0)
(0,0)
(6,0)
Point (0,0) is the optimal solution!
c=[-1 -1]

Graphical Solution of 2-variable LP Problems

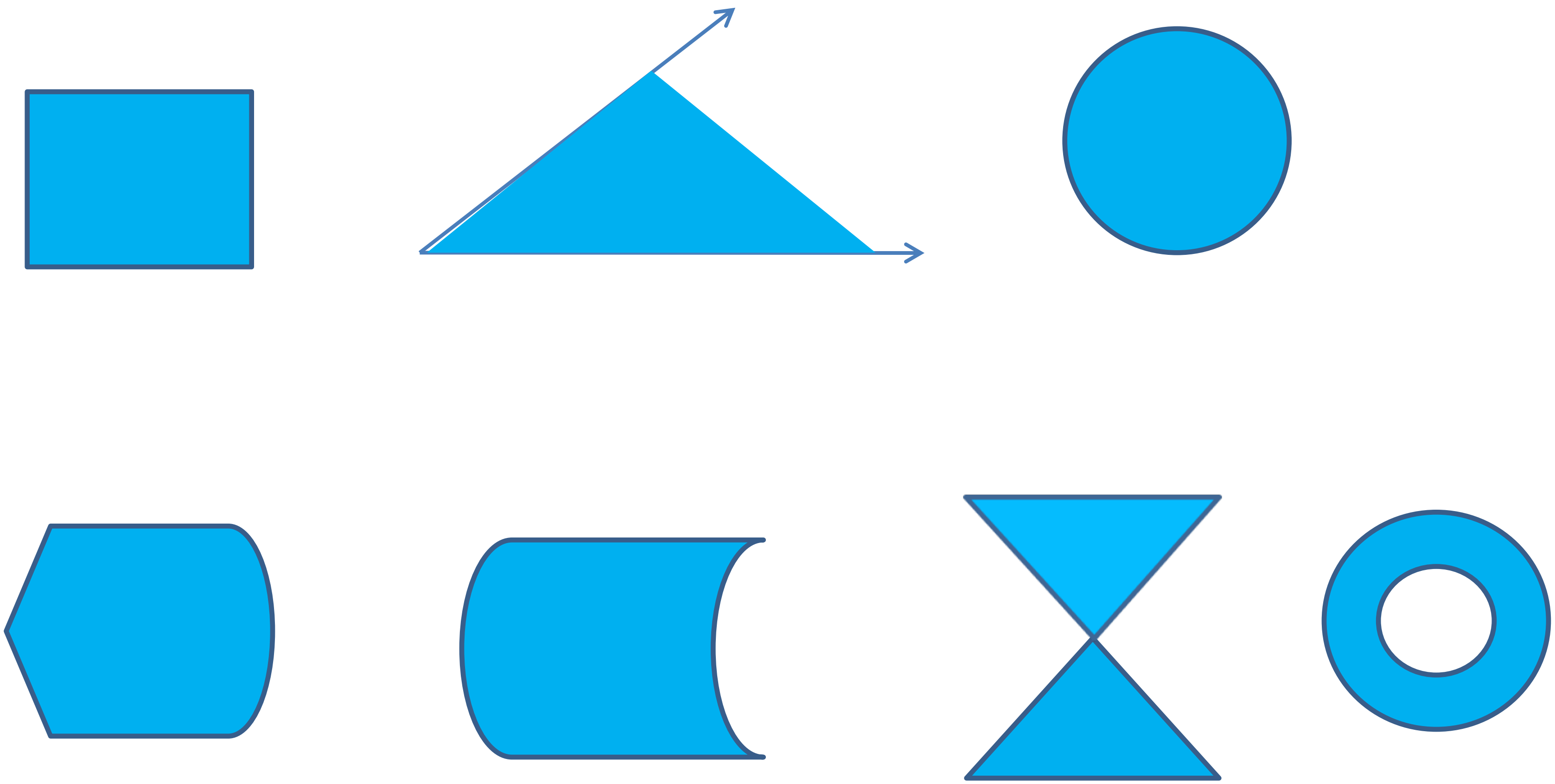
Have you noticed anything about the optimal points?

Convex Set:

A set of points S is a convex set if for any two points $x \in S$ and $y \in S$, their convex combination $\lambda x+(1-\lambda)y$ is also in S for all $\lambda \in [0,1]$.

In other words, a set is a convex set if the line segment joining any pair of points in S is wholly contained in S.

Are those Convex Sets?



Are those Convex Sets?

Feasible set of an LP $x=\{x\epsilon\ R^n: Ax\leq b,\ x\geq 0\}$

is a convex set.

Extreme Points (Corner Points):

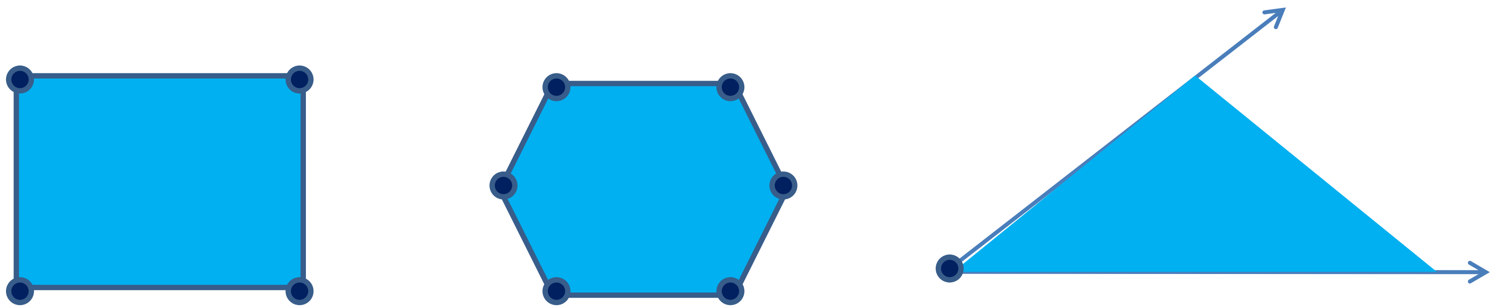
A point P of a convex set S is an extreme point (corner point) if it cannot be represented as a ***strict convex combination*** of two distinct points of S.

strict convex combination: x is a strict convex combination of x_1 and x_2 if

$$x=\lambda x_1+(1-\lambda)x_2 \text{ for some } \lambda \in (0,1).$$

Extreme Points (Corner Points):

In other words, for any convex set S, a point P in S is an extreme point (corner point) if each line segment that lies completely in S and contains the point P, has P as an endpoint of the line segment.



Graphical Solution of 2-variable LP Problems

Ex 2)

A company wishes to increase the demand for its product through advertising. Each minute of radio ad costs \$1 and each minute of TV ad costs \$2.

Each minute of radio ad increases the daily demand by 2 units and each minute of TV ad by 7 units.

The company would wish to place at least 9 minutes of daily ad in total. It wishes to increase daily demand by at least 28 units.

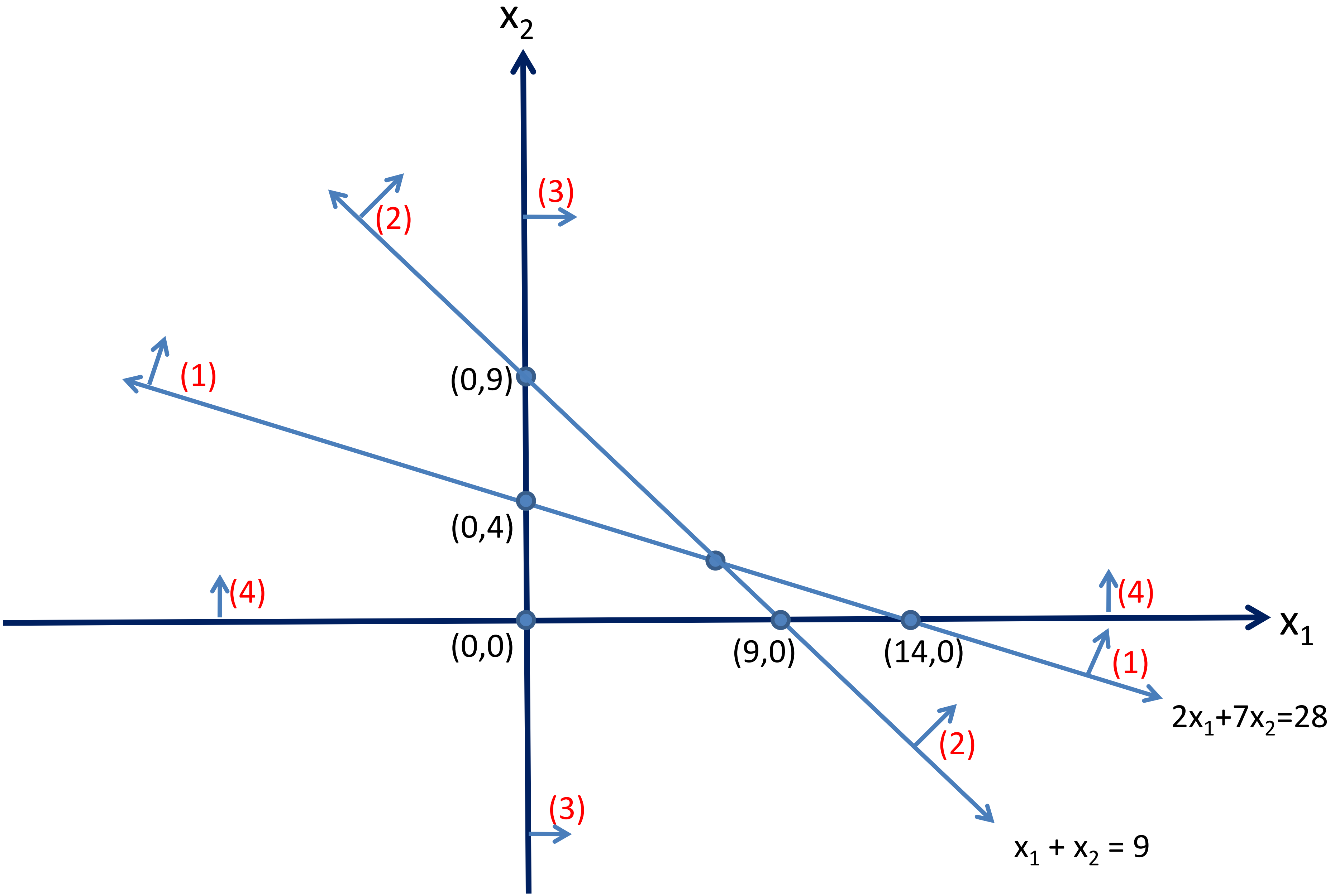
How can the company meet its advertising requirements at §minimum total cost?

Graphical Solution of 2-variable LP Problems

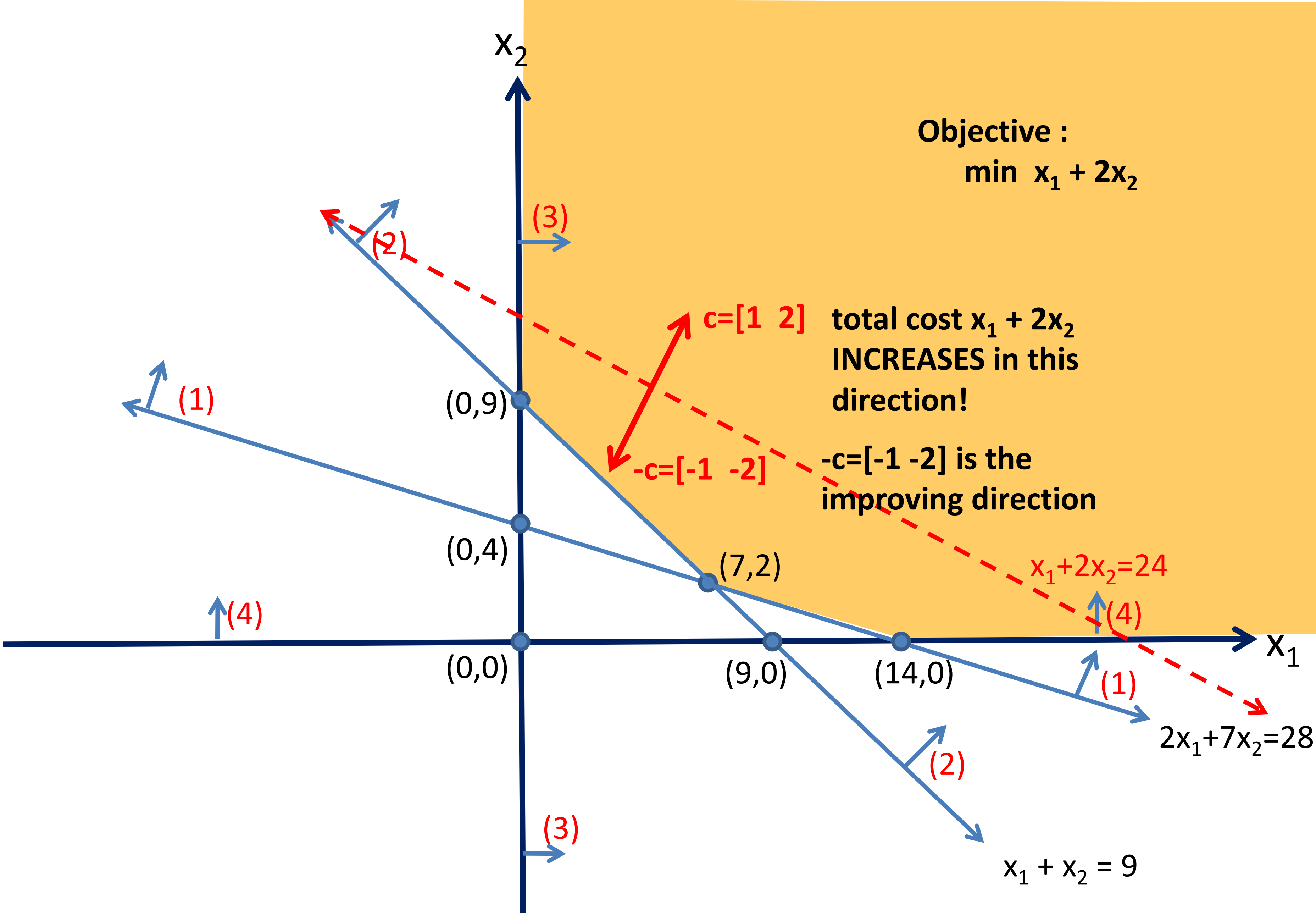
x_1: # of minutes of radio ads purchased

x_2: # of minutes of TV ads purchased

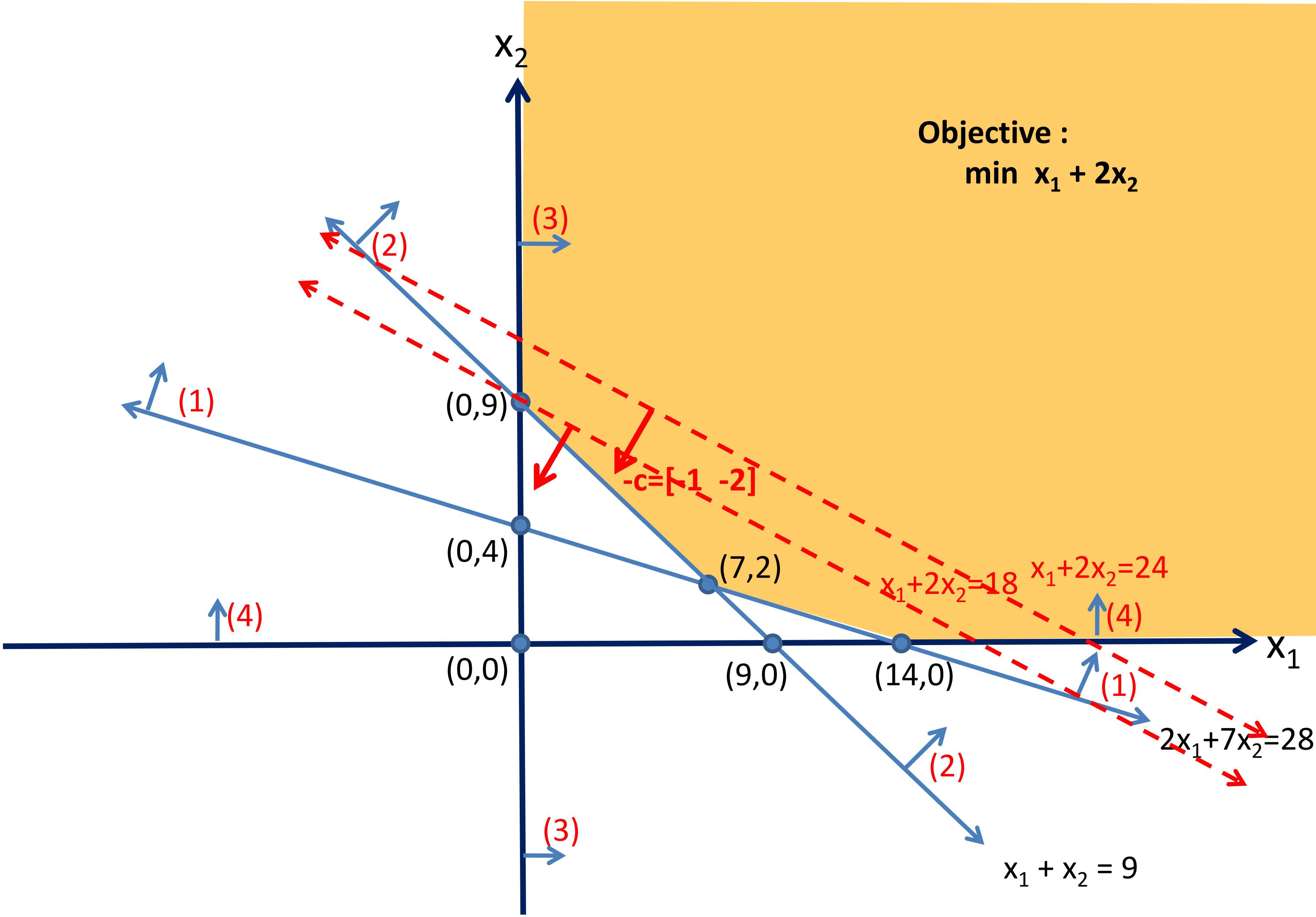
$$
\begin{aligned}
\min \quad & x_1 + 2x_2 \\
\text{s.t.} \quad & \\
& 2x_1 + 7x_2 \geq 28 && (1) \\
& x_1 + x_2 \geq 9 && (2) \\
& x_1, x_2 \geq 0 && (3), (4)
\end{aligned}
$$



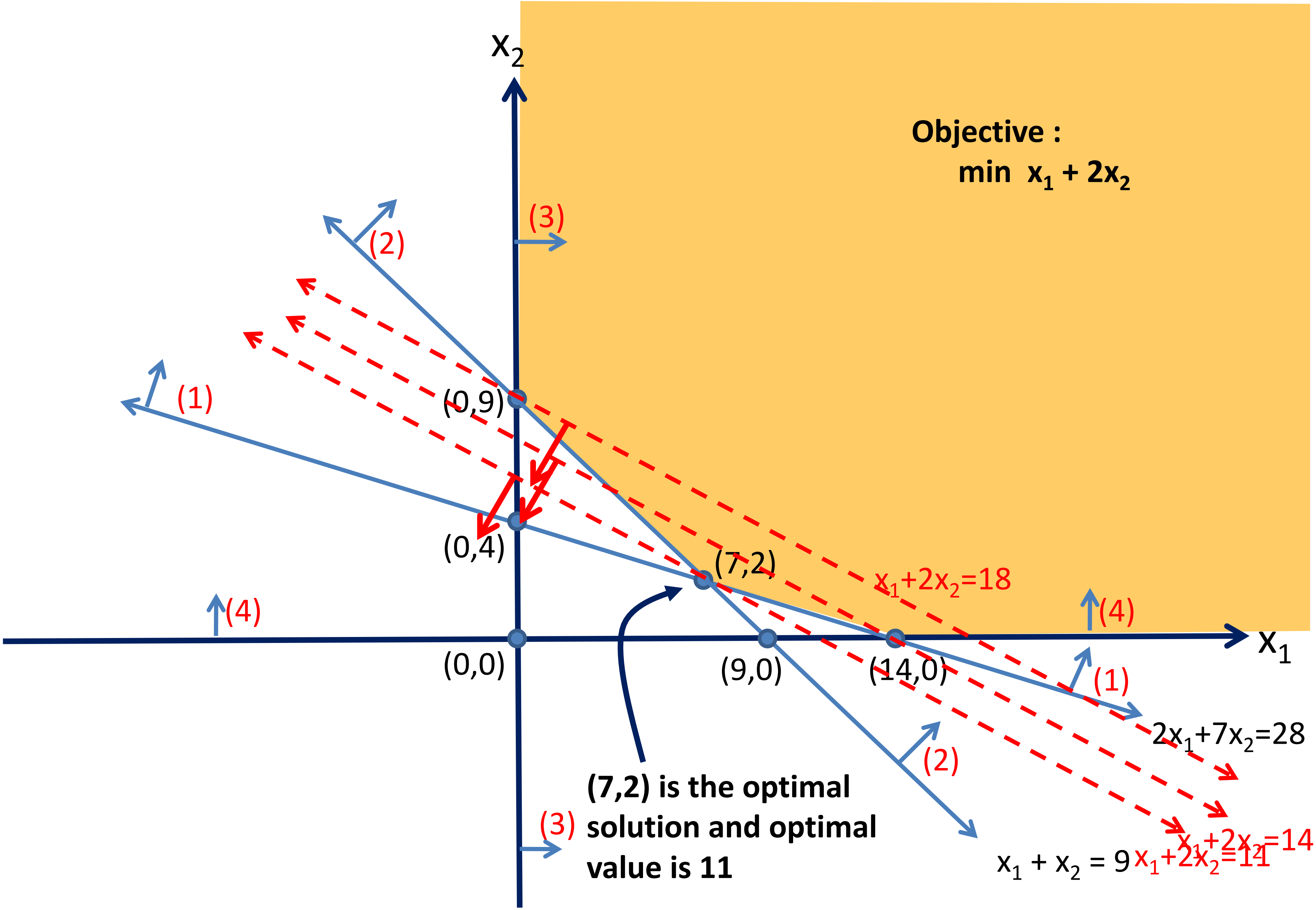
x_2
x_1
(0,9)
(0,4)
(0,0)
(9,0)
(14,0)
(1)
(2)
(3)
(4)
$2x_1+7x_2=28$
$x_1 + x_2 = 9$



x2
x1
Objective :
min x1 + 2x2
c=[1 2]
total cost x1 + 2x2
INCREASES in this
direction!
-c=[-1 -2]
-c=[-1 -2] is the
improving direction
x1+2x2=24
(0,9)
(0,4)
(7,2)
(0,0)
(9,0)
(14,0)
(1)
(2)
(3)
(4)
2x1+7x2=28
x1 + x2 = 9



Objective :
min $x_1 + 2x_2$
x_2
x_1
(1)
(2)
(3)
(4)
(0,9)
(0,4)
(0,0)
(7,2)
(9,0)
(14,0)
$-c=[-1\ \ -2]$
$x_1+2x_2=18$
$x_1+2x_2=24$
$2x_1+7x_2=28$
$x_1 + x_2 = 9$



x2
Objective :
min x1 + 2x2
(3)
(2)
(1)
(0,9)
(0,4)
(7,2)
x1+2x2=18
(4)
(4)
x1
(0,0)
(9,0)
(14,0)
(1)
2x1+7x2=28
(2)
(7,2) is the optimal solution and optimal value is 11
(3)
x1+2x2=14
x1 + x2 = 9
x1+2x2=11

Graphical Solution of 2-variable LP Problems

Ex 3)

Suppose that in the previous example, it costed $4 to place a minute of radio ad and $14 to place a minute of TV ad.

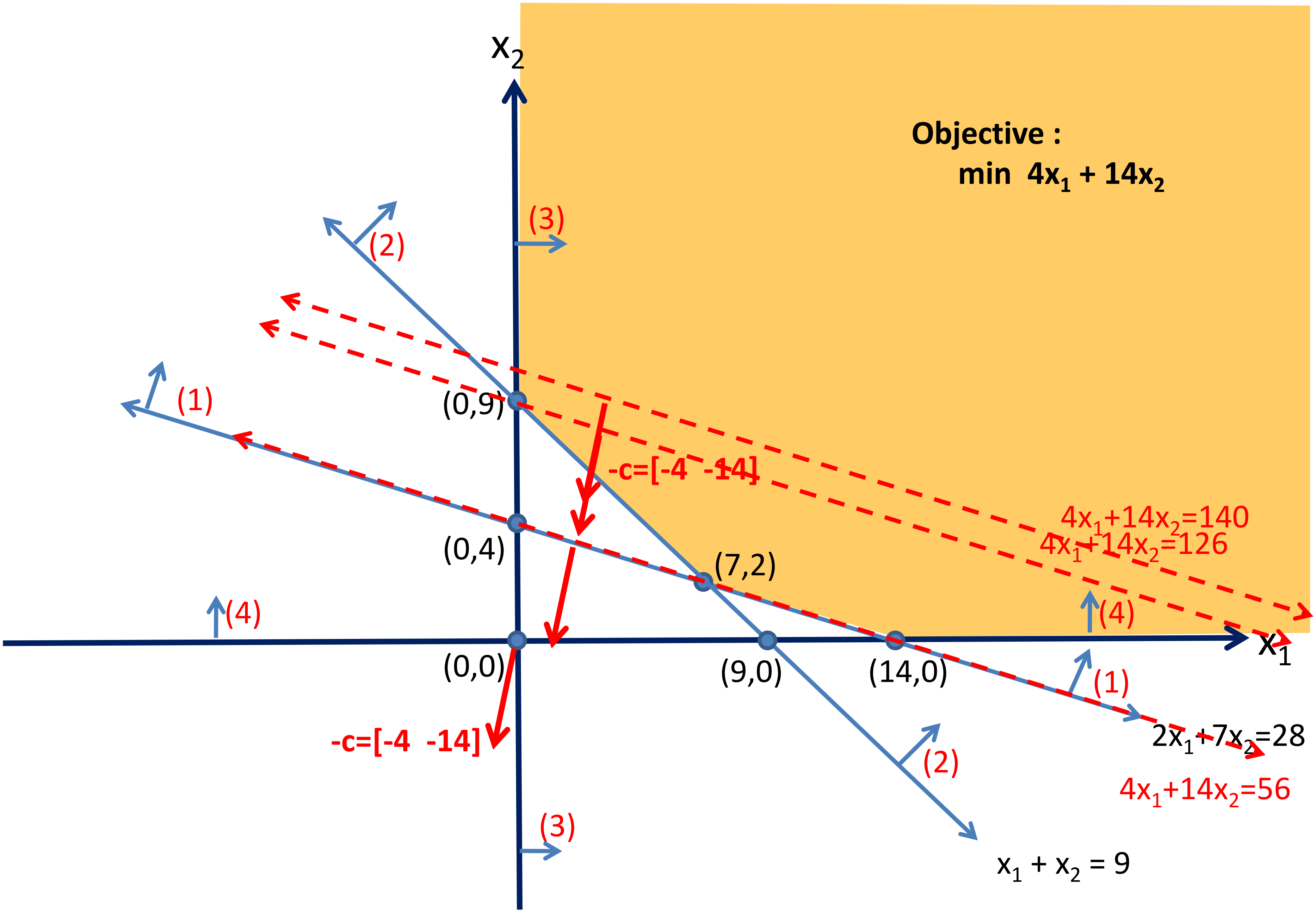
$$\min \quad 4x_1 + 14\,x_2$$

s.t.

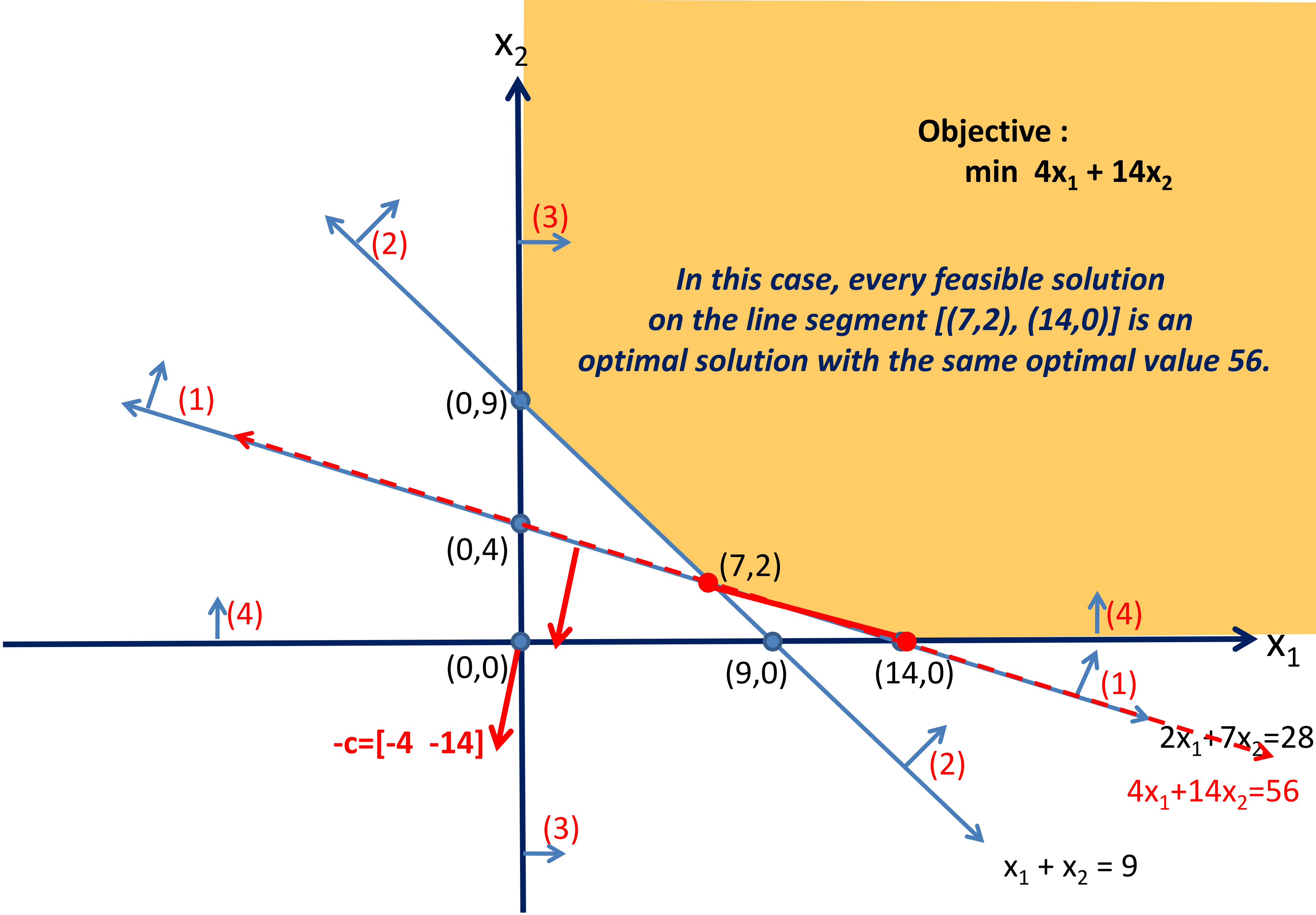
$$2x_1 + 7x_2 \geq 28 \qquad (1)$$

$$x_1 + x_2 \geq 9 \qquad (2)$$

$$x_1, x_2 \geq 0 \qquad (3), (4)$$



Objective :
min $4x_1 + 14x_2$
x_2
x_1
(1)
(2)
(3)
(4)
(0,9)
(0,4)
(0,0)
(7,2)
(9,0)
(14,0)
-c=[-4 -14]
-c=[-4 -14]
$4x_1+14x_2=140$
$4x_1+14x_2=126$
$2x_1+7x_2=28$
$4x_1+14x_2=56$
$x_1 + x_2 = 9$



Objective :
min 4x1 + 14x2
In this case, every feasible solution
on the line segment [(7,2), (14,0)] is an
optimal solution with the same optimal value 56.
x2
x1
(1)
(2)
(3)
(4)
(0,9)
(0,4)
(0,0)
(7,2)
(9,0)
(14,0)
-c=[-4 -14]
2x1+7x2=28
4x1+14x2=56
x1 + x2 = 9

Graphical Solution of 2-variable LP Problems

At point $A=\begin{bmatrix}7\\2\end{bmatrix}$, the objective function value is: $[4\ 14]\begin{bmatrix}7\\2\end{bmatrix}=56$,

At point $B=\begin{bmatrix}14\\0\end{bmatrix}$, the objective function value is: $[4\ 14]\begin{bmatrix}14\\0\end{bmatrix}=56$.

This is also true for any feasible point on the line segment [A,B]. We say that LP has **multiple** or **alternate optimal** solutions.

$$\text{Line Segment } [A,B]=\left\{\lambda\begin{bmatrix}7\\2\end{bmatrix}+(1-\lambda)\begin{bmatrix}14\\0\end{bmatrix}:\lambda\in[0,1]\right\}$$

Graphical Solution of 2-variable LP Problems

Ex 4)

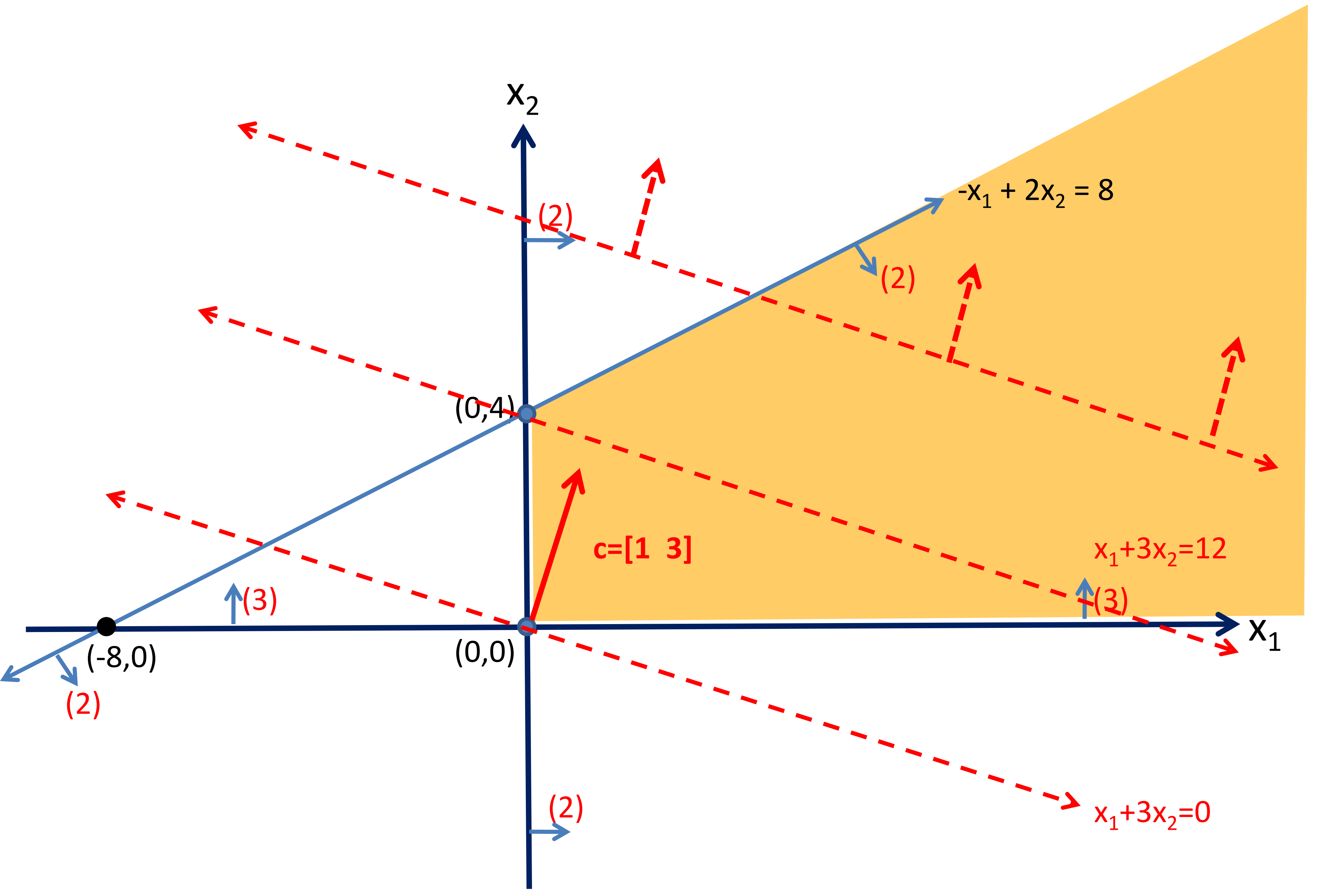
Consider the following LP problem:

max $\quad x_1 + 3\, x_2$

s.t.

$$-x_1 + 2x_2 \le 8 \qquad (1)$$

$$x_1, x_2 \ge 0 \qquad (2), (3)$$


x_2
x_1
$-x_1 + 2x_2 = 8$
(2)
(2)
(2)
(2)
(0,4)
(0,0)
(-8,0)
(3)
(3)
c=[1 3]
$x_1+3x_2=12$
$x_1+3x_2=0$

Unbounded LP Problems:

The objective function for this **maximization problem** can be increased by moving in the improving direction c as much as we want while still staying in the feasible region.
In this case, we say that the **LP is unbounded**.

For unbounded LP problems, there is no optimal solution and the optimal value is defined to be $+\infty$.

Unbounded LP Problems:

For the **minimization problem**, we say that the LP is unbounded if we can DECREASE the objective function value as much as we want while still staying in the feasible region.

In this case, the optimal value is defined to be **-∞**.

Graphical Solution of 2-variable LP Problems

Ex 5)

max $\quad x_1 + 3\,x_2$

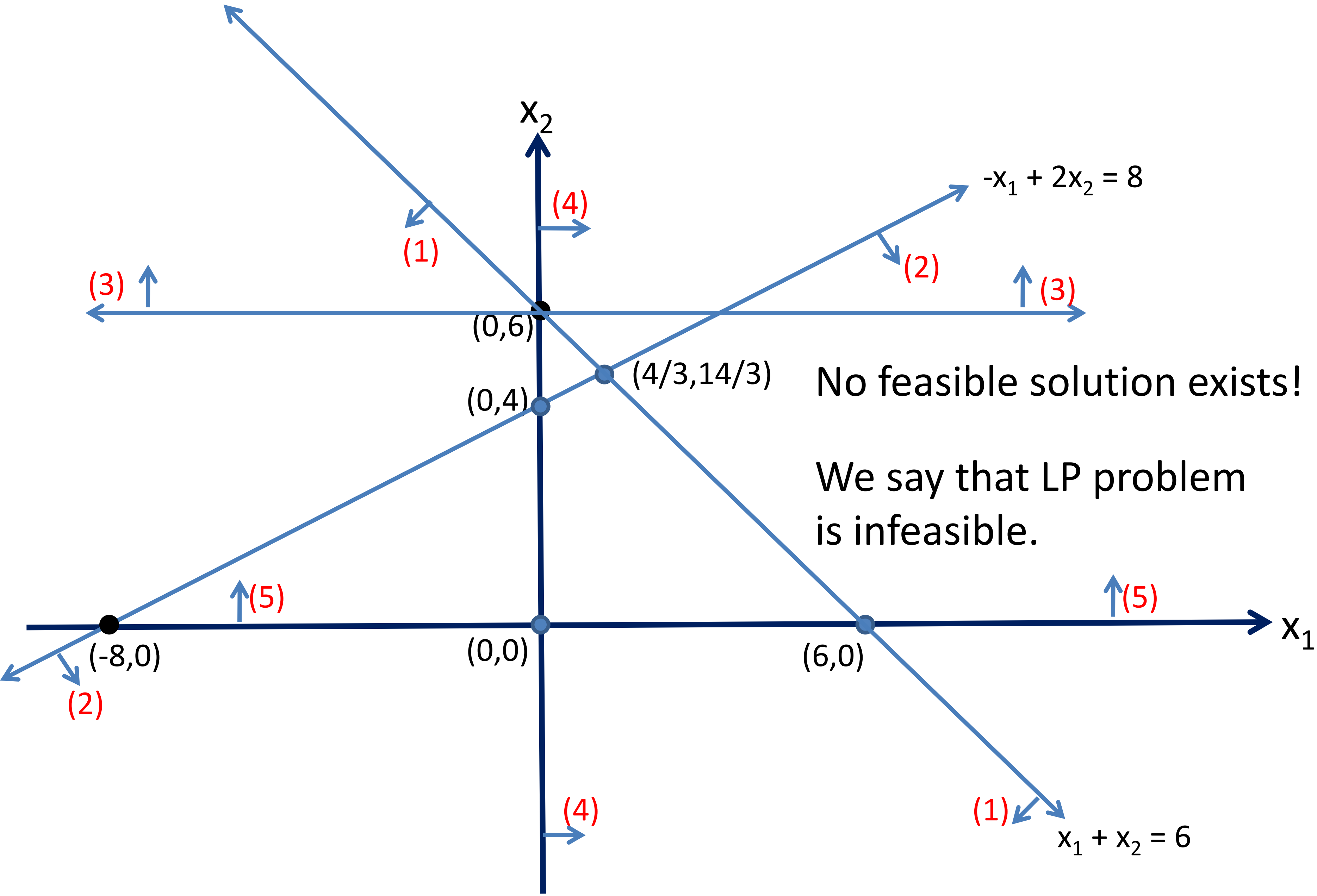
s.t.

$x_1 + \quad x_2 \leq 6 \qquad (1)$

$-x_1 + 2x_2 \leq 8 \qquad (2)$

$x_2 \geq 6 \qquad (3)$

$x_1, x_2 \geq 0 \qquad (4), (5)$



No feasible solution exists!

We say that LP problem is infeasible.

Every LP Problem falls into one of the 4 cases:

Case 1: The LP problem has a unique optimal solution (see Ex.1 and Ex.2)

Case 2: The LP problem has alternative or multiple optimal solution (see Ex.3). In this case, there are **infinitely many optimal solutions**.

Case 3: The LP problem is unbounded (see Ex.4). In this case, there is **no optimal solution**.

Case 4: The LP problem is infeasible (see Ex.5). In this case, there is no feasible solution.

Extreme Points (Corner Points):

Theorem: If the feasible set $\{x \in R^n: Ax \leq b, x \geq 0\}$ is not empty, that is if there is a feasible solution, then there is an extreme point.

Theorem: If an LP has an optimal solution, then there is an extreme (or corner) point of the feasible region which is an optimal solution to the LP

P.S: Not every optimal solution needs to be an extreme point! (Remember the alternate optimal solution case)

Graphical Solution of LP Problems:

We may graphically solve an LP with two decision variables as follows:

Step 1: Graph the feasible region.

Step 2: Draw an isoprofit line (for max problem) or an isocost line (for min problem).

Step 3: Move parallel to the isoprofit/isocost line in the improving direction. The last point in the feasible region that contacts an isoprofit/isocost line is an optimal solution to the LP.