Q.1)

Decision Variables :

X_{ij} = amount of land allocated by kibbutz j for crop i

i : 1 (Sugar beets), 2 (Cotton), 3 (Sorghum)

j : 1, 2, 3

Model

$$\text{Max } 1000(X_{11}+X_{12}+X_{13})+750(X_{21}+X_{22}+X_{23})+250(X_{31}+X_{32}+X_{33})$$

S.t

$$\left.\begin{array}{l} X_{11}+X_{12}+X_{13}\leq 600 \\ X_{21}+X_{22}+X_{23}\leq 500 \\ X_{31}+X_{32}+X_{33}\leq 325 \end{array}\right\}\text{ for max quota}$$

$$\left.\begin{array}{l} 3X_{11}+2X_{21}+X_{31}\leq 600 \\ 3X_{12}+2X_{22}+X_{32}\leq 800 \\ 3X_{13}+2X_{23}+X_{33}\leq 375 \end{array}\right\}\text{ for water Const.}$$

$$\left.\begin{array}{l} X_{11}+X_{21}+X_{31}\leq 400 \\ X_{12}+X_{22}+X_{32}\leq 600 \\ X_{13}+X_{23}+X_{33}\leq 300 \end{array}\right\}\text{ for usable land}$$

$$\left(\frac{X_{11}+X_{21}+X_{31}}{400}=\frac{X_{12}+X_{22}+X_{32}}{600}=\frac{X_{13}+X_{23}+X_{33}}{300}\right)$$

linearize the equation

$X_{ij}\geq 0$ $\quad i=1,2,3$, $j=1,2,3$

$$(X_{12}+X_{22}+X_{32})-2(X_{13}+X_{23}+X_{33})=0$$
$$3(X_{11}+X_{21}+X_{31})-2(X_{12}+X_{22}+X_{32})=0$$
$$3(X_{11}+X_{21}+X_{31})-4(X_{13}+X_{23}+X_{33})=0$$

Q.2)

$$P1 : \begin{cases} 2 \text{ labor h} \\ 1 \text{ lb raw} \end{cases} \longrightarrow \begin{matrix} 2 \text{ oz } A \\ 1 \text{ oz } B \end{matrix}$$

$$P2 : \begin{cases} 3 \text{ labor h} \\ 2 \text{ lb raw} \end{cases} \longrightarrow \begin{matrix} 3 \text{ oz } A \\ 2 \text{ oz } B \end{matrix}$$

Dec. Var

X_1 : amount of unit P1 is used

X_2 : amount of unit P2 is used

a : amount of oz A is produced

b_1 : amount of oz B that is sold

b_2 : amount of oz B that is disposed

Model

$$\max \ 16a + 14b_1 - 2b_2$$

s.t

$$a = 2X_1 + 3X_2$$

$$b_1 + b_2 = X_1 + 2X_2$$

$$2X_1 + 3X_2 \leq 60 \quad \text{(labor hour)}$$

$$X_1 + 2X_2 \leq 40 \quad \text{(raw material)}$$

$$b_1 \leq 20. \quad \text{(at most 20 oz of B can be so}$$

$$X_1, X_2, a, b_1, b_2 \geq 0$$

! you can also model without decision variables a and b_2, by putting $a = 2X_1 + 3X_2$ & $b_2 = X_1 + 2X_2 - b_1$.

Q.3) Dec. Var

C_i = Cash on hand at the end of month i, $i = 1, \dots, 4$
X_i = amount of tons purchased at the begining of month i, $i = 1, \dots, 4$
y_i = amount of tons sold at the end of month i, $i = 1, \dots, 4$
I_i = amount of tons on hand at month i, $i = 1, \dots, 4$ (after purchase, befor selling)

Parameters

P_i = price of buying at month i; $i = 1, \dots, 4$
S_i = price of selling at month i, $i = 1, \dots, 4$

Model

$$\max \quad C_4$$

s.t

$I_i = I_{i-1} + X_i - y_{i-1}$ $\quad i = 1, \dots, 4$ (Inventory balance)
$C_i = C_{i-1} - P_i X_i + S_i y_i$ $\quad i = 1, \dots, 4$ (Cash balance)
$I_i \leq 100$ $\quad i = 1, \dots, 4$ (Capacity Constraint)
$y_i \leq I_i$ $\quad i = 1, \dots, 4$ (Can't sell more Corn than you have)
$P_i X_i \leq C_{i-1}$ $\quad i = 1, \dots, 4$ (Can't spend more money than you have)
$I_0 = 50$
$C_0 = 1000$
$y_0 = 0$

$C_i, X_i, y_i, I_i \geq 0 \quad i = 1, \dots, 4$

Q.4) **Dec. Var.** : X_i : number of hours process i runs, $i=1,2,3$

a) max $\underbrace{9(2X_1+2X_3)}_{\text{rev of WB}} + \underbrace{10(X_1+3X_2-3X_3)}_{\text{rev of BC}} + \underbrace{24X_3}_{\text{rev of GST}} - \underbrace{(5X_1+4X_2+X_3)}_{\text{running Costs of processes}}$

$- \underbrace{2(2X_1+X_2)}_{\text{Cost of A}} - \underbrace{3(3X_1+3X_2+2X_3)}_{\text{Cost of B}}$

s.t

$2X_1+X_2 \leq 200$ (Sugar type A) (i)

$3X_1+3X_2+2X_3 \leq 300$ (Sugar type B) (ii)

$X_1+X_2+X_3 \leq 100$ (hour limitation) (iii)

$X_1+3X_2 \geq 3X_3$ (max BC that Can be used in process 3) (iv)

$X_1, X_2, X_3 \geq 0$ (V)

b) Instead of (i) and (ii), add the following one:

$(2X_1+X_2)+(3X_1+3X_2+2X_3) \leq 500 \longrightarrow 5X_1+4X_2+2X_3 \leq 500$

c) Add dec. var y : # of Wonka Box

$y=\min\left\{2X_1+2X_3,\ \frac{X_1+3X_2-3X_3}{2},\ X_3\right\} \rightarrow$! NOT linear, you MUST linearize in your model

Model max $54y-(5X_1+4X_2+X_3)-2(2X_1+X_2)-3(3X_1+3X_2+2X_3)$

s.t (i) – (v)

$y \leq 2X_1+2X_3$

$y \leq \frac{X_1+3X_2-3X_3}{2}$

$y \leq X_3$

$y \geq 0$

Q.5) <u>Parameters</u> : D_{ij} = demand of product i at period j $\quad i=1,2 \quad j=1,\dots,12$

<u>Dec. Var</u> : X_{ij} = amount of product i produced at period j $\quad i=1,2 \quad j=1,\dots,12$

I_{ij} = amount of product i stored at the end of period j $\quad i=1,2 \quad j=1,\dots,12$

<u>Model</u>

$$\min \underbrace{\sum_{j=1}^{5}(5X_{1j}+8X_{2j})}_{\text{Cost until June}} + \underbrace{\sum_{j=6}^{12}(4.5X_{1j}+7X_{2j})}_{\text{Cost after June}} + \sum_{j=1}^{12}(0.2I_{1j}+0.4I_{2j})$$

s.t

$$X_{1j}+X_{2j} \leq 120{,}000 \quad j=1,\dots,9$$
$$X_{1j}+X_{2j} \leq 150{,}000 \quad j=10,11,12$$
} Prod. Cap.

$$2I_{1j}+4I_{2j} \leq 150{,}000 \quad j=1,\dots,12$$ } Storage Cap.

$$X_{ij}+I_{i(j-1)} = D_{ij}+I_{ij} \quad i=1,2 \quad j=1,\dots,12$$

$$I_{i0}=0 \quad i=1,2$$

$$X_{ij},\ I_{ij} \geq 0 \quad i=1,2,\ j=1,\dots,12$$

Q.6) <u>Dec. Var</u> X_{ij} : amount of money invested in Option i at the beginning of year j $\quad i = \overset{A}{1}, \overset{B}{2}, \overset{C}{3}$

$j = 1, 2, 3$

S_j : uninvested Cash in the year j. $j = 1, \dots, 4$

<u>Model</u> Max $1.06S_4 + 1.3X_{12} + 1.5X_{32}$

s.t

$$100 = X_{11} + X_{21} + X_{31} + S_1 \quad \text{(year 1)}$$

$$0.1X_{11} + 0.2X_{21} + 1.06S_1 = X_{12} + X_{22} + X_{32} + S_2 \quad \text{(year 2)}$$

$$1.1X_{21} + 0.1X_{12} + 0.2X_{22} + 1.06S_2 = X_{23} + S_3 \quad \text{(year 3)}$$

$$1.3X_{11} + 1.5X_{31} + 1.1X_{22} + 0.2X_{23} + 1.06S_3 = S_4$$

$X_{ij} \geq 0 \quad \forall i \in \{1,2,3\}, \forall j \in \{1,2,3\}$, $\quad S_j \geq 0 \quad \forall j \in \{1,2,3,4\}$

$X_{ij} \leq 50 \quad \forall i \in \{1,2,3\}, \forall j \in \{1,2,3\}$

Q.7) a) $X = \begin{pmatrix} D \\ 0 \\ E \\ F \\ 0 \end{pmatrix}$. is current soln. and $Z = R$ is current obj. value

b) $X_1 + AX_2 + X_5 = D$ $\quad$ X_5 is nonbasic $\Rightarrow X_5 = 0$

$X_3 + BX_2 + 3X_5 = E$ $\quad$ X_3 becomes nonbasic : $X_3 = 0$

$X_4 + CX_2 + 0X_5 = F$ $\quad$ X_2 becomes basic

New solution

$X_2 = \frac{E}{B}$ $\quad X_1 = D - A\frac{E}{B}$ $\quad X_4 = F - C\frac{E}{B}$ $\Rightarrow X = \begin{pmatrix} D - A\frac{E}{B} \\ \frac{E}{B} \\ 0 \\ F - C\frac{E}{B} \\ 0 \end{pmatrix}$

$Z = R - \frac{PE}{B} \xrightarrow{\;!\;}$ Note that if $P \leq 0$ (maximization) objective value increases.

(If $P = 0$ obj. value remains same
$\therefore P$ cannot be positive)

C) Solution does not change since Min Ratio Test $= 0 (= \frac{F}{C})$

New Solution : $X_2 = \frac{F}{C}$, $X_1 = D - A\frac{F}{C}$ $\quad X_3 = E - B\frac{F}{C}$

$\Rightarrow X = \begin{pmatrix} D - A\frac{F}{C} \\ \frac{F}{C} \\ E - \frac{BF}{C} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} D \\ 0 \\ E \\ 0 \\ 0 \end{pmatrix}$ $\Rightarrow$ The solution is the same with the soln in part a.

Q.8) a) TRUE : Let $x, y \in S$ and z be a convex combination of x & y,

$z = \lambda x + (1-\lambda) y$ for some $\lambda \in [0,1]$. Since $x, y \in S$, we can say that $f(x) \leqslant 10$ and $f(y) \leqslant 10$. We also know that,

$f(z) = f(\lambda x + (1-\lambda) y) \leqslant \lambda f(x) - (1-\lambda) f(y)$ since f is a convex functio

Then $f(z) \leqslant \lambda \underbrace{f(x)}_{\leqslant 10} + (1-\lambda) \underbrace{f(y)}_{\leqslant 10} \leqslant 10$. So $f(z) \leqslant 10$ holds.

Thus, S is a convex set.

b) TRUE : Let $x, y \in S$. For the given statement to hold, we need to show that any convex combination of x & Y is also in set S.

Let $z = \lambda x + (1-\lambda) y$ for some $\lambda \in [0,1]$. We know that $Ax \leqslant b$ and $Ay \leqslant b$ hold since $x, y \in S$.

Let's multiply the first inequality by λ and the second by $(1-\lambda)$

$$\left.\begin{aligned} \lambda A x &\leqslant \lambda b \\ (1-\lambda) A y &\leqslant (1-\lambda) b \end{aligned}\right\} \xrightarrow[\text{sum!}]{\text{take their}} \lambda A x + (1-\lambda) A y \leqslant b\lambda + (1-\lambda) b \quad (*)$$

Rearrenging the terms of $(*)$ gives, $A(\underbrace{\lambda x + (1-\lambda) y}_{z}) \leqslant b$ OR $Az \leqslant b$

Hence, $z \in S$ and S is a convex set.

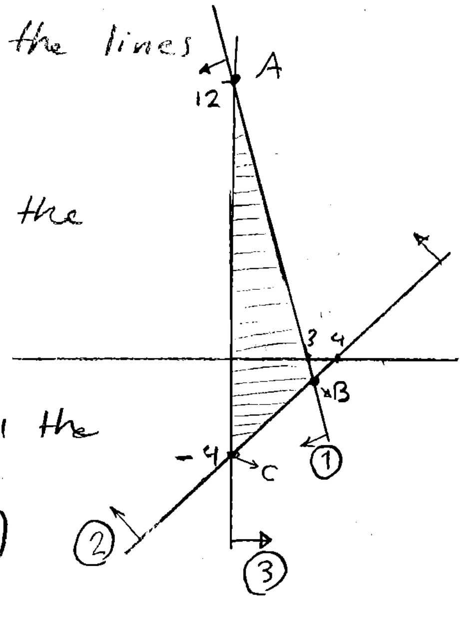
c) FALSE : Consider the following counter example : $S = \{(x_1, x_2) : x_2 \leqslant 2, -x_2 \leqslant$

Two nonintersecting lines as given above does not have an extreme p

d) FALSE : Consider the following counter example, $5 \in S$, $-5 \in S$ however, their midpoint 0 is not in S.

e) TRUE : We can show that S_1 and S_2 are convex sets as we show in part (b). Let $S = S_1 \cap S_2$ and $x, y \in S$. Since S_1 is convex and $x, y \in S_1$ we can say that $z = \lambda x + (1-\lambda) y \in S_1$. Similarly since S_2 is convex and $x, y \in S_2$, we can say $z \in S_2$. Thus $z \in S$ and S is a convex set.

Q. 9) a) To have alternative opt. Solutions $\vec{C} = (a,b)$ vector Should be prependicular to one of the lines

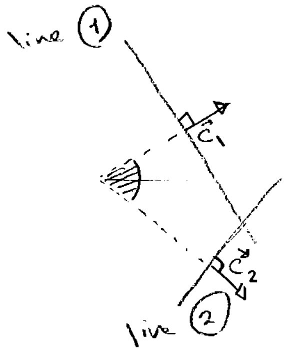


(i) $\vec{C_1} = (a,b)$ s.t $a = 4b$ $(a>0, b>0)$, all the points on the line 1 (between points A & B) are optimal.

(ii) $\vec{C_2} = (a,b)$ s.t $a = -b$ $(a>0, b<0)$, all the points on the line 2 (between points B & C) are optimal.

(iii) $\vec{C_3} = (a,b)$ s.t $a \leq 0$, $b = 0$, all the points on the line 3, (between A & C) are optimal

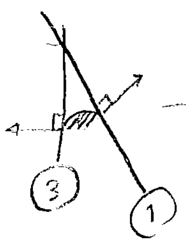
b) $\vec{C} = (1,0)$ makes B unique opt.



any vector between $\vec{C_1}$ and $\vec{C_2}$ makes B unique opt.

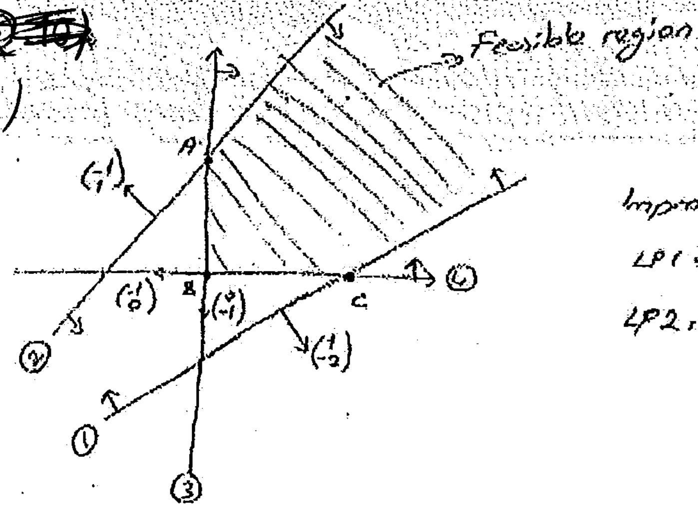
Note that the region Should be strict

c) $\vec{C} = (0,1)$ makes A unique opt. Soln.



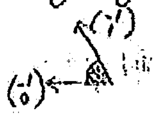
Any vector between $\vec{C_1}$ and $\vec{C_3}$ makes A unique opt. soln.

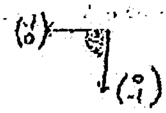
Q.10)

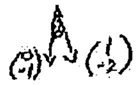


Improvement direction
LP1 : $-\underline{c}$ (minimization)
LP2 : $\underline{c}$ (maximization)

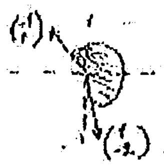
a) For LP1, for A to be a unique optimal $-\underline{c}$ should be in the following region



Similarly for B, $-c$ should be in 

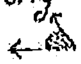
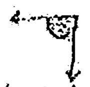
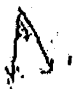
and for C, $-c$ should be in 

So for LP1 to be unbounded none of the ext. points should be optimal. And $-c$ should lie in the following region:



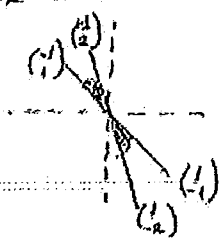
Remaining part of the regions are excluded.

Note that the regions should be strict, o.w it results with alternative opt. solutions.

Similarly for LP2, for A to be a unique opt, $\underline{c}$ should lie in  . for B in  and for C in .

So, for LP2 to be unbounded $\underline{c}$ should lie in the same region given above.

For both LP1 and LP2 to be unbounded, $\underline{c}$ and $-\underline{c}$ should lie in the shaded region given above. The following region is the solution where both $\underline{c}$ & $-\underline{c}$ are in the shaded region above.



[i.e., consider $c = \binom{0}{1}$. c is within the shaded region above, but $-c = \binom{0}{-1}$ is not. Thus $c = \binom{0}{1}$ does not make both LPs unbounded.]

Q.10) Cont'd

b) To have alternative opt. soln's, $-c$ should be perpendicular to one of the 4 lines. $\underline{-c}$ can be $(1,-2)$, $(-1,1)$, $(-1,0)$ or $(0,-1)$. Since all the directions for c makes LP2 unbounded, $\underline{c \text{ vector}}$ should be one of them: $(-1,2)$, $(1,-1)$ $(1,0)$, $(0,1)$

(i.e consider vector $(1,0)$. Then for LP1, all the points between A and B will be opt. solns while LP2 will be unbounded).

c) LP1 can have a unique optimal soln for different c vectors. For $c=(1,1) \to A$ is unique opt. soln for LP 1 and LP2 will be unbounded.

$c=(1,-\frac{2}{3}) \to B$ is unique opt soln for LP 1 and LP2 will be unbounded.

$c=(-1,3) \to C$ is unique opt soln for LP1 and LP2 will be unbounded.

(There are many! solns, you can find other examples)

Q.11) $x_2 = -x_2'$ and $x_3 = x_3' - x_3''$

Standard form:

$$\min\ z = 3x_1 + (-x_2')$$
$$\text{s.t}\quad (-x_2') - 3x_1 - e_1 = 3$$
$$x_1 + (-x_2') + s_2 = 4$$
$$2x_1 - (-x_2') + 3(x_3' - x_3'') = 3$$
$$x_1, x_2', x_3', x_3'', e_1, s_2 \geq 0$$

Q.12) $n=4$, $m=2$ $\quad \binom{n}{m} = \binom{4}{2} = 6$ (max # of basic soln exists)

☐ x_1, x_2 nonbasic
x_3, x_4 basic
$(x_1 = x_2 = 0)$

$5x_3 + 3x_4 = 15$
$x_4 = 10$
$\rightarrow x = \begin{pmatrix} 0 \\ 0 \\ -3 \\ 10 \end{pmatrix}$ Since $x_3 = -3 <$ it is not feasible (It is basic soln)

☐ x_1, x_3 nonbasic
x_2, x_4 basic
$(x_1 = x_3 = 0)$

$-x_2 + 3x_4 = 15$
$3x_2 + x_4 = 10$
$\rightarrow x = \begin{pmatrix} 0 \\ \frac{3}{2} \\ 0 \\ \frac{11}{2} \end{pmatrix} \rightarrow$ It is bfs

☐ x_1, x_4 nonbasic
x_2, x_3 basic
$(x_1 = x_4 = 0)$

$-x_2 + 5x_3 = 15$
$3x_2 = 10$
$\rightarrow x = \begin{pmatrix} 0 \\ \frac{10}{3} \\ \frac{11}{3} \\ 0 \end{pmatrix} \rightarrow$ It is bfs

☐ x_2, x_3 nonbasic
x_1, x_4 basic
$(x_2 = x_3 = 0)$

$3x_1 + 3x_4 = 15$
$x_1 + x_4 = 10$
$\rightarrow$ There is no basic soln

☐ x_2, x_4 nonbasic
x_1, x_3 basic
$(x_2 = x_4 = 0)$

$3x_1 + 5x_3 = 15$
$x_1 = 10$
$\rightarrow x = \begin{pmatrix} 10 \\ 0 \\ -3 \\ 0 \end{pmatrix}$ Since $x_3 = -3$ It is not feasible (It is basic soln)

☐ x_3, x_4 nonbasic
x_1, x_2 basic
$(x_3 = x_4 = 0)$

$3x_1 - x_2 = 15$
$x_1 + 3x_2 = 10$
$\rightarrow x = \begin{pmatrix} \frac{11}{2} \\ \frac{3}{2} \\ 0 \\ 0 \end{pmatrix} \rightarrow$ It is bfs

Q.13) Change Last constraint to $x_1 + x_2 \leq 4$ (in order to see degeneracy) and obj to max

ⓐ Standard form : $\min\ z = 5x_1 + 3x_2$

s.t

$$4x_1 + 2x_2 + s_1 = 12$$
$$4x_1 + x_2 + s_2 = 10$$
$$x_1 + x_2 + s_3 = 4$$
$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

Bfs: $x = \begin{pmatrix} 0 \\ 0 \\ 12 \\ 10 \\ 4 \end{pmatrix}$ (for $x_1 = 0$ and $x_2 = 0$)

ⓑ

	z	x_1 ↓	x_2	s_1	s_2	s_3	RHS
	1	−5	−3	0	0	0	0
s_1	0	4	2	1	0	0	12
← s_2	0	4	1	0	1	0	10
s_3	0	1	1	0	0	1	4

Apply MRT: $\left\{ \frac{12}{4}, \underline{\underline{\frac{10}{4}}}, \frac{4}{1} \right\}$

s_2 leaves

			↓				
	1	0	$-7/4$	0	$5/4$	0	12.5
← s_1	0	0	1	1	−1	0	2
x_1	0	1	$1/4$	0	$1/4$	0	2.5
s_3	0	0	$3/4$	0	$-1/4$	1	1.5

MRT: $\left\{ \underline{\frac{2}{1}}, \frac{2.5}{(1/4)}, \underline{\frac{1.5}{(3/4)}} \right\}$

Same choose arbitrarily one of them.

s_1 leaves.

	1	0	0	1	0	1	16
x_2	0	0	1	1	−1	0	2
x_1	0	1	0	$-1/4$	$1/2$	0	2
s_3	0	0	0	$-3/4$	$1/2$	1	0

There is no negative value in row 0.

So the soln. is opt.

$z^* = 16$ (opt soln. value)

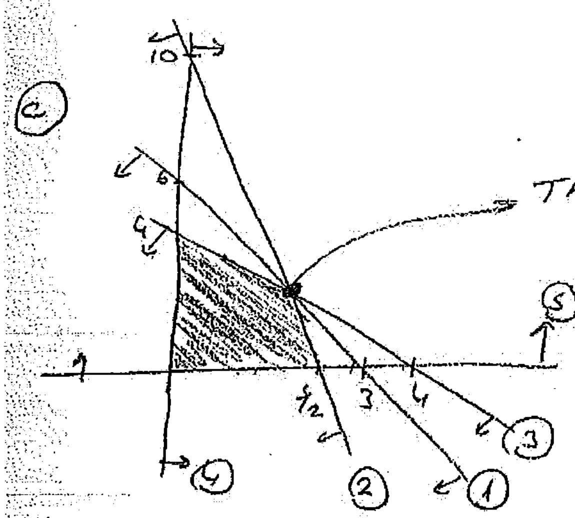
$x^* = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Q.13) Cont'd.

Even if s_3 is basic variable, at the optimal sol'n, it is zero. So the solution $x^* = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ is a degenerate bfs.

(Also the tie of the MRT at the second step implies that at the next step there will be a degenerate bfs)

(c)



Three lines meet at the same point which corresponds to $\begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

The path of the simplex iterations