

IE 400 2019-2020 Fall Study Set 2 Solutions

Q1)

Parameters f_i : fixed cost of plant i , $i=1,2,3,4$
 c_{ij} : cost of producing car j at plant i , $i=1,2,3,4$
 $j=1,2,3$

Dec. Var $y_i = \begin{cases} 1 & \text{if plant } i \text{ is used} \\ 0 & \text{o.w.} \end{cases} \quad i=1,2,3,4$
 $x_{ij} = \begin{cases} 1 & \text{if car } j \text{ is produced at plant } i \\ 0 & \text{o.w.} \end{cases} \quad i=1,2,3,4$
 $j=1,2,3$

Model:

$$\min \sum_{i=1}^4 f_i y_i + \sum_{i=1}^4 \sum_{j=1}^3 c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^3 x_{ij} \leq 1 \quad \forall i=1,2,3,4$$

$$\sum_{i=1}^4 x_{ij} = 1 \quad \forall j=1,2,3$$

$$y_3 + y_4 \leq 1 + y_1$$

$$x_{ij} \leq y_i \quad \forall i=1,2,3,4, j=1,2,3$$

$$x_{ij} \in \{0,1\} \quad \forall i=1,2,3,4, j=1,2,3$$

$$y_j \in \{0,1\} \quad \forall j=1,2,3$$

Q2)

④ Parameters

cb_i : cost of producing a ton of steam in boiler i , $i=1,2,3$
 ct_j : " " processing " " " " turbine j , $j=1,2,3$
 $minb_i$: min amount of steam that can be produced in boiler i ,
 $maxb_i$: max " " " " " " " " $i=1,2,3$
 $mint_j$: min " " " " " " " " processed in turbine j
 $maxt_j$: max " " " " " " " " $j=1,2,3$

Dec. Var

$x_i = \begin{cases} 1 & \text{if boiler } i \text{ is used} \\ 0 & \text{o.w} \end{cases} \quad i=1,2,3$

y_i : amount of steam that is produced in boiler i , $i=1,2,3$

$t_j = \begin{cases} 1 & \text{if turbine } j \text{ is used} \\ 0 & \text{o.w} \end{cases} \quad j=1,2,3$

z_j : amount of steam that is processed in turbine j , $j=1,2,3$

Model

$$\begin{aligned} \min \quad & \sum_{i=1}^3 cb_i y_i + \sum_{j=1}^3 ct_j z_j \\ \text{s.t.} \quad & minb_i x_i \leq y_i \leq maxb_i x_i \quad \forall i=1,2,3 \quad (\text{production limitations of boilers}) \\ & mint_j t_j \leq z_j \leq maxt_j t_j \quad \forall j=1,2,3 \quad (\text{process limitations of turbines}) \\ & 4z_1 + 5z_2 + 6z_3 = 800 \quad (\text{required production}) \\ & \sum_{i=1}^3 y_i \geq \sum_{j=1}^3 z_j \quad (\text{produced steam should be larger than processed}) \\ & x_i \in \{0,1\} \quad y_i \geq 0 \quad \forall i=1,2,3 \\ & t_j \in \{0,1\} \quad z_j \geq 0 \quad \forall j=1,2,3 \end{aligned}$$

Q3)

If we assume the extra translator can handle both tourist groups simultaneously:

x_i : the number of Italian translators who start to work at period i , $i=1,2,3,4,5,6$

y_i : the number of German translators who start to work at period i , $i=1,2,3,4,5,6$

z_i : the number of extra translators who work at period i , $i=1,2,3,4,5,6$

$$\min 320 X_i + 280 y_i + 240 Z_i$$

st.

$$x_6 + x_1 + z_1 \geq 5$$

$$x_1 + x_2 + z_2 \geq 3$$

$$x_2 + x_3 + z_3 \geq 8$$

$$x_3 + x_4 + z_4 \geq 8$$

$$x_4 + x_5 + z_5 \geq 11$$

$$x_5 + x_6 + z_6 \geq 4$$

$$y_6 + y_1 + z_1 \geq 4$$

$$y_1 + y_2 + z_2 \geq 4$$

$$y_2 + y_3 + z_3 \geq 7$$

$$y_3 + y_4 + z_4 \geq 7$$

$$y_4 + y_5 + z_5 \geq 13$$

$$y_5 + y_6 + z_6 \geq 4$$

$$x_i, y_i, z_i \geq 0 \text{ for } i=1,2,3,4,5,6 \text{ and integer}$$

If we assume the extra translator is needed for each group separately:

x_{ij} : the number of regular translators who start to work at period i and translates language j , $i=1,2,3,4,5,6$ $j=1,2$ (1 Italian, 2 German)

y_{ij} : the number of extra translators who work at period i and translates language j , $i=1,2,3,4,5,6$ $j=1,2$ (1 Italian, 2 German)

$$\min 320 X_{i1} + 280 X_{i2} + 240 y_{ij}$$

st.

$$x_{61} + x_{11} + y_{11} \geq 5$$

$$x_{11} + x_{21} + y_{21} \geq 3$$

$$x_{21} + x_{31} + z_{31} \geq 8$$

$$x_{31} + x_{41} + y_{41} \geq 8$$

$$x_{41} + x_{51} + y_{51} \geq 11$$

$$x_{51} + x_{61} + y_{61} \geq 4$$

$$x_{62} + x_{12} + y_{12} \geq 4$$

$$x_{12} + x_{22} + y_{22} \geq 4$$

$$x_{22} + x_{32} + y_{32} \geq 7$$

$$x_{32} + x_{42} + y_{42} \geq 7$$

$$x_{42} + x_{52} + y_{52} \geq 13$$

$$x_{52} + x_{62} + y_{62} \geq 4$$

$$x_{ij}, y_{ij} \geq 0 \text{ for } i=1,2,3,4,5,6 \text{ } j=1,2 \text{ and integer}$$

Q4)

Decision Variables:

$P_{N,i}$: Units produced at month i under normal shift

$P_{E,i}$: Units produced at month i under extended shift

$X_i: \begin{cases} 1 & \text{if } P_{E,i} > 0 \\ 0 & \text{if otherwise} \end{cases} \quad (i=1, \dots, 6)$

I_i : inventory level at the end of month i

Parameters:

C_N : Cost of normal shift (£100,000 per month)

C_E : Cost of extended shift (£180,000 per month)

U_N : max capacity of normal shift (5,000 units per month)

U_E : " " " extended " (7,500 units per month)

C_c : Cost of changing from normal to extended shift (£15,000)

h : holding cost (£2 per unit per month)

I_0 : initial stock (3,000 units)

D_i : Demand for month i ($i=1, \dots, 6$)

Model:

$$\min \sum P_{N,i} C_N + P_{E,i} C_E X_i + C_c X_i + h I_i$$

$$\text{s.t. } P_{N,i} + P_{E,i} + I_{i-1} \geq D_i \quad \forall i=1 \dots 6$$

$$P_{E,i} \leq M X_i \quad \forall i=1 \dots 6$$

$$P_{E,i} + P_{N,i} \geq 2,000$$

$$P_{N,6} + P_{E,6} - D_6 + I_5 \geq 2,000$$

$$0 \leq P_{N,i} \leq U_N$$

$$0 \leq P_{E,i} \leq U_E$$

$$0 \leq X_i \leq 1$$

$$I_i \geq 0$$

$$I_i = P_{N,i} + P_{E,i} + I_{i-1} - D_i$$

$$P_{N,i}, P_{E,i}, X_i, I_i \text{ integer}$$

Q5)

Dec. Var

$$x_{ij} = \begin{cases} 1 & \text{if } i^{\text{th}} \text{ ast. is assigned to } j^{\text{th}} \text{ task} \\ 0 & \text{o.w} \end{cases} \quad i=1..4, j=1..4$$

$$\max \sum_{i=1}^4 \sum_{j=1}^4 p_{ij} \cdot x_{ij}$$

s.t

$$\sum_{i=1}^4 x_{ij} = 1 \quad \forall j=1..4 \quad (\text{each post should have a single ast.})$$

$$\sum_{j=1}^4 x_{ij} = 1 \quad \forall i=1..4 \quad (\text{each ast. should be assigned to a single post})$$

$$x_{ij} \in \{0, 1\} \quad \forall i=1..4, \forall j=1..4$$

Q6)

Dec. Var : given

Model : $\min \sum_{i=1}^M x_i$

$$\text{s.t} \quad \sum_{i=1}^M y_{ik} = d_k \quad \forall k=1..n$$

$$\sum_{k=1}^n y_{ik} \cdot p_k \leq p \cdot x_i \quad \forall i=1..M$$

$$x_i \in \{0, 1\} \quad \forall i=1..M$$

$$y_{ik} \geq 0 \text{ and integer} \quad \forall i=1..M, k=1..n$$

Q7)

④ Convert into standard form and add artificial variable.

$$\begin{aligned} \min \quad & -3x_1 + 4x_2 \\ \text{s.t.} \quad & x_1 + x_2 + s_1 = 4 \\ & 2x_1 + 3x_2 - e_1 + a_1 = 18 \\ & x_1, x_2, s_1, e_1, a_1 \geq 0 \end{aligned}$$

Phase I: change obj. function with $\min a_1$ (some const.)

	x_1	x_2	s_1	e_1	a_1	RHS
z	0	0	0	0	-1	0
1	2	3	0	-1	0	18
s_1	1	1	1	0	0	4
a_1	2	3	0	-1	1	18
1	-1	0	-3	-1	0	6
x_2	0	1	1	0	0	4
a_1	0	-1	0	-1	1	6

There does not exist any positive value in row zero. So the tableau is optimal for Phase I.

Since a_1 is still basic, the original problem is infeasible.

Q8)

a) $A < 0$ $B = 0$ C any $D \geq 0$ $E = 1$ F any

b) $A > 0$ $B = 0$ $C \leq 0$ $D \geq 0$ $E = 1$ F any

c) A any $B = 0$ C any $D = 0$ $E = 1$ F any

d) $A = 0$ $B = 0$ $C \leq 0$ $D \geq 0$ $E = 1$ F any

Q9)

$$1) \quad x_1 - x_2 + 3x_3 + 5x_4 = 15$$

$$x_1 + x_2 + x_4 = 9$$

$$x_i \geq 0 \quad i=1,2,3,4$$

$$n=4 \quad m=2 \quad \binom{n}{m} = \binom{4}{2} = 6 \quad \# \text{ of basic solutions}$$

• x_1, x_2 : basic

$$\begin{aligned} x_3, x_4 : \text{nonbasic} & \Rightarrow \begin{aligned} x_1 - x_2 &= 15 \\ x_1 + x_2 &= 9 \end{aligned} \Rightarrow x = \begin{pmatrix} 12 \\ -3 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

$$(x_3=0, x_4=0)$$

This is a basic solution but not feasible since $x_2 = -3 < 0$

• x_1, x_3 : basic

$$\begin{aligned} x_2, x_4 : \text{nonbasic} & \Rightarrow \begin{aligned} x_1 + 3x_3 &= 15 \\ x_1 &= 9 \end{aligned} \Rightarrow x = \begin{pmatrix} 9 \\ 0 \\ 2 \\ 0 \end{pmatrix} \text{ This's a bfs.} \end{aligned}$$

$$(x_2=0, x_4=0)$$

• x_1, x_4 : basic

$$\begin{aligned} x_2, x_3 : \text{nonbasic} & \Rightarrow \begin{aligned} x_1 + 5x_4 &= 15 \\ x_1 + x_4 &= 9 \end{aligned} \Rightarrow x = \begin{pmatrix} 15/2 \\ 0 \\ 0 \\ 3/2 \end{pmatrix} \text{ It's a bfs.} \end{aligned}$$

$$(x_2=0, x_3=0)$$

• x_2, x_3 : basic

$$\begin{aligned} x_1, x_4 : \text{nonbasic} & \Rightarrow \begin{aligned} -x_2 + 3x_3 &= 15 \\ x_2 &= 9 \end{aligned} \Rightarrow x = \begin{pmatrix} 0 \\ 9 \\ 8 \\ 0 \end{pmatrix} \text{ It's a bfs.} \end{aligned}$$

$$(x_1=0, x_4=0)$$

• x_2, x_4 : basic

$$\begin{aligned} x_1, x_3 : \text{nonbasic} & \Rightarrow \begin{aligned} -x_2 + 5x_4 &= 15 \\ x_2 + x_4 &= 9 \end{aligned} \Rightarrow x = \begin{pmatrix} 0 \\ 5 \\ 0 \\ 4 \end{pmatrix} \text{ It's a bfs.} \end{aligned}$$

$$(x_1=0, x_3=0)$$

• x_3, x_4 : basic

$$\begin{aligned} x_1, x_2 : \text{nonbasic} & \Rightarrow \begin{aligned} 3x_3 + 5x_4 &= 15 \\ x_4 &= 9 \end{aligned} \Rightarrow x = \begin{pmatrix} 0 \\ 0 \\ -10 \\ 9 \end{pmatrix} \end{aligned}$$

$$(x_1=0, x_2=0)$$

This is basic but not feasible since $x_3 < 0$

Q10)

(4) First convert to standard form

$$\begin{aligned} \max \quad & -x_1 - 3x_2' + (x_3' - x_3'') - M a_2 \\ \text{s.t.} \quad & x_1 + x_2' + (x_3' - x_3'') + s_1 = 4 \\ & 2x_1 - x_2' + (x_3' - x_3'') - e_2 + a_2 = 10 \\ & x_1, x_2', x_3', x_3'', s_1, e_2 \geq 0 \end{aligned}$$

$$\begin{aligned} x_2 &= -x_2' \\ x_3 &= x_3' - x_3'' \end{aligned}$$

	z	x_1	x_2'	x_3'	x_3''	s_1	e_2	a_2	RHS
$\leftarrow$	s_1	1	1	1	-1	1	0	0	4
$\leftarrow$	a_2	2	-1	1	-1	0	-1	1	10
	z	0	$3M+2$	$M-2$	$-M+2$	$2M-1$	M	0	$-2M-4$
$\leftarrow$	x_1	0	1	1	-1	1	0	0	4
$\leftarrow$	a_2	0	-3	-1	1	-2	-1	1	2
	z	0	8	0	0	3	2	$M-2$	-8
$\leftarrow$	x_1	0	1	-2	0	-1	-1	1	6
$\leftarrow$	x_3''	0	0	-3	-1	1	-2	-1	2

$\rightarrow x^* = \begin{pmatrix} x_1 \\ x_2' \\ x_3' \\ x_3'' \\ s_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} \quad z^* = -8$

$\exists$ no neg row 0 value. So the tableau is final. Since a_2 is not in the basis, remove the column a_2 . the remaining tableau is ^{for} optimal soln.

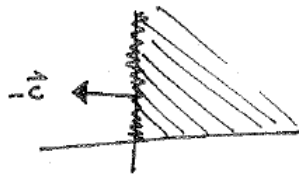
$$x_2 = -x_2' = 0$$

$$x_3 = x_3' - x_3'' = 0 - 2 = -2$$

Q11)

- ② a) Every LP with an unbounded feasible region is unbounded.

FALSE,

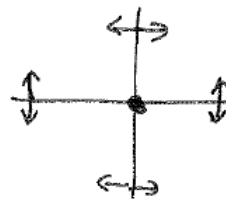


For the following
min x_1
 $z^* = 0$

- ② b) A linear programming problem cannot have exactly one feasible solution.

FALSE

$$\begin{aligned} \min & 0 \\ \text{s.t.} & x_1 \leq 0 \\ & x_1 \geq 0 \\ & x_2 \leq 0 \\ & x_2 \geq 0 \end{aligned}$$



(0,0) is the
feasible region
for this LP

c) **TRUE:** Since the objective value of phase-I (w) is bounded, phase-I will always yield an optimal tableau.

d) **FALSE:** Since Big-M and 2-Phase methods make the same sequence of pivots, if Big-M stops with infeasibility (at least one artificial variable is positive and there is no entering variable) then phase-I will also stop with infeasibility and won't move to phase-II.

e) f) **FALSE:** counter example:

A Degenerate LP

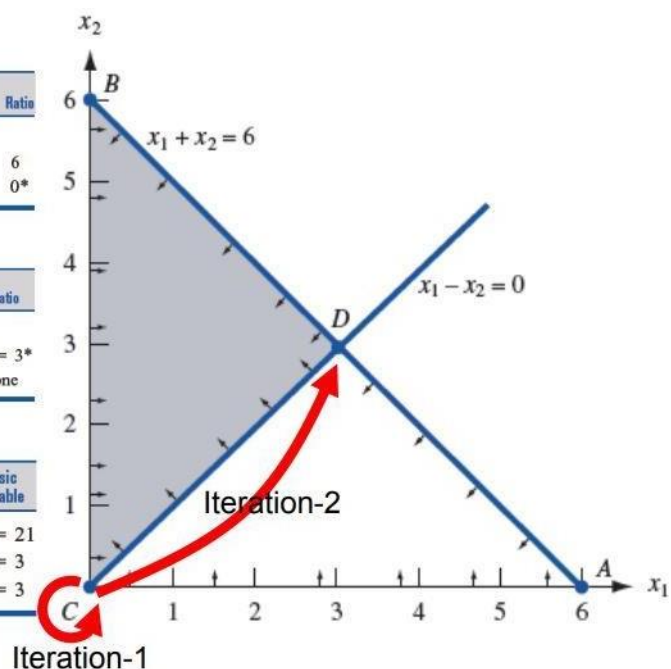
z	x_1	x_2	s_1	s_2	rhs	Basic Variable	Ratio
1	-5	-2	0	0	0	$z = 0$	
0	1	1	1	0	6	$s_1 = 6$	6
0	①	-1	0	1	0	$s_2 = 0$	0*

First Tableau for (16)

z	x_1	x_2	s_1	s_2	rhs	Basic Variable	Ratio
1	0	-7	0	5	0	$z = 0$	
0	0	②	1	-1	6	$s_1 = 6$	$\frac{6}{2} = 3^*$
0	1	-1	0	1	0	$x_1 = 0$	None

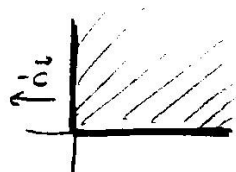
Optimal Tableau for (16)

z	x_1	x_2	s_1	s_2	rhs	Basic Variable
1	0	0	3.5	1.5	21	$z = 21$
0	0	1	0.5	-0.5	3	$x_2 = 3$
0	1	0	0.5	0.5	3	$x_1 = 3$



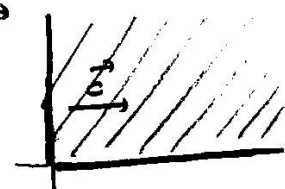
g) TRUE: By definition, feasible region of an LP is convex. Again by definition convex combination of any two points in a convex set is also convex.

h) TRUE: Consider the feasible region below for a minimization problem.



$$\begin{array}{ll} \min & cx \\ \text{s.t.} & Ax \geq b \end{array}$$

by multiplying c with -1



unbounded

Q12)

④ First convert into standard form

$$\begin{aligned} \max \quad & 5x_1 + 3x_2 + x_3 \\ \text{s.t.} \quad & x_1 + x_2 + 3x_3 + s_1 = 6 \\ & 5x_1 + 3x_2 + 6x_3 + s_2 = 15 \\ & x_i \geq 0 \quad (i=1,2,3), s_i \geq 0 \quad (i=1,2) \end{aligned}$$

	$\downarrow$	z	x_1	x_2	x_3	s_1	s_2	RHS
	$\downarrow$	1	-5	-3	-1	0	0	0
s_1	0	1	1	3	1	0	0	6
$\leftarrow s_2$	0	5	3	6	0	1	0	15

	$\downarrow$	z	x_1	x_2	x_3	s_1	s_2	RHS
	$\downarrow$	1	0	0	5	0	1	15
s_1	0	1	0	$\frac{2}{5}$	$\frac{3}{5}$	1	$-\frac{1}{5}$	3
$\leftarrow s_2$	0	1	0	$\frac{3}{5}$	$\frac{3}{5}$	0	$\frac{1}{5}$	3

x_2 enters (since $-5 < 0$)
(problem is max)

Apply MRT and $\min\{\frac{6}{1}, \frac{15}{3}\}$
so s_2 leaves

Since there is no negative row 0 coefficient, the solution is optimal

$x^* = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ with opt. soln. value = 15

If there exists '0' in the row 0, there may be alternative optimal soln. (0's for nonbasic variables)

So x_2 enters, apply MRT and $\min\{\frac{3}{2/5}, \frac{3}{3/5}\}$, x_1 leaves

	z	x_1	x_2	x_3	s_1	s_2	RHS
	1	0	0	5	0	1	15
s_1	0	$\frac{2}{5}$	0	1	1	$-\frac{1}{5}$	1
s_2	0	$\frac{3}{5}$	1	2	0	$\frac{1}{5}$	5

$\rightarrow$ optimal soln

$x^* = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}$ with opt. soln. value = 15

If you again enter x_1 , you will turn back to previous tableau!

Q13)

- (a) FALSE; this cannot be true for a maximization problem because a newly added constraint either does not change the feasible region or it makes it narrower. Hence the objective function may stay the same or decrease but there cannot be a constraint to improve the optimal solution.
- (b) TRUE; $\binom{m+1}{m} = m+1$
- (c) FALSE; it can have "at most" m positive variables. If it is less than m variables then that is called degeneracy.
- (d) FALSE; because the set P is a convex set and the points on the convex combination of two optimal solutions also become optimal.
- (e) FALSE; due to degeneracy, an extreme point of P may correspond to more than one basis.

Q14)

Converting the solution into standard form:

$$\begin{array}{llllllll}
 \text{max} & 2x_1 & +x_2 & +6x_3 & -4x_4 & & & \\
 \text{st.} & x_1 & +2x_2 & +4x_3 & -x_4 & +x_5 & & =6 \\
 & 2x_1 & +3x_2 & -x_3 & +x_4 & & +x_6 & =12 \\
 & x_1 & & +x_3 & +x_4 & & +x_7 & =6 \\
 & x_1, & x_2, & x_3, & x_4, & x_5, & x_6, & x_7 \geq 0
 \end{array}$$

$$A = \begin{bmatrix} 1 & 2 & 4 & -1 \\ 2 & 3 & -1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

For basic variables x_1, x_2, x_4 the basis $B = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$$B^{-1} = \begin{bmatrix} 3/4 & -1/2 & 5/4 \\ -1/4 & 1/2 & -3/4 \\ -3/4 & 1/2 & -1/4 \end{bmatrix}$$

$$c_B = [2 \ 1 \ -4] \quad b = \begin{bmatrix} 6 \\ 12 \\ 6 \end{bmatrix} \quad c = [2 \ 1 \ 6 \ -4]$$

Remember the matrix notation

	z	x	x ₅	
z	1	$c_B B^{-1} A - c$	$c_B B^{-1}$	$c_B B^{-1} b$
x _B	0	$B^{-1} A$	B^{-1}	$B^{-1} b$

$$B^{-1} A = \begin{bmatrix} 1 & 0 & 19/4 & 0 \\ 0 & 1 & -9/4 & 0 \\ 0 & 0 & -15/4 & 1 \end{bmatrix}$$

$$c_B B^{-1} A = [2 \ 1 \ 89/4 \ -4]$$

$$c_B B^{-1} A - c = [0 \ 0 \ 65/4 \ 0]$$

$$c_B B^{-1} = [17/4 \ -10/4 \ 11/4] \quad c_B B^{-1} b = [12] \quad B^{-1} b = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

Putting these values into the matrix:

	z	x ₁	x ₂	x ₃	x ₄	x ₅	x ₆	x ₇	RHS
z	1	0	0	65/4	0	17/4	-10/4	11/4	12
x ₁	0	1	0	19/4	0	3/4	-2/4	5/4	6
x ₂	0	0	1	-9/4	0	-1/4	2/4	-3/4	0
x ₄	0	0	0	-15/4	1	-3/4	2/4	-1/4	0

Since we have negative entry in Row 0, this tableau is not optimal.

x₆ enters and x₂ leaves the basis. New tableau: