

IE 400 2019 Spring Midterm Study Set Solutions

1. (30 points) A cargo plane has three compartments for storing cargo: front, center and rear. These compartments have the following limits on both weight and space:

Compartment	Weight capacity (tons) w_j	Space capacity (cubic meters) s_j
Front	10	6800
Center	16	8700
Rear	8	5300

The following four cargoes are available for shipment on the next flight:

Cargo	Weight (tons) w_i	Volume (cubic meters/ton) v_i	Profit (TL/ton) p_i
C1	18	480	310
C2	15	650	380
C3	23	580	350
C4	12	390	285

We assume

- that each cargo can be split into whatever proportions/fractions we desire for shipment
- that each cargo can be split between 2 or more compartments if we so desire

The aim is to determine how much (if any) of each cargo C1, C2, C3 and C4 should be accepted and how to distribute each among the compartments so that the total profit for the flight is maximized.

- a) (8 pts) Let us define the utilization rate of a cargo compartment as the ratio of the total cargo weight carried in that compartment to the weight capacity of that compartment (for example, if a total of 8 tons of cargo is carried in the front compartment, the utilization rate will be $8/10=0.8$). Formulate this problem as a linear programming problem under the constraint that the utilization rate of each compartment must be the same to maintain the balance of the plane.

Decision Variables:

x_{ij} = amount of cargo i distributed to compartment j in tons

Model: $i=1,2,3,4$ $j=1,2,3$
(C1 C2 C3 C4) (F C R)

$$\max \sum_{i=1}^4 \sum_{j=1}^3 p_i x_{ij}$$

$$\text{st} \quad (a) \quad \sum_{j=1}^3 x_{ij} \leq w_i \quad \forall i=1,2,3,4 \quad (\text{available cargoes})$$

$$(b) \quad \sum_{i=1}^4 v_i x_{ij} \leq s_j \quad \forall j=1,2,3 \quad (\text{space capacity for each compartment})$$

$$(c) \quad \sum_{i=1}^4 x_{ij} \leq w_j \quad \forall j=1,2,3 \quad (\text{weight capacity for each compartment})$$

$$(d) \quad \frac{\sum_{i=1}^4 x_{i1}}{w_1} = \frac{\sum_{i=1}^4 x_{i2}}{w_2} = \frac{\sum_{i=1}^4 x_{i3}}{w_3} \quad (\text{utilization rate of each compartment is equal})$$

$$x_{ij} \geq 0 \quad \forall i=1,2,3,4 \quad \forall j=1,2,3$$

- b) (8 pts) Formulate this problem as a linear programming problem under the constraint that the differences between the utilization rates of any two compartments should not be more than 0.1 to maintain the balance of the plane.

Decision Variables:

Model: Replace (d) with

$$\left\{ \begin{array}{l} \left| \frac{\sum_{i=1}^4 x_{i1}}{w_1} - \frac{\sum_{i=1}^4 x_{i2}}{w_2} \right| \leq 0.1 \\ \left| \frac{\sum_{i=1}^4 x_{i1}}{w_1} - \frac{\sum_{i=1}^4 x_{i3}}{w_3} \right| \leq 0.1 \\ \left| \frac{\sum_{i=1}^4 x_{i2}}{w_2} - \frac{\sum_{i=1}^4 x_{i3}}{w_3} \right| \leq 0.1 \end{array} \right. \quad \text{linearize} \Rightarrow$$

$$\begin{array}{l} \frac{\sum x_{i1}}{w_1} - \frac{\sum x_{i2}}{w_2} \leq 0.1 \\ \frac{\sum x_{i2}}{w_2} - \frac{\sum x_{i1}}{w_1} \leq 0.1 \\ \frac{\sum x_{i1}}{w_1} - \frac{\sum x_{i3}}{w_3} \leq 0.1 \\ \frac{\sum x_{i3}}{w_3} - \frac{\sum x_{i1}}{w_1} \leq 0.1 \\ \text{OR} \\ -0.1 \leq \frac{\sum x_{i2}}{w_2} - \frac{\sum x_{i3}}{w_3} \leq 0.1 \end{array}$$

- c) (4 pts) Consider the two LPs in parts a) and b). Which one is more likely to have a larger optimal value? Please justify your answer.

(b) is more likely to have a larger optimal value since feasible region in (b) includes feasible region in (a).

- d) (10 pts) If cargo can be delivered only in integer amounts (in tons) and if it is not possible to split the cargo between compartments, how would your model in part b) change? Your new model should still be a linear one.

Decision Variables:

x_{ij} = amount shipped in tons from cargo i in compartment j

$y_{ij} = \begin{cases} 1 & \text{if cargo } i \text{ is shipped in compartment } j \\ 0 & \text{o.w.} \end{cases}$

Model:

All constraints in (b) +

$$x_{ij} \leq w_j y_{ij} \quad \forall i=1, \dots, 4 \quad j=1, 2, 3$$

$$\sum_{j=1}^3 y_{ij} \leq 1 \quad \forall i=1, 2, 3, 4$$

$$y_{ij} \in \{0, 1\}$$

$$x_{ij} \text{ integer} \quad \forall i, j$$

Q. 2)

Parameters : d_j, L_k, c_k are given

Dec. var : $x_k = \begin{cases} 1 & \text{if truck } k \text{ is used} \\ 0 & \text{o.w} \end{cases} \quad k=1..4$

$y_{jk} = \begin{cases} 1 & \text{if cust } j \text{ is visited by truck } k \\ 0 & \text{o.w} \end{cases} \quad \begin{matrix} j=1..10 \\ k=1..4 \end{matrix}$

z_{jk} = amount of good is carried by truck k
for cust j , $j=1..10, k=1..4$

Model

$$\min \sum_{k=1}^4 c_k x_k$$

s.t.

$$\sum_{k=1}^4 z_{jk} = d_j \quad \forall j=1..10 \quad (\text{demand satisfaction})$$

$$\sum_{j=1}^{10} z_{jk} \leq L_k x_k \quad \forall k=1..4 \quad (\text{truck capacity})$$

$$z_{jk} \leq d_j y_{jk} \quad \begin{matrix} \forall j=1..10 \\ k=1..4 \end{matrix} \quad (\text{relation})$$

$$y_{jk} \leq x_k \quad \begin{matrix} \forall j=1..10 \\ k=1..4 \end{matrix} \quad (\text{constraints})$$

$$x_k \in \{0,1\} \quad \forall k=1..4$$

$$y_{jk} \in \{0,1\} \quad z_{jk} \geq 0 \quad \forall j=1..10, k=1..4$$

(a) $\sum_{k=1}^{10} y_{jk} \leq 5 \quad \forall k=1..4$ (c) $y_{3k} \leq y_{4k} + y_{5k} \quad \forall k=1..4$

(b) $y_{1k} + y_{2k} \leq 1 \quad \forall k=1..4$ (d) $y_{2k} = y_{6k} \quad \forall k=1..4$

(e) either $y_{14} + y_{24} + y_{34} \geq 1$ (I)
or $y_{54} + y_{64} + y_{74} \leq 2$ (II)

Not linear
You must linearize
in your model.

* Add a binary var $w = \begin{cases} 1 & \text{if (I) holds} \\ 0 & \text{if (II) holds} \end{cases}$

$$y_{14} + y_{24} + y_{34} \geq w$$

$$y_{54} + y_{64} + y_{74} \leq 2(1-w) + 3w$$

$$w \in \{0,1\}$$

Q. 3)

5) Dec. Var

$$y_i = \begin{cases} 1 & \text{if } i^{\text{th}} \text{ constraint is satisfied} \\ 0 & \text{o.w.} \end{cases}$$

$$z_1 = \begin{cases} 1 & \text{if at least one of 7, 8, 9 is satisfied} \\ 0 & \text{o.w.} \end{cases}$$

$$z_2 = \begin{cases} 1 & \text{if at least two of 10, 11, 12, 13 is satisfied} \\ 0 & \text{o.w.} \end{cases}$$

Model $\min 0$
s.t

$$\sum_{j=1}^n a_{ij} x_j \leq b_i + M(1 - y_i) \quad \forall i = 1 \dots 20$$

$$\sum_{j=1}^n a_{ij} x_j \geq b_i - M y_i \quad \forall i = 1 \dots 20$$

$$\sum_{i=1}^{20} y_i \geq 10$$

$$y_1 \leq y_4$$

$$y_2 + y_3 \leq 1$$

$$y_5 = y_6$$

$$y_7 + y_8 + y_9 \geq z_1$$

$$y_{10} + y_{11} + y_{12} + y_{13} \geq 2z_2$$

$$z_1 + z_2 \geq 1$$

$$z_1, z_2, y_i \in \{0, 1\} \quad \forall i = 1 \dots 20$$

(since we only searching for feasibility of the problem. i.e. if there exists a feasible solution then the objective will be zero)

if $y_i = 1$

$$\sum a_{ij} x_j \leq b_i \quad (\text{const. is satisfied})$$

$$\sum a_{ij} x_j \geq b_i - M \rightarrow (\text{adds } -M \text{ to r.h.s. to restrict: redundant})$$

if $y_i = 0$

$$\sum a_{ij} x_j \leq b_i + M \rightarrow \text{redundant}$$

$$\sum a_{ij} x_j \geq b_i \rightarrow (\text{const. not satisfied})$$

Q. 4)

Solution: x_i : # workers employed on line i

$$y_i: \begin{cases} 1 & \text{if line } i \text{ is used} \\ 0 & \text{o.w} \end{cases}$$

LP is:

$$\min z = 1000y_1 + 2000y_2 + 500x_1 + 900x_2$$

s.t.

$$20x_1 + 50x_2 \geq 120$$

$$30x_1 + 35x_2 \geq 150$$

$$40x_1 + 45x_2 \geq 200$$

$$x_1 \leq 7y_1$$

(each line can have 7 workers at a time)

$$x_2 \leq 7y_2$$

$$x_1, x_2 \geq 0; y_1, y_2 \in \{0, 1\}$$

Q. 5)

Soln:Let x_i : Amount of product produced on machine i . where $i=1, 2, 3, 4$.

$$y_i: \begin{cases} 1 & \text{if machine } i \text{ is used for production.} \\ 0 & \text{o.w} \end{cases}$$

$$\min 1000y_1 + 920y_2 + 800y_3 + 700y_4 + 20x_1 + 24x_2 + 16x_3 + 28x_4$$

s.t.

$$x_1 + x_2 + x_3 + x_4 \geq 2000 \quad \text{(demand const.)}$$

$$x_1 \leq 9000y_1$$

$$x_2 \leq 1000y_2$$

$$x_3 \leq 1200y_3$$

$$x_4 \leq 1600y_4$$

$$\left\{ \begin{array}{l} \text{Capacity} \\ \text{constraints} \end{array} \right\}$$

$$x_1, x_2, x_3, x_4 \geq 0, y_1, y_2, y_3, y_4 \in \{0, 1\}$$

 x_1, x_2, x_3, x_4 are integers.

Q. 6)

Let C be the set of countries, N_i be the set of neighbors of country i . Say there are m countries and m colors.

Define $x_{ij} = \begin{cases} 1 & \text{if country } i \text{ is colored by } j \\ 0 & \text{o/w} \end{cases}$

$y_j = \begin{cases} 1 & \text{if color } j \text{ is used} \\ 0 & \text{o/w} \end{cases}$

$$\min \sum_{j=1}^m y_j$$

s.t.

$$\sum_{j=1}^m x_{ij} = 1 \quad \forall i \in C$$

$$x_{ij} + x_{kj} \leq y_j \quad \forall i \in C, k \in N_i \quad j=1, \dots, m$$

$$x_{ij} \in \{0,1\} \quad \forall i \in C, j=1, \dots, m$$

$$y_j \in \{0,1\} \quad j=1, \dots, m$$

Q. 7)

Let's first convert it to the standard form:

$$\min \quad 3x_1 + 2x_2 + 4x_3$$

s.t.

$$2x_1 + x_2 + 3x_3 = 60$$

$$3x_1 + 3x_2 + 5x_3 - e_1 = 120$$

$$x_1, x_2, x_3, e_1 \geq 0$$

Adding the artificial variables for each row, we will end up with following LP:

$$\min \quad 3x_1 + 2x_2 + 4x_3$$

s.t.

$$2x_1 + x_2 + 3x_3 + a_1 = 60$$

$$3x_1 + 3x_2 + 5x_3 - e_1 + a_2 = 120$$

$$x_1, x_2, x_3, e_1, a_1, a_2 \geq 0$$

Two-Phase Method:

The phase-I problem is the following:

$$\begin{array}{ll}\min & a_1 + a_2 \\ \text{s.t.} & \\ & 2x_1 + x_2 + 3x_3 + a_1 = 60 \\ & 3x_1 + 3x_2 + 5x_3 - e_1 + a_2 = 120 \\ & x_1, x_2, x_3, e_1, a_1, a_2 \geq 0\end{array}$$

Now we need to solve phase-I LP:

Tableau 0:

z	x ₁	x ₂	x ₃	e ₁	a ₁	a ₂	RHS
1	0	0	0	0	-1	-1	0
0	2	1	3	0	1	0	60
0	3	3	5	-1	0	1	120

The simplex tableau above is not in proper format since the row zero coefficients of basic variables a₁ and a₂ are not zero. We add the first and second row to row zero to make these coefficients zero:

Tableau 0:

z	x ₁	x ₂	x ₃	e ₁	a ₁	a ₂	RHS
1	5	4	8	-1	0	0	180
0	2	1	3	0	1	0	60
0	3	3	5	-1	0	1	120

→ 60/3
120/5

Now we can start simplex iterations: since it is a minimization problem we are looking for positive coefficients to decide on entering variables.

x₃ enters and a₁ leaves:

Tableau 1:

z	x ₁	x ₂	x ₃	e ₁	a ₁	a ₂	RHS
1	-1/3	4/3	0	-1	-8/3	0	20
0	2/3	1/3	1	0	1/3	0	20
0	-1/3	4/3	0	-1	-5/3	1	20

→ 20/(1/3)
20/(4/3)

x₂ enters and a₂ leaves:

Tableau 2:

z	x ₁	x ₂	x ₃	e ₁	a ₁	a ₂	RHS
1	0	0	0	0	-1	-1	0
0	3/4	0	1	1/4	3/4	-1/4	15
0	-1/4	1	0	-3/4	-5/4	3/4	15

Both a₁ and a₂ left the basis so we can delete their columns from the tableau and move onto phase-II. We write the original objective function coefficients to the tableau:

Tableau 0:

z	x ₁	x ₂	x ₃	e ₁	RHS
1	-3	-2	-4	0	0
0	3/4	0	1	1/4	15
0	-1/4	1	0	-3/4	15

Again, notice that this tableau is not in proper format, we need to make row zero coefficients of basic variables x₂ and x₃ zero:

Tableau 0:

z	x ₁	x ₂	x ₃	e ₁	RHS
1	-1/2	0	0	-1/2	90
0	3/4	0	1	1/4	15
0	-1/4	1	0	-3/4	15

Stop. The tableau is optimal since there is no positive entry in row zero.

The optimal solution is x₁^{*}=0, x₂^{*}=x₃^{*}=15 with objective value 90.

Now let's solve the same problem with Big-M method.

Big-M method:

The LP is the following:

$$\begin{aligned}
 \min \quad & 3x_1 + 2x_2 + 4x_3 + Ma_1 + Ma_2 \\
 \text{s.t.} \quad & 2x_1 + x_2 + 3x_3 + a_1 = 60 \\
 & 3x_1 + 3x_2 + 5x_3 - e_1 + a_2 = 120 \\
 & x_1, x_2, x_3, e_1, a_1, a_2 \geq 0
 \end{aligned}$$

Tableau 0:

z	x ₁	x ₂	x ₃	e ₁	a ₁	a ₂	RHS
1	-3	-2	-4	0	-M	-M	0
0	2	1	3	0	1	0	60
0	3	3	5	-1	0	1	120

Let's put the tableau in the proper format first:

Tableau 0:

z	x ₁	x ₂	x ₃	e ₁	a ₁	a ₂	RHS
1	5M-3	4M-2	8M-4	-M	0	0	180M
0	2	1	3	0	1	0	60
0	3	3	5	-1	0	1	120

→ 60/3
120/5

x₃ enters and a₁ leaves:

Tableau 1:

z	x ₁	x ₂	x ₃	e ₁	a ₁	a ₂	RHS
1	-(M+1)/3	(4M-2)/3	0	-M	(4-8M)/3	0	80 + 20M
0	2/3	1/3	1	0	1/3	0	20
0	-1/3	4/3	0	-1	-5/3	1	20

→ 20/(1/3)
20/(4/3)

x₂ enters and a₂ leaves:

Tableau 2:

z	x ₁	x ₂	x ₃	e ₁	a ₁	a ₂	RHS
1	-1/2	0	0	-1/2	1/2-M	1/2-M	90
0	3/4	0	1	1/4	3/4	-1/4	15
0	-1/4	1	0	-3/4	-5/4	3/4	15

Stop. The tableau is optimal since there is no positive entry in row zero.

The optimal solution is $x_1^*=0$, $x_2^*=x_3^*=15$ with objective value 90.

Q. 8)

First Tableau:

x_1	x_2	e_1	e_2	x_3	x_4	RHS
-2	-3	0	0	-M	-M	0
2	1	-1	0	1	0	4
1	-1	0	-1	0	1	1

x_3 and x_4 are basic variables but their row zero coefficients are not zero so we need to add the first and the second row to row zero after multiplying by M:

x_1	x_2	e_1	e_2	x_3	x_4	RHS
$3M-2$	-3	-M	-M	0	0	$5M$
2	1	-1	0	1	0	4
1	-1	0	-1	0	1	1

Then, x_1 enters x_4 leaves.

Second Tableau:

x_1	x_2	s_1	e_1	x_3	x_4	RHS
0	0	0	0	-1	-1	0
1	1	1	0	0	0	3
2	1	0	-1	1	0	4
1	1	0	0	0	1	3

x_3 and x_4 are basic variables but their row zero coefficients are not zero so we need to add the second and the third row to row zero:

x_1	x_2	s_1	e_1	x_3	x_4	RHS
3	2	0	-1	0	0	7
1	1	1	0	0	0	3
2	1	0	-1	1	0	4
1	1	0	0	0	1	3

Then, x_1 enters x_3 leaves.

Third Tableau:

x_1	x_2	x_3	s_1	s_2	e_1	RHS
-2	-1	-1	0	0	0	0
$1/3$	$2/3$	0	1	0	$1/3$	1
2	1	0	0	1	0	3
$2/3$	$1/3$	1	0	0	$-1/3$	1

x_3 is a basic variable but its row zero coefficient is not zero so we need to add the third row to row zero:

x_1	x_2	x_3	s_1	s_2	e_1	RHS
-4/3	-2/3	0	0	0	-1/3	1
1/3	2/3	0	1	0	1/3	1
2	1	0	0	1	0	3
2/3	1/3	1	0	0	-1/3	1

Then, x_1 enters and s_2 leaves (or x_3 leaves).

Q. 9)

a) (5 pts) The tableau is final and there exists a unique optimal solution.

$$A > 0 \quad B, C, D \text{ any} \\ E > 0$$

b) (5 pts) The simplex method determines an unbounded solution from this tableau.

$$A < 0 \quad \text{Ratio test does not return a finite value.} \\ \Rightarrow B, C \leq 0 \quad E > 0 \quad D \text{ any}$$

c) (5 pts) The current bfs is degenerate (not necessarily optimal).

$$E = 0 \quad A, B, C, D \text{ anything}$$

d) (5 pts) The current solution is optimal, there are alternative optimal solutions but no alternative optimal bfs.

$$A = 0 \quad \text{Ratio test is not finite} \\ E > 0 \quad B, C \leq 0 \quad D \text{ any}$$

e) (10 pts) Find a specific set of values for unknowns A-E satisfying the general conditions in part d). Give the current optimal bfs and an alternative optimal solution that is not a bfs.

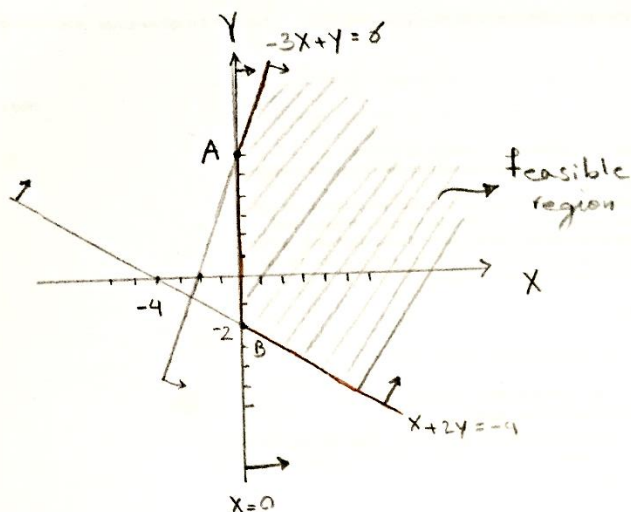
$$A = 0 \quad B = 0 \quad C = 0 \quad D = 0 \quad E = 0$$

x_1 enters no leaving variable

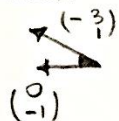
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \\ 1 \\ 0 \end{pmatrix} + x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{is an alternative optimal solution for any } x_1 > 0$$

Q. 10)

a)



b) Let assume we want that point $A = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$ be unique optimal solution, then $-C = (-a, -b)$ should lie in the following region



, e.g. $-C = (-4, 1) \Rightarrow a = 4$
 $b = -1$

the optimal solution is $\begin{pmatrix} 0 \\ 6 \end{pmatrix}$ and the optimal solution value is $4 \times 0 + (-1) \times 6 = -6$

c) $-C$ should lie in the following region

e.g. $-C = (1, 1) \Rightarrow a = -1$
 $b = -1$



d) There are two possible cases $\Rightarrow$

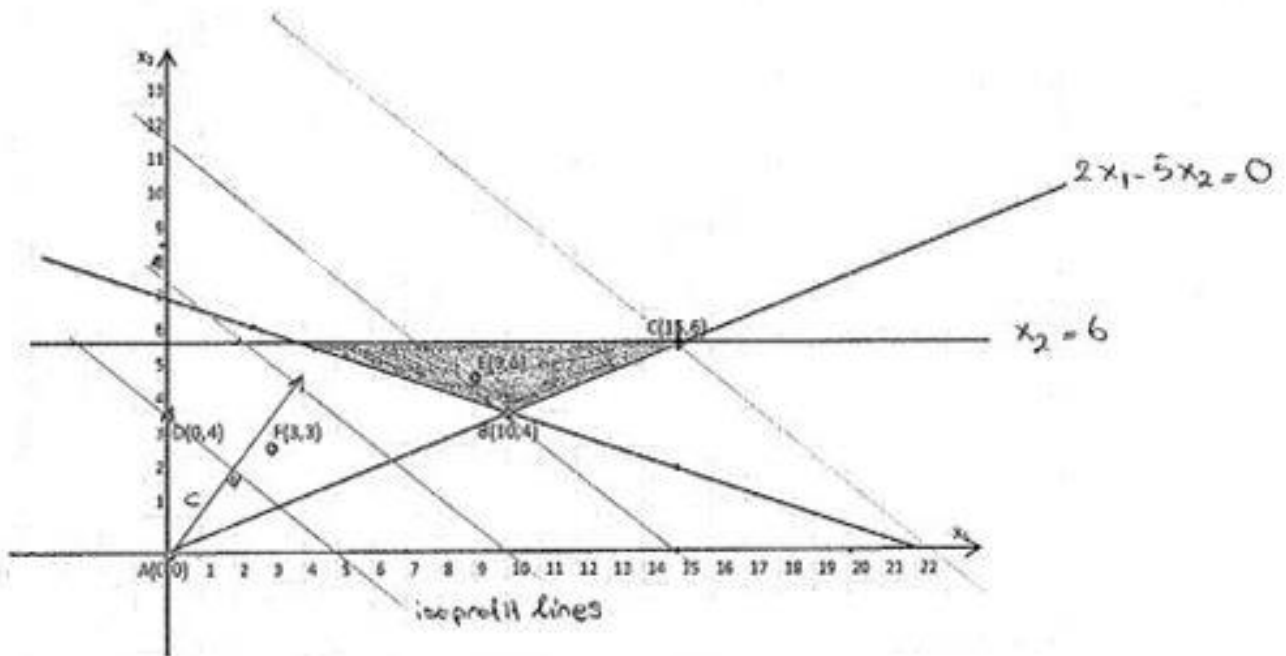
for Case 1, point A is the only optimal extreme point and for Case 2, point B is the only optimal extreme point

e) $\min ax + b(y'' - y')$
 $-x - 2(y'' - y') + s_1 = 4$
 $-3x + (y'' - y') + s_2 = 6$
 $x, y'', y', s_1, s_2 \geq 0$

f) $\begin{pmatrix} x \\ y' \\ y'' \\ s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 4 \\ 6 \end{pmatrix}$ b.f.s

x, y', y'' are nonbasic
 s_1, s_2 are basic

Q. 11)



The shaded area is the feasible region, the initial bfs can determined using this region.

Standard form:

$$Z = 4x_1 + 5x_2$$

$$x_1 + 3x_2 - e_1 + s_1 = 22$$

$$x_2 + s_2 = 6$$

$$2x_1 - 5x_2 + s_3 = 0$$

$$x_1, x_2, e_1, s_1, s_2, s_3 \geq 0$$

Simplex tableau:

Basic Var.	x1	↓ x2	e1	s1	s2	s3	RHS
z	-4	-5	0	0	0	0	0
s1	1	3	-1	1	0	0	22
← s2	0	↓ 1	0	0	1	0	6
s3	2	-5	0	0	0	1	0
z	-4	0	0	0	5	0	30
← s1	1	0	-1	1	-3	0	4
x2	0	1	0	0	↓ 1	0	6
s3	2	0	0	0	5	↓ 1	30
z	0	0	-4	4	-7	0	46
x1	1	0	-1	1	-3	0	4
x2	0	1	0	0	1	0	6
← s3	0	0	↓ 2	-2	11	1	22
z	0	0	-30/11	30/11	0	7/11	60
x1	1	0	-5/11	5/11	0	3/11	10
x2	0	1	-2/11	2/11	0	-1/11	4
← s2	0	0	2/11	-2/11	1	1/11	2
z	0	0	0	0	15	2	90
x1	1	0	0	0	5/2	1/2	15
x2	0	1	0	0	1	0	6
e1	0	0	1	-1	11/2	1/2	11

Tableau stops with optimality.

$$x^* = (x_1=15, x_2=6) \quad z^*=90$$