IE 400 Principles of Engineering Management

The Simplex Algorithm-I:
Set 3

- So far, we have studied how to solve two-variable LP problems graphically.
- However, most real life problems have more than two variables!
- Therefore, we need to have another method to solve LPs with more than two variables.
- We are going to study **The Simplex Algorithm** which is quite useful in solving very large LP problems.
- Today, The Simplex Algorithm is used to solve LP problems in many industrial applications that involve thousands of variables and constraints.

- LP problems can have both equality and inequality constraints.
- LP problems can have nonnegative and urs (unrestricted in sign) variables.
- To use the Simplex Method, LP problems should be converted to ***Standard Form LP***.

Standard Form LP:

  - Why? We know from 2-variables that extreme points are potential optimal solutions
  - This will be true in higher dimensions as well
  - We need an ALGEBRAIC way of characterizing extreme points. We can’t draw the feasible region in higher dimensions
  - Standard form LPs will provide an easy way to do this characterization

- **All constraints are equalities, with constant nonnegative right-hand sides (RHS),**
- **All variables are nonnegative.**
- **Any LP can be brought into standard form!**

Simplex Method:

- Start with an extreme point
- Move to a neighboring extreme point in the improving direction
- Stop if all neighbors are no better

- **Simple Greedy Logic**

How to find a feasible extreme point?

How to go to a neighbor?

Converting LP into Standard Form:

Ex: Consider the following LP problem:

$$\max \quad x_1 + 3\, x_2$$

s.t.

$$2x_1 + 3x_2 + x_3 \le 5 \qquad (1)$$

$$4x_1 + x_2 + 2x_3 = -11 \qquad (2)$$

$$3x_1 + 4x_2 + 2x_3 \ge 8 \qquad (3)$$

$x_1 \ge 0$, $x_2 \le 0$, x_3 urs (free)

Converting LP into Standard Form:

a) Define a "slack" variable for each of the "≤" constraint to convert the inequality constraint into an equality constraint:

$$s_1 = 5 - 2x_1 - 3x_2 - x_3 \,,\ s_1 \geq 0 \qquad (1)$$

So that, the first constraint becomes:

$$2x_1 + 3x_2 + x_3 + s_1 = 5 \qquad (1)$$

Converting LP into Standard Form:

b) Multiply the second constraint by -1 to get a nonnegative right hand side value, i.e. replace

$$4x_1 + x_2 + 2x_3 = -11 \qquad (2)$$

with:

$$-4x_1 - x_2 - 2x_3 = 11 \qquad (2)$$

Converting LP into Standard Form:

c) Define an "excess (surplus)" variable for each of the "≥" constraints to convert the inequality constraint into an equality constraint:

$$e_3 = 3x_1 + 4x_2 + 2x_3 - 8, \; e_3 \geq 0 \qquad (3)$$

so that, the third constraint becomes:

$$3x_1 + 4x_2 + 2x_3 - e_3 = 8 \qquad (3)$$

Converting LP into Standard Form:

d) Variable x_2 has a reverse sign restriction:

Replace x_2 with $-x_2'$ throughout.

If x_2 is nonpositive then $-x_2'$ will be nonnegative

Converting LP into Standard Form:

e) Variable x_3 is unrestricted in sign:

Replace x_3 with $x_3' - x_3''$ and force both x_3' and x_3''

to both be nonnegative

Converting LP into Standard Form:

$$\max \quad x_1 + 3x_2$$

s.t.

$$2x_1 + 3x_2 + x_3 \le 5 \quad (1)$$

$$4x_1 + x_2 + 2x_3 = -11 \quad (2)$$

$$3x_1 + 4x_2 + 2x_3 \ge 8 \quad (3)$$

$$x_1 \ge 0,\ x_2 \le 0,\ x_3 \text{ urs}$$

$$\max \quad z = x_1 - 3x_2'$$

s.t.

$$2x_1 - 3x_2' + x_3' - x_3'' + \mathbf{s_1} = 5 \quad (1)$$

$$-4x_1 + x_2' - 2x_3' + 2x_3'' = 11 \quad (2)$$

$$3x_1 - 4x_2' + 2x_3' - 2x_3'' - \mathbf{e_3} = 8 \quad (3)$$

$$x_1 \ge 0,\ x_2' \ge 0,\ x_3' \ge 0,\ x_3'' \ge 0,$$
$$s_1 \ge 0,\ e_3 \ge 0.$$

To Convert an LP into Standard Form:

- each inequality constraint is converted into an equality constraint by adding or subtracting nonnegative slack/excess variables,
- an inequality (equality) can be multiplied by -1 to get nonnegative RHS,
- unrestricted variable can be represented as the difference of two new nonnegative variables.

If x_i is urs, then let $x_i = x_i' - x_i''$ where $x_i', x_i'' \geq 0$.

Replace every occurrence of x_i with $x_i' - x_i''$ and add sign restrictions $x_i', x_i'' \geq 0$.

- For sign restriction $x_k \leq 0$, let $x_k' = -x_k$ and replace every occurrence of x_k with $-x_k'$ and add the sign restriction $x_k' \geq 0$.

Standard Form LP:

Suppose that we converted an LP with *m* constraints into a standard form. Also assume that after the conversion, we have *n* variables as $x_1, x_2, x_3, \ldots, x_n$.

Standard Form LP:

Suppose that we converted an LP with *m* constraints into a standard form. Also assume that after the conversion, we have *n* variables as x_1, x_2, x_3,..., x_n.

$$\max\ (\min)\quad c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

$$\text{s.t.}$$

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad & \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

$$x_i \geq 0 \quad (i = 1, 2, \ldots, n)$$

Standard Form LP:

If we define:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

and

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Standard Form LP:

If we define:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

and

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

LP can be written as the system of equations :

$$\Rightarrow \quad Ax=b$$

Standard Form LP:

- For Ax=b to have a solution, rank (A|b)=rank (A).
- We also assume that all redundant constraints are removed, so rank(A)=m.

- i.e.

$$
\begin{aligned}
x_1 + 2x_2 + x_3 &= 4 \quad (1)\\
x_2 - x_3 &= 1 \quad (2)\\
2x_1 + 6x_2 &= 10 \quad (3)
\end{aligned}
$$

Standard Form LP:

- For $Ax=b$ to have a solution, rank $(A|b)$=rank (A).
- We also assume that all redundant constraints are removed, so rank$(A)=m$.

  i.e.

$$x_1 + 2x_2 + x_3 = 4 \quad (1)$$

$$x_2 - x_3 = 1 \quad (2)$$

$$2x_1 + 6x_2 = 10 \quad (3)$$

Constraint (3) can be written as a linear combination of (1) and (2):

2 [(1)+(2)] = (3)

Remove the redundant constraint

Standard Form LP:

- Before proceeding any further with the discussion of the simplex algorithm, we should define the concept of a basic solution to a linear system.
- **Basic Solution:** A solution to Ax=b is called a basic solution if it is obtained by setting ***n-m*** variables equal to 0 and solving for the remaining ***m*** variables whose columns are linearly independent.

Standard Form LP:

- The ***n-m*** variables whose values are set to 0 are called **nonbasic variables**.
- The remaining ***m*** variables are called **basic variables**.

Standard Form LP:

For $Ax=b$, where A is a $m\times n$ matrix, $\text{rank}(A)=m$, $b\geq 0$,

- pick m linearly independent columns from A,
- rearrange A such that these chosen columns are the first m columns in A.

Standard Form LP:

For $Ax=b$, where A is a $m \times n$ matrix, $\text{rank}(A)=m$, $b \geq 0$,

- pick m linearly independent columns from A,
- rearrange A such that these chosen columns are the first m columns in A.

Since elementary column and row operations do not change the system of linear equations, matrix A can be brought into a form $A=[B_{m \times m} \ \ N_{m \times (n-m)}]$, where $B_{m \times m}$ is invertible.

Standard Form LP:

Let $x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}$ be the corresponding partition in x.

$$Ax = b \;\equiv\; \begin{bmatrix} B & N \end{bmatrix} \begin{pmatrix} x_B \\ x_N \end{pmatrix} = b \;\Rightarrow Bx_B + Nx_N = b$$

Standard Form LP:

Let $x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}$ be the corresponding partition in x.

$$(Ax = b) \equiv \left(\begin{bmatrix} B & N \end{bmatrix} \begin{pmatrix} x_B \\ x_N \end{pmatrix} = b \right) \Rightarrow Bx_B + Nx_N = b$$

If we set $x_N = 0$, then $x_B = B^{-1}b$ will be a unique solution.

For every such B choice, $\exists$ a unique solution $\begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}$.

Standard Form LP:

$$\text{If a basic solution } x = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix} \geq 0,$$

then x is called a basic feasible solution (bfs).

Basic Solutions:

- Consider the system of equations:

$$\begin{aligned} x_1 + x_2 + x_3 \quad &= 6 \\ x_2 + x_4 &= 3 \end{aligned}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Basic Solutions:

- Consider the system of equations:

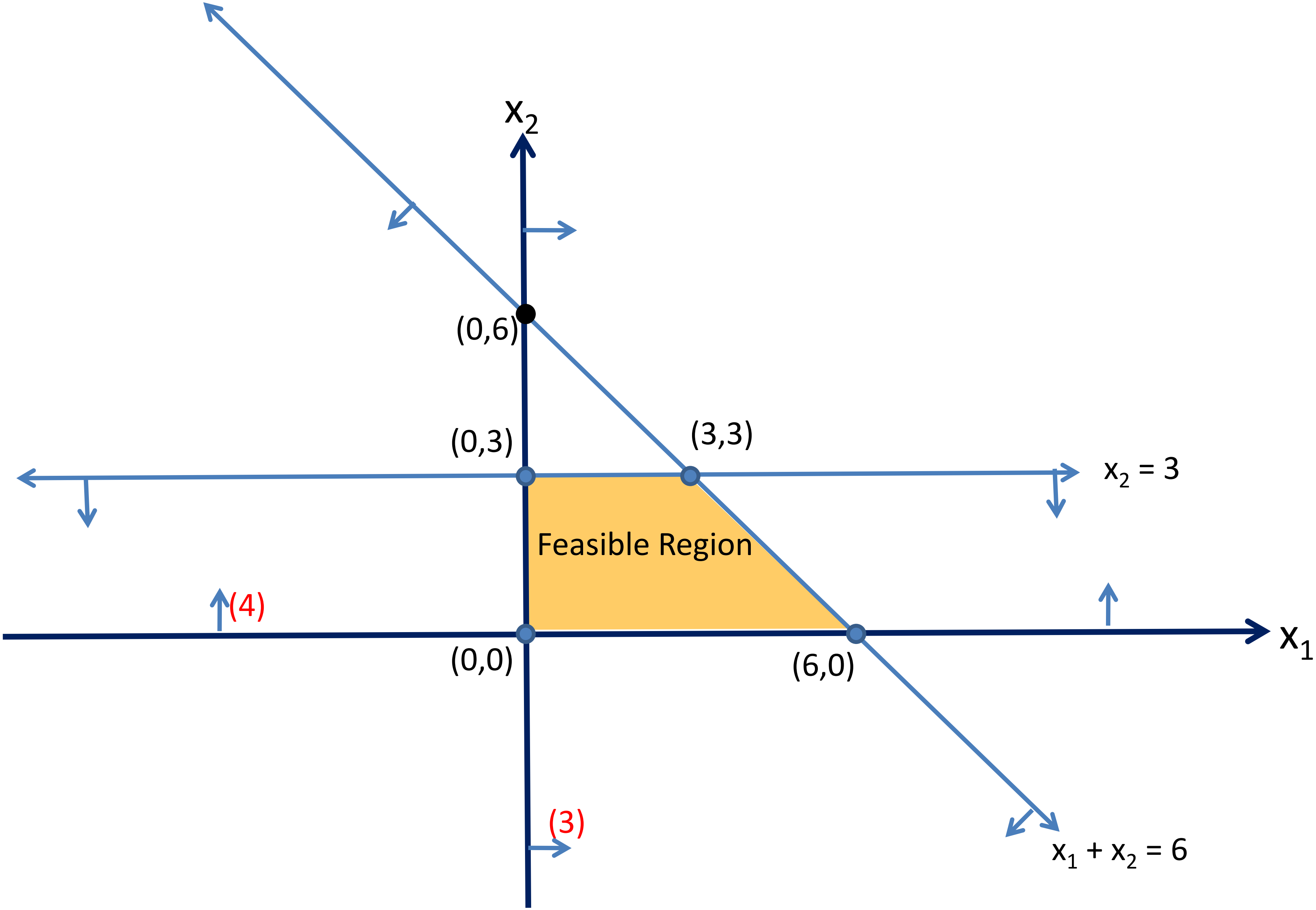
$$
\begin{aligned}
x_1 + x_2 &\leq 6 \\
x_2 &\leq 3 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

(a)

- Converting into a standard form LP:

$$
\begin{aligned}
x_1 + x_2 + x_3 \phantom{{}+x_4} &= 6 \\
x_2 \phantom{{}+x_3} + x_4 &= 3 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}
$$

(b)



x_2
$(0,6)$
$(0,3)$
$(3,3)$
$x_2 = 3$
Feasible Region
(4)
$(0,0)$
$(6,0)$
x_1
(3)
$x_1 + x_2 = 6$

Basic Solutions:

- Consider the system of equations:

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 6 \\
x_2 + x_4 &= 3
\end{aligned}
$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$\Rightarrow$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \qquad b = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

Basic Solutions:

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

1. Let us chose:

$$B = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\text{The columns are linearly independent}} \quad x_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad x_N = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

Basic Solutions:

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$$\text{Setting } x_N = 0 \text{ and solving for } x_B : \; Bx_B = b \Rightarrow x_B = B^{-1}b$$

Basic Solutions:

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Setting $x_N = 0$ and solving for x_B : $Bx_B = b \Rightarrow x_B = B^{-1}b$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1}\begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

Basic Solutions:

$$\text{Since } x_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \geq 0, \quad x = \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 0 \\ 0 \end{pmatrix} \text{ is a bfs.}$$

Basic Solutions:

Let us consider different ways of forming B:

2.

$$\underbrace{B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{\text{The columns are Linearly dependent!}} \quad x_B = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \quad x_N = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix}$$

Basic Solutions:

Let us consider different ways of forming B:

2.

$$B = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{\text{The columns are Linearly dependent!}} \quad x_B = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \quad x_N = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix}$$

Hence the choice of $x_B = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$ cannot be a basic solution.

Basic Solutions:

3. Let us chose:

$$B = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{The columns are linearly independent}} \quad x_B = \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} \quad x_N = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$$

Basic Solutions:

3. Let us chose:

$$B = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\substack{\text{The columns are} \\ \text{linearly independent}}} \quad x_B = \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} \quad x_N = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$$

Setting $x_N = 0$ and solving for x_B :

$$x_B = B^{-1}b = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} \Rightarrow x = \begin{pmatrix} 6 \\ 0 \\ 0 \\ 3 \end{pmatrix} \text{is a bfs.}$$

Basic Solutions:

4. Let us chose:

$$B = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{\text{The columns are linearly independent}} \qquad x_B = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \qquad x_N = \begin{pmatrix} x_1 \\ x_4 \end{pmatrix}$$

Basic Solutions:

4. Let us chose:

$$B = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{\text{The columns are linearly independent}} \qquad x_B = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \qquad x_N = \begin{pmatrix} x_1 \\ x_4 \end{pmatrix}$$

Setting $x_N = 0$ and solving for x_B:

$$x_B = B^{-1}b = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \Rightarrow x = \begin{pmatrix} 0 \\ 3 \\ 3 \\ 0 \end{pmatrix} \text{ is a bfs.}$$

Basic Solutions:

5. Let us chose:

$$B = \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{\text{The columns are linearly independent}} \quad x_B = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} \quad x_N = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$$

Basic Solutions:

5. Let us chose:

$$B = \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{\text{The columns are linearly independent}} \quad x_B = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} \quad x_N = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$$

$$\text{Setting } x_N = 0 \text{ and solving for } x_B :$$

$$x_B = B^{-1}b = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix} \Rightarrow x = \begin{pmatrix} 0 \\ 6 \\ 0 \\ -3 \end{pmatrix},$$

Basic Solutions:

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$$B = \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{\text{The columns are linearly independent}} \qquad x_B = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} \qquad x_N = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$$

$$\text{Setting } x_N = 0 \text{ and solving for } x_B :$$

$$x_B = B^{-1}b = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix} \Rightarrow x = \begin{pmatrix} 0 \\ 6 \\ 0 \\ -3 \end{pmatrix},$$

This basic solution is not feasible!

Basic Solutions:

6. Let us chose:

$$B = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{The columns are linearly independent}} \qquad x_B = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \qquad x_N = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Basic Solutions:

6. Let us chose:

$$B = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{The columns are linearly independent}} \quad x_B = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \quad x_N = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

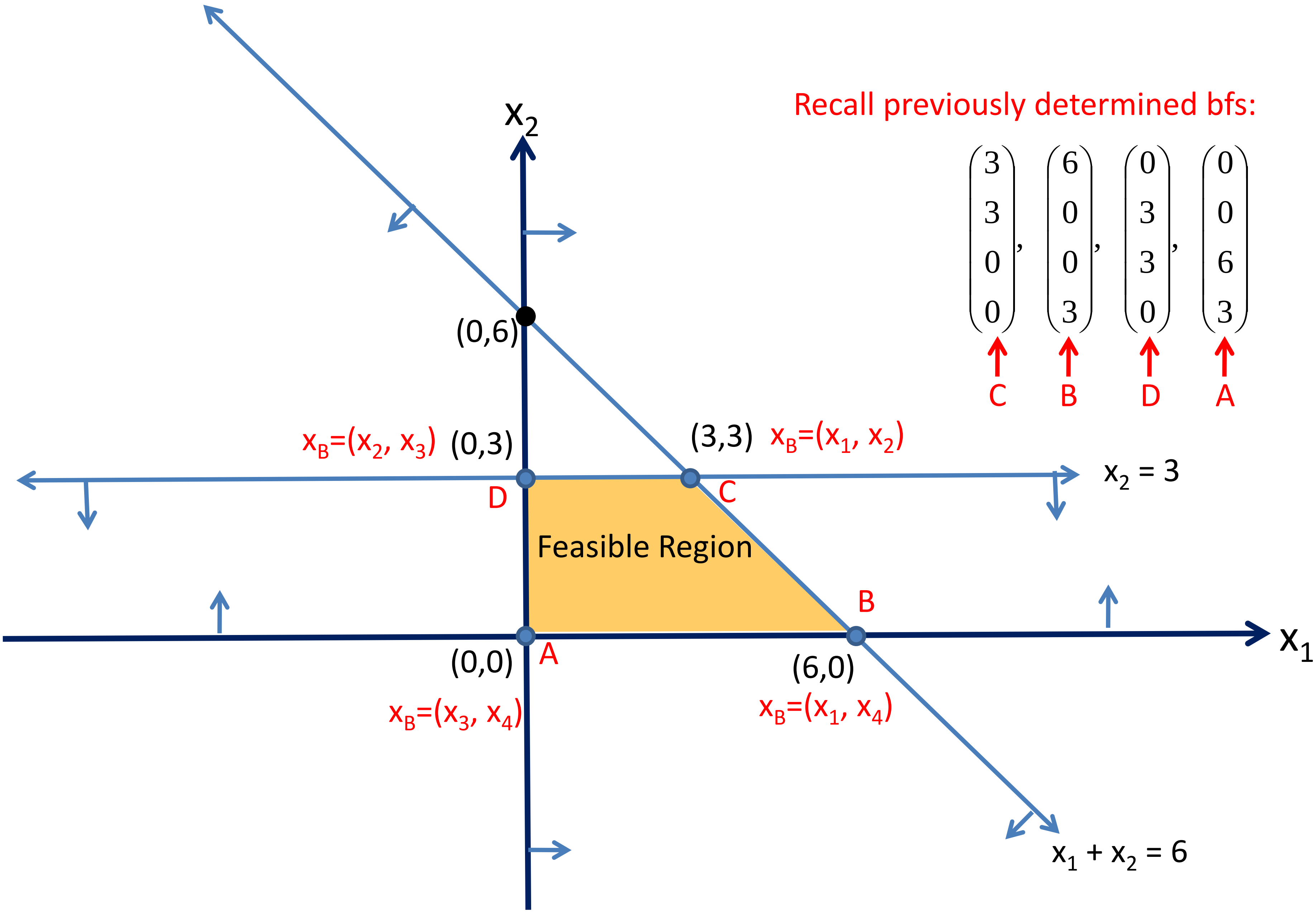
Setting $x_N = 0$ and solving for x_B :

$$x_B = B^{-1}b = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} \Rightarrow x = \begin{pmatrix} 0 \\ 0 \\ 6 \\ 3 \end{pmatrix} \text{ is a bfs.}$$

Basic Feasible Solutions:

So, this system of equations has 4 basic feasible solutions:

$$\begin{pmatrix} 3 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ 0 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 6 \\ 3 \end{pmatrix}$$


Recall previously determined bfs:
$\begin{pmatrix}3\\3\\0\\0\end{pmatrix}, \begin{pmatrix}6\\0\\0\\3\end{pmatrix}, \begin{pmatrix}0\\3\\3\\0\end{pmatrix}, \begin{pmatrix}0\\0\\6\\3\end{pmatrix}$
C B D A
x_2
x_1
(0,6)
$x_B=(x_2, x_3)$ (0,3)
(3,3) $x_B=(x_1, x_2)$
$x_2 = 3$
D
C
Feasible Region
B
(0,0)
A
(6,0)
$x_B=(x_3, x_4)$
$x_B=(x_1, x_4)$
$x_1 + x_2 = 6$

Basic Feasible Solutions:

$$\text{The number of basic feasible solutions} \leq \binom{n}{m}$$

Theorem:

A point in the feasible region of an LP is an extreme point if and only if it is a basic feasible solution to the LP.

Fundamental Theorem of LP:

- If feasible set of an LP is non-empty, then there is at least one bfs.
- If an LP has an optimal solution, then there is a bfs which is optimal.

This Theorem Implies:

- Finding an optimal solution to an LP problem is equivalent to finding the best bfs!

$$\text{The number of bfs} \leq \binom{n}{m}$$

- Therefore, we should search for the best bfs.

- One way is to enumerate all bfs and choose the one that gives he best objective function value.

- One way is to enumerate all bfs and choose the one that gives he best objective function value.

- However, enumerating all bfs can be very expensive!

$$\text{For example, for } n = 20 \text{ and } m = 10, \begin{pmatrix} n \\ m \end{pmatrix} \text{ is } 184756!$$

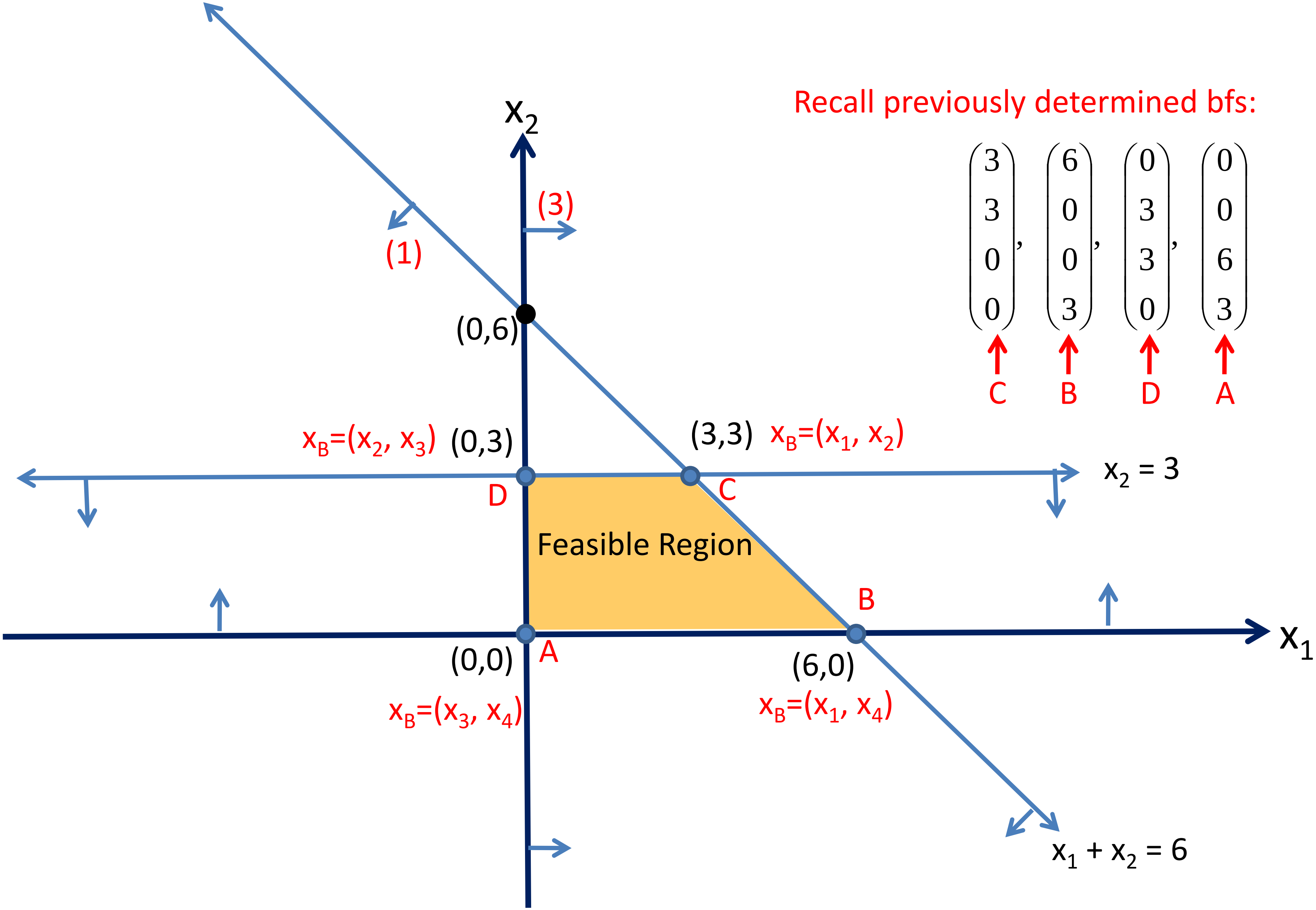
- Simplex Algorithm does this in a clever way. Usually it finds an optimal solution within *3m* enumeration.

Neighboring extreme points (bfs solutions):

- For any LP with m constraints, two bfs are said to be **adjacent** if their set of basic variables have m-1 basic variables in common.

  (Intuitively, two bfs are adjacent if they both lie on the same edge of the boundary of the feasible region)

- Simplex Algorithm goes from an extreme point to an adjacent extreme point with a better objective value.



Recall previously determined bfs:
3 3 0 0, 6 0 0 3, 0 3 3 0, 0 0 6 3
C B D A
x2
(3)
(1)
(0,6)
xB=(x2, x3) (0,3)
(3,3) xB=(x1, x2)
x2 = 3
D
C
Feasible Region
B
x1
(0,0)
A
(6,0)
xB=(x3, x4)
xB=(x1, x4)
x1 + x2 = 6

General Description of the Simplex Algorithm:

1. Convert the LP problem into a standard form LP.

2. Obtain a bfs to the LP. This bfs is called the **initial bfs**. In general, the most recent bfs is called the **current bfs**. Therefore, at the beginning the initial bfs is the current bfs.

General Description of the Simplex Algorithm:

3. Determine if the current bfs is an optimal solution or not.
4. If the current bfs is not optimal, then find an adjacent bfs with a better objective function value (one nonbasic variable becomes basic and one basic variable becomes nonbasic).
5. Go to Step 3.

The Simplex Algorithm:

- To begin the simplex algorithm, convert the LP into a standard form,

- Convert the objective function $z=c_1x_1+c_2x_2+\dots+c_nx_n$ to the row-0 format:

$$z-c_1x_1-c_2x_2-\dots-c_nx_n=0$$

The Simplex Algorithm:

max $z = x_1 + 3x_2$

s.t.

$$x_1 + x_2 \leq 6 \quad (1)$$

$$-x_1 + 2x_2 \leq 8 \quad (2)$$

$$x_1, x_2 \geq 0$$

max z

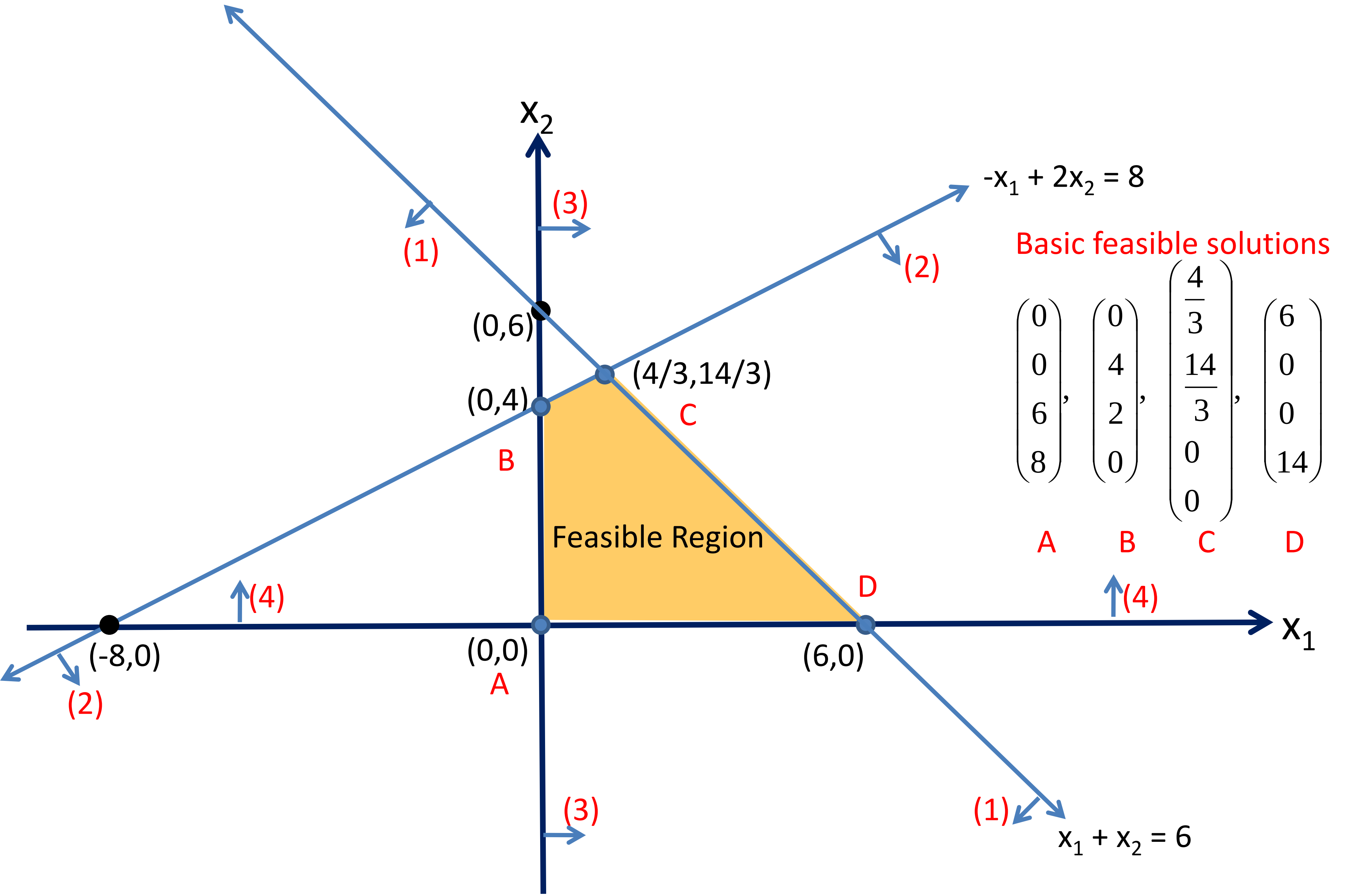
s.t.

$$z - x_1 - 3x_2 = 0 \quad (0)$$

$$x_1 + x_2 + s_1 = 6 \quad (1)$$

$$-x_1 + 2x_2 + s_2 = 8 \quad (2)$$

$$x_1, x_2, s_1, s_2 \geq 0$$



x_2
x_1
$-x_1 + 2x_2 = 8$
$x_1 + x_2 = 6$
(1)
(2)
(3)
(4)
(0,6)
(0,4)
(4/3,14/3)
(-8,0)
(0,0)
(6,0)
A
B
C
D
Feasible Region
Basic feasible solutions
$\begin{pmatrix}0\\0\\6\\8\end{pmatrix}, \begin{pmatrix}0\\4\\2\\0\end{pmatrix}, \begin{pmatrix}\frac{4}{3}\\\frac{14}{3}\\0\\0\end{pmatrix}, \begin{pmatrix}6\\0\\0\\14\end{pmatrix}$
A B C D

The Simplex Algorithm

Canonical Form 0:

Row	**Basic Variable**											**RHS**
0		z	-	x_1	-	$3x_2$					=	0
1				x_1	+	x_2	+	s_1			=	6
2			-	x_1	+	$2x_2$			+	s_2	=	8

A system of linear equations in which each equation has a variable <u>with a coefficient 1</u> in that equation (and <u>a zero coefficient</u> in all other equations) is said to be in *canonical form*.

If the RHS of each constraint in a canonical form is nonnegative, a basic feasible solution can be obtained by inspection.

The Simplex Algorithm

Recall that the Simplex Algorithm begins with an initial bfs and attempts to find better ones. After obtaining a canonical form, we search for the **initial bfs**.

By inspection, if we set $x_1=x_2=0$, we can solve for the values of s_1 and s_2 by setting s_i equal to the RHS of row i.

$BV=\{s_1,s_2\}$ and $NBV=\{x_1,x_2\}$

The Simplex Algorithm

You may also verify the calculations for the initial basic feasible solution by:

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad x_N = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad x_B = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$$

The columns are
linearly independent

$$\text{Setting } \; x_N = 0 \text{ and solving for } x_B : \; Bx_B = b \Rightarrow x_B = B^{-1}b$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix} \Rightarrow \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{pmatrix} 6 \\ 8 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} 6 \\ 8 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$$

The Simplex Algorithm

Notice that each basic variable is associated with the row of the canonical form in which the basic variable has a coefficient of 1.

To perform the simplex algorithm, we also need a basic variable **(not necessarily nonnegative!)** for row 0.

Observe that variable z appears in row 0 with a coefficient of 1, and z does not appear in any other row. Therefore, **we use z as basic variable for row 0**.

The Simplex Algorithm

Let us denote the initial canonical form as canonical form 0.
With this convention, the basic and nonbasic variables
for the canonical form 0 are BV=$\{z, s_1, s_2\}$ and NBV=$\{x_1, x_2\}$.

For this basic feasible solution, $z=0$, $s_1=6$, $s_2=8$, $x_1=0$, $x_2=0$.

The Simplex Algorithm

In summary, the **canonical form**:

- LP has equality constraints and nonnegativity constraints ,
- There is one **basic** variable for each equality constraint,
- The column for the basic variable for constraint i has a 1 in constraint i and 0’s elsewhere,
- The remaining variables are called **nonbasic**.

The Simplex Algorithm

Canonical Form 0:

Row	Basic Variable												RHS
0		z	$-$	x_1	$-$	$3x_2$						$=$	0
1				x_1	$+$	x_2	$+$	s_1				$=$	6
2			$-$	x_1	$+$	$2x_2$			$+$	s_2		$=$	8

The Simplex Algorithm

Canonical Form 0:

Row	**Basic Variable**												**RHS**
0	$z=0$	z	-	x_1	-	$3x_2$						=	0
1	$s_1=6$			x_1	+	x_2	+	s_1				=	6
2	$s_2=8$		-	x_1	+	$2x_2$			+	s_2		=	8

Is this current basic feasible solution optimal?

To answer this question, we should determine whether there is any way that z can be increased by increasing some nonbasic variable from its current value of zero while holding all other nonbasic variables at their current values of zero ***(To reach an adjacent bfs)***.

The Simplex Algorithm

Canonical Form 0:

Row	Basic Variable											RHS
0	$z=0$	z	-	x_1	-	$3x_2$					=	0
1	$s_1=6$			x_1	+	x_2	+	s_1			=	6
2	$s_2=8$		-	x_1	+	$2x_2$			+	s_2	=	8

The Simplex Algorithm

Canonical Form 0:

Row	**Basic Variable**												**RHS**
0	$z=0$	z	-	x_1	-	$3x_2$						=	0
1	$s_1=6$			x_1	+	x_2	+	s_1				=	6
2	$s_2=8$		-	x_1	+	$2x_2$			+	s_2		=	8

Row 0 : $z - x_1 - 3\,x_2 = 0$

- if x_1 is increased by 1 unit, z increases by 1 unit
- if x_2 is increased by 1 unit, z increases by 3 units

So we choose x_2 as the **"entering variable".** If x_2 is to increase from its current value of zero, it has to become a **basic variable**.

The Simplex Algorithm

- For a max. problem, the entering variable has a negative coefficient in row 0. Usually we choose the variable with the most negative coefficient to be the entering variable (ties may be broken in an arbitrary fashion).

- x_2 will become basic. Therefore, one basic variable should become nonbasic. This will be the **"leaving variable"**.

The Simplex Algorithm

- Increasing x_2 may cause a basic variable to become negative. We look at how increasing x_2 (while holding x_1=0) changes the values of current set of basic variables:

$$\underbrace{x_1}_{=0} + x_2 + s_1 = 6 \qquad\Rightarrow\qquad x_2 + s_1 = 6$$

$$\underbrace{-x_1}_{=0} + 2x_2 + s_2 = 8 \qquad\Rightarrow\qquad 2x_2 + s_2 = 8$$

The Simplex Algorithm

As $s_1 \geq 0$, $s_1 = 6 - x_2 \geq 0$ $\Rightarrow$ $x_2 \leq 6$

As $s_2 \geq 0$, $s_2 = 8 - 2x_2 \geq 0$ $\Rightarrow$ $x_2 \leq 4$

So, x_2 can be at most 4 (otherwise s_2 would become negative!)

The Simplex Algorithm

Observe that for any row in which the entering variable has a positive coefficient, the row's basic variable becomes negative if the entering variable exceeds:

$$\frac{\text{Right Hand Side of row}}{\text{Coefficient of entering variable in row}}$$

The Simplex Algorithm

Ratio Test: When entering a variable into the basis, for every *row i* in which the entering variable has a *positive coefficient*, we compute the ratio:

$$\frac{\text{RHS of row i}}{\text{Coefficient of entering variable in row i}},$$

and determine the smallest one.

The smallest ratio is the largest value of the entering variable that will keep all the current basic variables nonnegative.

The Simplex Algorithm

Canonical Form 0:

Row	Basic Variable												RHS
0	$z=0$	z	$-$	x_1	$-$	$3x_2$						$=$	0
1	$s_1=6$			x_1	$+$	x_2	$+$	s_1				$=$	6
2	$s_2=8$		$-$	x_1	$+$	$2x_2$			$+$	s_2		$=$	8

If $x_2=4$, then $s_2=0$ and $s_1=2$.

Therefore s_2 is the **leaving variable** and **becomes nonbasic**.

New basic variables = $\{s_1,x_2\}$; and new nonbasic variables = $\{x_1,s_2\}$
Hence, new $z = x_1+3x_2=12$.

The Simplex Algorithm

Canonical Form 0:

entering variable

Row	Basic Variable		RHS
0	$z=0$	$z - x_1 - 3x_2 = $	0
1	$s_1=6$	$x_1 + x_2 + s_1 = $	6
2	$\boxed{s_2}=8$	$-x_1 + 2x_2 + s_2 = $	8

leaving variable

Always make the entering variable a basic variable in a row that wins the ratio test (ties may be broken arbitrarily).

The Simplex Algorithm

Canonical Form 0:

entering variable

Row	**Basic Variable**											**RHS**
0	z	z	-	x_1	-	$3x_2$					=	0
1	$s_1=6$			x_1	+	x_2	+	s_1			=	6
2	$\boxed{s_2}=8$		-	x_1	+	$2x_2$			+	s_2	=	8

leaving variable

Always make the entering variable a basic variable in a row that wins the ratio test (ties may be broken arbitrarily).

The Simplex Algorithm

Canonical Form 0:

Row	Basic Variable					**entering variable**						RHS
0	z	z	-	x_1	-	$3x_2$					=	0
1	s_1			x_1	+	x_2	+	s_1			=	6
2	$\boxed{s_2}=8$		-	x_1	+	$2x_2$			+	s_2	=	8

leaving variable

Always make the entering variable a basic variable in a row that wins the ratio test (ties may be broken arbitrarily).

The Simplex Algorithm

Row	Basic Variable											RHS
0	z	z	$-$	x_1	$-$	$3x_2$					$=$	0
1	s_1			x_1	$+$	x_2	$+$	s_1			$=$	6
2			$-$	x_1	$+$	$2x_2$			$+$	s_2	$=$	8

Always make the entering variable a basic variable in a row that wins the ratio test (ties may be broken arbitrarily).

The Simplex Algorithm

Row	Basic Variable											RHS
0	z	z	-	x_1	-	$3x_2$					=	0
1	s_1			x_1	+	x_2	+	s_1			=	6
2	$\boxed{x_2}$		-	x_1	+	$2x_2$			+	s_2	=	8

entering variable

Always make the entering variable a basic variable in a row that wins the ratio test (ties may be broken arbitrarily).

The Simplex Algorithm

Row	Basic Variable		RHS
0	z	$z - x_1 - 3x_2 =$	0
1	s_1	$x_1 + x_2 + s_1 =$	6
2	$\boxed{x_2}$	$- x_1 + 2x_2 + s_2 =$	8

entering variable

Always make the entering variable a basic variable in a row that wins the ratio test (ties may be broken arbitrarily).

To make x_2 a basic variable in row 2, we use elementary row operations to make x_2 has a coefficient of 1 in row 2 and coefficient of 0 in all other rows.

The Simplex Algorithm

Row	**Basic Variable**											**RHS**
0	z	z	-	x_1	-	$3x_2$					=	0
1	s_1			x_1	+	x_2	+	s_1			=	6
2	$\boxed{x_2}$		-	x_1	+	$2x_2$			+	s_2	=	8

entering variable

Always make the entering variable a basic variable in a row that wins the ratio test (ties may be broken arbitrarily).

To make x_1 a basic variable in row 2, we use elementary row operations to make x_1 has a coefficient of 1 in row 2 and coefficient of 0 in all other rows. This procedure is called **pivoting** on row 2.

The Simplex Algorithm

Row	Basic Variable		RHS
0	z	$z - x_1 - 3x_2 =$	0
1	s_1	$x_1 + x_2 + s_1 =$	6
2	$\boxed{x_2}$	$- x_1 + 2x_2 + s_2 =$	8

entering variable

Pivoting: Purpose is to rewrite the original problem in an equivalent form where columns corresponding to basic variables form an identity matrix. This allows us to determine the values of entering and leaving variables in the new solution.

The Simplex Algorithm

Row	Basic Variable											RHS
0	z	z	-	x_1	-	$3x_2$					=	0
1	s_1			x_1	+	x_2	+	s_1			=	6
2	x_2		-	x_1	+	$2x_2$			+	s_2	=	8

We may perform elementary row operations step by step, starting from the pivot row, one row at a time.

The Simplex Algorithm

Row	**Basic Variable**											**RHS**
0	z	z	-	x_1	-	$3x_2$					=	0
1	s_1			x_1	+	x_2	+	s_1			=	6
2	x_2		-	x_1	+	$2x_2$			+	s_2	=	8

To make x_2 has a coefficient of 1 in row 2:

- multiply row 2 by 0.5

The Simplex Algorithm

Row	Basic Variable											RHS
0	z	z	-	x_1	-	$3x_2$					=	0
1	s_1			x_1	+	x_2	+	s_1			=	6
2	x_2		-	$0.5\ x_1$	+	x_2			+	$0.5\ s_2$	=	4

To make x_2 has a coefficient of 1 in row 2:

- multiply row 2 by 0.5

The Simplex Algorithm

Row	Basic Variable											RHS
0	z	z	-	x_1	-	$3x_2$					=	0
1	s_1			x_1	+	x_2	+	s_1			=	6
2	x_2		-	$0.5\ x_1$	+	x_2			+	$0.5\ s_2$	=	4

To make x_2 has a coefficient of 0 in row 1:

- replace row 1 with row 1 – row 2

The Simplex Algorithm

Row	**Basic Variable**											**RHS**
0	z	z	-	x_1	-	$3x_2$					=	0
1	s_1			$1.5\ x_1$			+	s_1	-	$0.5\ s_2$	=	2
2	x_2		-	$0.5\ x_1$	+	x_2			+	$0.5\ s_2$	=	4

To make x_2 has a coefficient of 0 in row 1:

- replace row 1 with row 1 – row 2

The Simplex Algorithm

Row	Basic Variable											RHS
0	z	z	-	x_1	-	$3x_2$					=	0
1	s_1			$1.5\ x_1$			+	s_1	-	$0.5\ s_2$	=	2
2	x_2		-	$0.5\ x_1$	+	x_2			+	$0.5\ s_2$	=	4

To make x_2 has a coefficient of 0 in row 0:

- replace row 0 with row 0 + 3 (row 2)

The Simplex Algorithm

Row	Basic Variable												RHS
0	z	z	-	$2.5\ x_1$					+	$1.5\ s_2$	=		12
1	s_1			$1.5\ x_1$			+	s_1	-	$0.5\ s_2$	=		2
2	x_2		-	$0.5\ x_1$	+	x_2			+	$0.5\ s_2$	=		4

To make x_2 has a coefficient of 0 in row 0:

- replace row 0 with row 0 + 3 (row 2)

The Simplex Algorithm

Canonical Form 1:

Row	**Basic Variable**										**RHS**
0	z	z	$-$	$2.5\,x_1$					$+$	$1.5\,s_2$	$=$	12
1	s_1			$1.5\,x_1$			$+$	s_1	$-$	$0.5\,s_2$	$=$	2
2	x_2		$-$	$0.5\,x_1$	$+$	x_2			$+$	$0.5\,s_2$	$=$	4

Is this bfs optimal?

No, increasing the nonbasic variable x_1 will increase z!

So *x_1 is the entering variable*. Also note that x_1 is the variable with the most negative coefficient in row 0.

The Simplex Algorithm

We perform the ratio test to find the leaving variable:

Since $s_2 = 0$, the system is:

$1.5\,x_1 + s_1 = 2$, and $s_1 \geq 0$ $\Rightarrow$ $s_1 = 2 - 1.5\,x_1 \geq 0$

$\Rightarrow$ $x_1 \leq 4/3$

$-0.5\,x_1 + x_2 = 4$, and $x_2 \geq 0$ $\Rightarrow$ $x_2 = 4 + 0.5\,x_1 \geq 0$

$\Rightarrow$ $x_1 \geq -8$

So, $x_1 = 4/3$ and s_1 becomes the leaving variable

The Simplex Algorithm

The new bfs is:

$$x_1 = 4/3,$$
$$x_2 = 14/3,$$
$$s_1 = 0,$$
$$s_2 = 0,$$
$$z = 46/3.$$

Now, keep the above results in mind and let us have a look at the pivot of the simplex algorithm:

The Simplex Algorithm

Canonical Form 1:

entering variable

Row	**Basic Variable**		**RHS**
0	z	$z \; - \; 2.5\,x_1 \; + \; 1.5\,s_2 \; =$	12
1	$\boxed{s_1}$	$1.5\,x_1 \; + \; s_1 \; - \; 0.5\,s_2 \; =$	2
2	x_2	$- \; 0.5\,x_1 \; + \; x_2 \; + \; 0.5\,s_2 \; =$	4

leaving variable

The Simplex Algorithm

entering variable

Row	Basic Variable											**RHS**
0	z	z	$-$	$2.5\,x_1$					$+$	$1.5\,s_2$	$=$	12
1	$\boxed{}$			$1.5\,x_1$			$+$	s_1	$-$	$0.5\,s_2$	$=$	2
2	x_2		$-$	$0.5\,x_1$	$+$	x_2			$+$	$0.5\,s_2$	$=$	4

leaving variable

The Simplex Algorithm

Canonical Form 2:

Row	**Basic Variable**											**RHS**
0	z	z	-	$2.5\ x_1$					+	$1.5\ s_2$	=	12
1	x_1			$1.5\ x_1$			+	s_1	-	$0.5\ s_2$	=	2
2	x_2		-	$0.5\ x_1$	+	x_2			+	$0.5\ s_2$	=	4

To make x_1 a basic variable in row 1, we use elementary row operations to make x_1 has a coefficient of 1 in row 1 and coefficient of 0 in all other rows.

The Simplex Algorithm

Canonical Form 2:

Row	Basic Variable												RHS
0	z	z	$-$	$2.5\ x_1$					$+$	$1.5\ s_2$	$=$		12
1	x_1			$1.5\ x_1$			$+$	s_1	$-$	$0.5\ s_2$	$=$		2
2	x_2		$-$	$0.5\ x_1$	$+$	x_2			$+$	$0.5\ s_2$	$=$		4

To make x_1 has a coefficient of 1 in row 1:

- multiply row 1 by 2/3

The Simplex Algorithm

Canonical Form 2:

Row	Basic Variable		RHS
0	z	$z - 2.5x_1 + 1.5s_2 =$	12
1	x_1	$x_1 + 2/3\,s_1 - 1/3\,s_2 =$	4/3
2	x_2	$-0.5x_1 + x_2 + 0.5s_2 =$	4

To make x_1 has a coefficient of 1 in row 1:

- multiply row 1 by 2/3

The Simplex Algorithm

Canonical Form 2:

Row	Basic Variable											RHS
0	z	z	-	$2.5\ x_1$					+	$1.5\ s_2$	=	12
1	x_1			x_1			+	$2/3\ s_1$	-	$1/3\ s_2$	=	4/3
2	x_2		-	$0.5\ x_1$	+	x_2			+	$0.5\ s_2$	=	4

The Simplex Algorithm

Canonical Form 2:

Row	**Basic Variable**											**RHS**
0	z	z	-	$2.5\ x_1$					+	$1.5\ s_2$	=	12
1	x_1			x_1			+	$2/3\ s_1$	-	$1/3\ s_2$	=	4/3
2	x_2		-	$0.5\ x_1$	+	x_2			+	$0.5\ s_2$	=	4

To make x_1 has a coefficient of 0 in row 2:

- multiply row 1 by 0.5 and add to the row 2

The Simplex Algorithm

Canonical Form 2:

Row	Basic Variable										RHS
0	z	z	-	$2.5\,x_1$				+	$1.5\,s_2$	=	12
1	x_1			x_1		+	$2/3\,s_1$	-	$1/3\,s_2$	=	4/3
2	x_2				x_2	+	$1/3\,s_1$	+	$1/3\,s_2$	=	14/3

To make x_1 has a coefficient of 0 in row 2:

- multiply row 1 by 0.5 and add to the row 2

The Simplex Algorithm

Canonical Form 2:

Row	Basic Variable										RHS
0	z	z	$-$	$2.5\ x_1$				$+$	$1.5\ s_2$	$=$	12
1	x_1			x_1		$+$	$2/3\ s_1$	$-$	$1/3\ s_2$	$=$	$4/3$
2	x_2				x_2	$+$	$1/3\ s_1$	$+$	$1/3\ s_2$	$=$	$14/3$

The Simplex Algorithm

Canonical Form 2:

Row	**Basic Variable**											**RHS**
0	z	z	$-$	$2.5\ x_1$				$+$	$1.5\ s_2$	$=$	12
1	x_1			x_1		$+$	$2/3\ s_1$	$-$	$1/3\ s_2$	$=$	$4/3$
2	x_2				x_2	$+$	$1/3\ s_1$	$+$	$1/3\ s_2$	$=$	$14/3$

To make x_1 has a coefficient of 0 in row 0:

- multiply row 1 by 2.5 and add to the row 0

The Simplex Algorithm

Canonical Form 2:

Row	Basic Variable									RHS
0	z	z			$+$	$5/3\ s_1$	$+$	$2/3\ s_2$	$=$	$46/3$
1	x_1		x_1		$+$	$2/3\ s_1$	$-$	$1/3\ s_2$	$=$	$4/3$
2	x_2			x_2	$+$	$1/3\ s_1$	$+$	$1/3\ s_2$	$=$	$14/3$

To make x_1 has a coefficient of 0 in row 0:

- multiply row 1 by 2.5 and add to the row 0

The Simplex Algorithm

Canonical Form 2:

Row	**Basic Variable**									**RHS**
0	z	z			+	$5/3\ s_1$	+	$2/3\ s_2$	=	$46/3$
1	x_1		x_1		+	$2/3\ s_1$	-	$1/3\ s_2$	=	$4/3$
2	x_2			x_2	+	$1/3\ s_1$	+	$1/3\ s_2$	=	$14/3$

This result is the same as we had calculated before!

The Simplex Algorithm

Previously calculated new bfs was:

$$x_1 = 4/3,$$
$$x_2 = 14/3,$$
$$s_1 = 0,$$
$$s_2 = 0,$$
$$z = 46/3.$$

The Simplex Algorithm

Canonical Form 2:

Row	Basic Variable									RHS
0	z	z			+	$5/3\ s_1$	+	$2/3\ s_2$	=	$46/3$
1	x_1		x_1		+	$2/3\ s_1$	-	$1/3\ s_2$	=	$4/3$
2	x_2			x_2	+	$1/3\ s_1$	+	$1/3\ s_2$	=	$14/3$

Is this bfs optimal?

YES! Because increasing nonbasic variables s_1 and s_2 will decrease z (Also note that there is no variable in row 0 with a negative coefficient!)

Representing Simplex Tableaus

- Instead of writing each variable in every constraint, we can use a shorthand display called a ***simplex tableau***.

Representing Simplex Tableaus

Canonical Form 0:

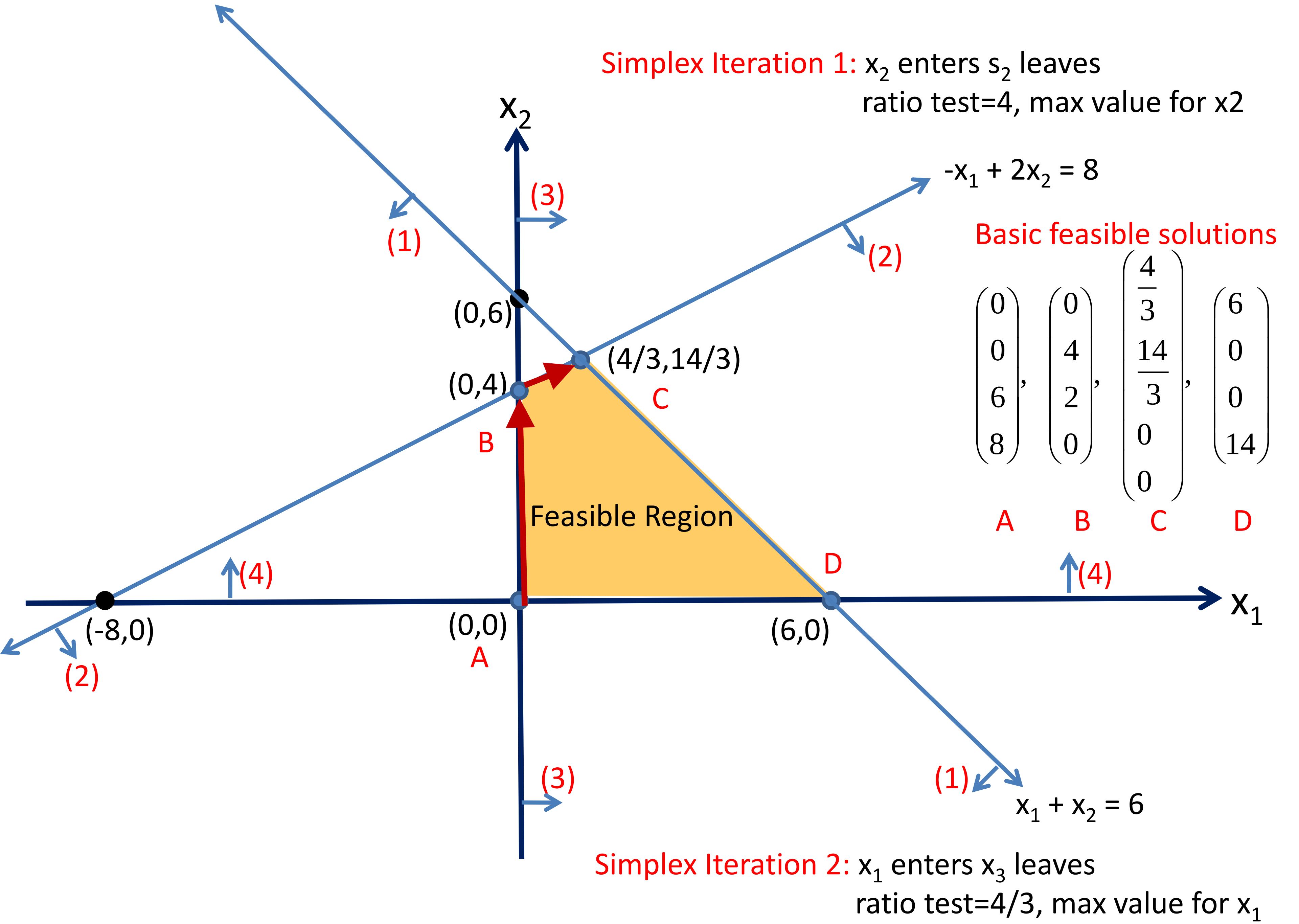
Row	**Basic Variable**										**RHS**
0	$z=0$	z	-	x_1	-	$3x_2$					=	0
1	$s_1=6$			x_1	+	x_2	+	s_1			=	6
2	$s_2=8$		-	x_1	+	$2x_2$			+	s_2	=	8

For example, canonical form 0 could be written as

Row	**Basic Variable**	**z**	x_1	x_2	s_1	s_2	**RHS**
0	$z=0$	1	-1	-3	0	0	0
1	$s_1=6$	0	1	1	1	0	6
2	$s_2=8$	0	-1	2	0	1	8

Representing Simplex Tableaus

- With this format, it is very easy to spot the basic variables. We just look for columns with a column of identity matrix underneath.



Simplex Iteration 1: x_2 enters s_2 leaves
ratio test=4, max value for x2
x_2
$-x_1 + 2x_2 = 8$
(3)
(1)
(2)
Basic feasible solutions
$\begin{pmatrix}0\\0\\6\\8\end{pmatrix}, \begin{pmatrix}0\\4\\2\\0\end{pmatrix}, \begin{pmatrix}\frac{4}{3}\\\frac{14}{3}\\0\\0\end{pmatrix}, \begin{pmatrix}6\\0\\0\\14\end{pmatrix}$
A B C D
(0,6)
(4/3,14/3)
(0,4)
C
B
Feasible Region
D
(4)
(4)
x_1
(-8,0)
(0,0)
A
(6,0)
(2)
(3)
(1)
$x_1 + x_2 = 6$
Simplex Iteration 2: x_1 enters x_3 leaves
ratio test=4/3, max value for x_1

Representing Simplex Tableaus

Simplex Tableau-0:

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
z=0	1	-1	-3	0	0	0
s_1=6	0	1	1	1	0	6
s_2=8	0	-1	2	0	1	8

Note that z will always be a basic variable. Therefore, we won't be mentioning it unless it is necessary.
In addition, since row 0 corresponds to the objective function, it is indicated separately in the simplex tableau.

Also note that, since we are in canonical form, the basic variable of a row will be equal to the RHS of that row.

Representing Simplex Tableaus

Simplex Tableau-0:

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
$z=0$	1	-1	-3	0	0	0
$s_1=6$	0	1	1	1	0	6
$s_2=8$	0	-1	2	0	1	8

Now let us summarize what we have done so far!

Representing Simplex Tableaus

Simplex Tableau-0:

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
z	1	-1	-3	0	0	0
	0	1	1	1	0	6
	0	-1	2	0	1	8

For a max problem:

We start with an initial bfs in the canonical form above (if there are *m* slack variables, we use them as basic variables)

Representing Simplex Tableaus

Simplex Tableau-0:

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
z	1	-1	-3	0	0	0
s_1	0	1	1	1	0	6
s_2	0	-1	2	0	1	8

For a max problem:

We start with an initial bfs in the canonical form above (if there are m slack variables, we use them as basic variables)

Representing Simplex Tableaus

Simplex Tableau-0:

entering variable

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
z	1	-1	-3	0	0	0
s_1	0	1	1	1	0	6
s_2	0	-1	2	0	1	8

Entering Variable: Choose a variable with the most negative coefficient in row 0.

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Representing Simplex Tableaus

Simplex Tableau-0:

entering variable ↓ (x_2)

Basic Variable	z	x_1	x_2	s_1	s_2	RHS
z	1	-1	-3	0	0	0
s_1	0	1	1	1	0	6
s_2	0	-1	2	0	1	8

Leaving Variable: compute ratios:

$$\frac{\text{RHS of row } i}{\text{Coefficient of entering variable in row } i},$$

The smallest positive ratio wins and the basic variable of the winning row leaves the basis.

Representing Simplex Tableaus

Simplex Tableau-0:

entering variable: x_2

Basic Variable	z	x_1	x_2	s_1	s_2	RHS	Ratio Test
z	1	-1	-3	0	0	0	
s_1	0	1	1	1	0	6	6
$\boxed{s_2}$	0	-1	2	0	1	8	$\boxed{4}$

leaving variable: s_2

pivot term: 2

Leaving Variable: compute ratios:

$$\frac{\text{RHS of row } i}{\text{Coefficient of entering variable in row } i},$$

The smallest positive ratio wins and the basic variable of the winning row leaves the basis.

Representing Simplex Tableaus

Simplex Tableau-0:

entering variable

Basic Variable	z	x_1	x_2	s_1	s_2	RHS	Ratio Test
z	1	-1	-3	0	0	0	
s_1	0	1	1	1	0	6	6
$\boxed{s_2}$	0	-1	2	0	1	8	$\boxed{4}$

leaving variable

pivot term

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s_1	0	1	1	1	0	6	6
$\boxed{x_2}$	0	-1	2	0	1	8	4

pivot term

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$\boxed{x_2}$	0	-1	2	0	1	8

pivot term

Pivot: Transform the tableau so that the new basic variable (entering variable) has 1 in the row of the leaving variable (pivot row) and 0 in other rows.

Representing Simplex Tableaus

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Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
z	1	-1	-3	0	0	0
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s_1	0	1	1	1	0	6
$\boxed{x_2}$	0	-1/2	1	0	1/2	4

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Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
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$\boxed{x_2}$	0	-1/2	1	0	1/2	4

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Pivot: Transform the tableau so that the new basic variable (entering variable) has 1 in the row of the leaving variable (pivot row) and 0 in other rows.

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Simplex Tableau-0:

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
z						
s_1	0	3/2	0	1	-1/2	2
$\boxed{x_2}$	0	-1/2	1	0	1/2	4

pivot term

Pivot: Transform the tableau so that the new basic variable (entering variable) has 1 in the row of the leaving variable (pivot row) and 0 in other rows.

Representing Simplex Tableaus

Simplex Tableau-0:

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
z	1	-5/2	0	0	3/2	12
s_1	0	3/2	0	1	-1/2	2
$\boxed{x_2}$	0	-1/2	1	0	1/2	4

pivot term

Pivot: Transform the tableau so that the new basic variable (entering variable) has 1 in the row of the leaving variable (pivot row) and 0 in other rows.

Representing Simplex Tableaus

Simplex Tableau-1:

enter ↓ (x_1)

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
z	1	-5/2	0	0	3/2	12
s_1	0	3/2	0	1	-1/2	2
x_2	0	-1/2	1	0	1/2	4

Representing Simplex Tableaus

Simplex Tableau-1:

enter

Basic Variable	z	x_1	x_2	s_1	s_2	RHS	Ratio Test
z	1	-5/2	0	0	3/2	12	
leave $\boxed{s_1}$	0	3/2	0	1	-1/2	2	$\boxed{4/3}$
x_2	0	-1/2	1	0	1/2	4	no ratio

Representing Simplex Tableaus

Simplex Tableau-1:

Basic Variable	**z**	enter x_1	x_2	s_1	s_2	**RHS**	**Ratio Test**
z	1	-5/2	0	0	3/2	12	
leave	0	3/2	0	1	-1/2	2	$\boxed{4/3}$
x_2	0	-1/2	1	0	1/2	4	no ratio

Representing Simplex Tableaus

Simplex Tableau-1:

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**	**Ratio Test**
z	1	-5/2	0	0	3/2	12	
x_1	0	3/2	0	1	-1/2	2	4/3
x_2	0	-1/2	1	0	1/2	4	no ratio

Representing Simplex Tableaus

Simplex Tableau-1:

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
z	1	-5/2	0	0	3/2	12
x_1	0	3/2	0	1	-1/2	2
x_2	0	-1/2	1	0	1/2	4

Representing Simplex Tableaus

Simplex Tableau-2:

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
z	1	0	0	5/3	2/3	46/3
x_1	0	1	0	2/3	-1/3	4/3
x_2	0	0	1	1/3	1/3	14/3

Representing Simplex Tableaus

Simplex Tableau-2:

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
z	1	0	0	5/3	2/3	46/3
x_1	0	1	0	2/3	-1/3	4/3
x_2	0	0	1	1/3	1/3	14/3

Can we iterate more?

No, because all row 0 coefficients are nonnegative. We stop here. The current bfs is optimal!

Summary of the Simplex Algorithm

(For a max problem)

1) Convert the LP into the standard form and then obtain the canonical form

2) Find an initial bfs (if there are *m* slack variables, use them as basic variables).

   If all nonbasic variables have nonnegative coefficients in row 0, then the current LP is optimal.

   If there are any variables with a negative coefficient, then we should decide the entering variable.

   Entering variable: choose a variable with the most negative coefficient in row 0 to enter the basis.

Summary of the Simplex Algorithm

3) For any row in which the entering variable has a *positive coefficient*, compute the ratios:

$$\frac{\text{RHS of row i}}{\text{Coefficient of the entering variable in row i}}$$

The smallest ratio wins (ties may be broken arbitrarily) and the basic variable of the winning row leaves the basis.

4) **Pivot:** Transform the tableau so that the new basic variable (entering variable) has coefficient of 1 in the row of the leaving variable (pivot row) and coefficient of 0 in all other rows. In the end, we get a tableau with a new canonical form.

Summary of the Simplex Algorithm

After the pivoting, note that:

$$\text{New pivot row} = \frac{1}{\begin{pmatrix} \text{coefficient of the entering} \\ \text{variable in the pivot row} \end{pmatrix}} \cdot (\text{old pivot row})$$

$$\text{new row i} = \text{old row i} - \begin{pmatrix} \text{coefficient of the entering} \\ \text{variable in the pivot row} \end{pmatrix} \cdot (\text{new pivot row})$$

Summary of the Simplex Algorithm

5) **Repeat steps 1,2,3, and 4** until all row 0 coefficients becomes nonnegative. If each nonbasic variable has a nonnegative coefficient in a canonical form's row 0 (remember that basic variables have coefficient of 0 in row 0 of a canonical form), then the canonical form is optimal. ***We stop here and the current bfs is optimal!***