

Q1)

Parameters f_i : fixed cost of plant i , $i=1,2,3,4$
 c_{ij} : cost of producing car j at plant i , $i=1,2,3,4$
 $j=1,2,3$

Dec. Var $y_i = \begin{cases} 1 & \text{if plant } i \text{ is used} \\ 0 & \text{o.w.} \end{cases} \quad i=1,2,3,4$
 $x_{ij} = \begin{cases} 1 & \text{if car } j \text{ is produced at plant } i \\ 0 & \text{o.w.} \end{cases} \quad i=1,2,3,4$
 $j=1,2,3$

Model: $\min \sum_{i=1}^4 f_i y_i + \sum_{i=1}^4 \sum_{j=1}^3 c_{ij} x_{ij}$

$$\text{s.t.} \quad \sum_{j=1}^3 x_{ij} \leq 1 \quad \forall i=1,2,3,4$$

$$\sum_{i=1}^4 x_{ij} = 1 \quad \forall j=1,2,3$$

$$y_3 + y_4 \leq 1 + y_1$$

$$x_{ij} \leq y_i \quad \forall i=1,2,3,4, j=1,2,3$$

$$x_{ij} \in \{0,1\} \quad \forall i=1,2,3,4, j=1,2,3$$

$$y_j \in \{0,1\} \quad \forall j=1,2,3$$

Q2)

④ Parameters

cb_i : cost of producing a ton of steam in boiler i , $i=1,2,3$

ct_j : " " processing " " " " turbine j , $j=1,2,3$

$minb_i$: min amount of steam that can be produced in boiler i ,

$maxb_i$: max " " " " " " " " $i=1,2,3$

$mint_j$: min " " " " " " " " processed in turbine j

$maxt_j$: max " " " " " " " " $j=1,2,3$

Dec. Var

$x_i = \begin{cases} 1 & \text{if boiler } i \text{ is used} \\ 0 & \text{o.w} \end{cases} \quad i=1,2,3$

y_i : amount of steam that is produced in boiler i , $i=1,2,3$

$t_j = \begin{cases} 1 & \text{if turbine } j \text{ is used} \\ 0 & \text{o.w} \end{cases} \quad j=1,2,3$

z_j : amount of steam that is processed in turbine $j=1,2,3$

Model

$$\min \sum_{i=1}^3 cb_i y_i + \sum_{j=1}^3 ct_j z_j$$

s.t

$$minb_i x_i \leq y_i \leq maxb_i x_i \quad \forall i=1,2,3 \quad \begin{matrix} \text{(production limitations)} \\ \text{of boilers} \end{matrix}$$

$$mint_j t_j \leq z_j \leq maxt_j t_j \quad \forall j=1,2,3 \quad \begin{matrix} \text{(process limitations)} \\ \text{of turbines} \end{matrix}$$

$$4z_1 + 5z_2 + 6z_3 = 800 \quad \text{(required production)}$$

$$\sum_{i=1}^3 y_i \geq \sum_{j=1}^3 z_j \quad \begin{matrix} \text{(produced steam should} \\ \text{be larger than processed)} \end{matrix}$$

$$x_i \in \{0,1\} \quad y_i \geq 0 \quad \forall i=1,2,3$$

$$t_j \in \{0,1\} \quad z_j \geq 0 \quad \forall j=1,2,3$$

Q3)

If we assume the extra translator can handle both tourist groups simultaneously:

x_i : the number of Italian translators who start to work at period i , $i=1,2,3,4,5,6$

y_i : the number of German translators who start to work at period i , $i=1,2,3,4,5,6$

z_i : the number of extra translators who work at period i , $i=1,2,3,4,5,6$

$$\min 320 X_i + 280 Y_i + 240 Z_i$$

st.

$$X_6 + X_1 + Z_1 \geq 5$$

$$X_1 + X_2 + Z_2 \geq 3$$

$$X_2 + X_3 + Z_3 \geq 8$$

$$X_3 + X_4 + Z_4 \geq 8$$

$$X_4 + X_5 + Z_5 \geq 11$$

$$X_5 + X_6 + Z_6 \geq 4$$

$$Y_6 + Y_1 + Z_1 \geq 4$$

$$Y_1 + Y_2 + Z_2 \geq 4$$

$$Y_2 + Y_3 + Z_3 \geq 7$$

$$Y_3 + Y_4 + Z_4 \geq 7$$

$$Y_4 + Y_5 + Z_5 \geq 13$$

$$Y_5 + Y_6 + Z_6 \geq 4$$

$$x_i, y_i, z_i \geq 0 \text{ for } i=1,2,3,4,5,6 \text{ and integer}$$

If we assume the extra translator is needed for each group separately:

x_{ij} : the number of regular translators who start to work at period i and translates language j ,
 $i=1,2,3,4,5,6$ $j=1,2$ (1 Italian, 2 German)

y_{ij} : the number of extra translators who work at period i and translates language j , $i=1,2,3,4,5,6$
 $j=1,2$ (1 Italian, 2 German)

$$\min 320 X_{i1} + 280 X_{i2} + 240 Y_{ij}$$

st.

$$X_{61} + X_{11} + y_{11} \geq 5$$

$$X_{11} + X_{21} + y_{21} \geq 3$$

$$X_{21} + X_{31} + Z_{31} \geq 8$$

$$X_{31} + X_{41} + y_{41} \geq 8$$

$$X_{41} + X_{51} + y_{51} \geq 11$$

$$X_{51} + X_{61} + y_{61} \geq 4$$

$$X_{62} + X_{12} + y_{12} \geq 4$$

$$X_{12} + X_{22} + y_{22} \geq 4$$

$$X_{22} + X_{32} + y_{32} \geq 7$$

$$X_{32} + X_{42} + y_{42} \geq 7$$

$$X_{42} + X_{52} + y_{52} \geq 13$$

$$X_{52} + X_{62} + y_{62} \geq 4$$

$$x_{ij}, y_{ij} \geq 0 \text{ for } i=1,2,3,4,5,6 \text{ } j=1,2 \text{ and integer}$$

Q4)

Decision variables:

$P_{N,i}$: units produced at month i under normal shift
 $P_{E,i}$: units produced at month i under extended shift
 x_i : 1 if $P_{N,i} > 0$
 y_i : 1 if $P_{E,i} > 0$
 t_i : 1 if changes from normal shift to extended shift
 I_i : inventory level at the end of month i

Parameters:

C_N : cost of normal shift (100,000 per month)
 C_E : cost of extended shift (180,000 per month)
 C_C : cost of changing from normal shift to extended shift
 h : holding cost (2 per unit per month)
 I_0 : initial stock (3,000 units)
 D_i : demand for month i

Model:

$$\begin{aligned} \min \sum_{i=1}^6 x_i C_N + y_i C_E + t_i C_C + h I_i \\ \text{s.t } x_i + y_i \leq 1, \quad \forall i = 1, \dots, 6 \\ x_{i-1} + y_i \leq 1 + t_i, \quad \forall i = 1, \dots, 6 \\ x_0 = 1 \\ 2000x_i \leq P_{N,i} \leq 5000x_i, \quad \forall i = 1, \dots, 6 \\ 2000y_i \leq P_{E,i} \leq 7500y_i, \quad \forall i = 1, \dots, 6 \\ P_{N,i} + P_{E,i} + I_{i-1} = D_i + I_i, \quad \forall i = 1, \dots, 6 \\ I_0 = 3000 \\ I_6 = 2000 \\ x_i, y_i, t_i \in \{0, 1\}, \quad \forall i = 1, \dots, 6 \\ P_{N,i}, P_{E,i}, I_i \geq 0, \quad \forall i = 1, \dots, 6 \\ P_{N,i}, P_{E,i}, I_i : \text{integer}, \quad \forall i = 1, \dots, 6 \end{aligned}$$

Q5)

Dec. Var

$$x_{ij} = \begin{cases} 1 & \text{if } i^{\text{th}} \text{ ast. is assigned to } j^{\text{th}} \text{ task} \\ 0 & \text{o.w} \end{cases} \quad i=1..4, j=1..4$$

$$\max \sum_{i=1}^4 \sum_{j=1}^4 p_{ij} \cdot x_{ij}$$

s.t

$$\sum_{i=1}^4 x_{ij} = 1 \quad \forall j=1..4 \quad (\text{each post should have a single ast.})$$

$$\sum_{j=1}^4 x_{ij} = 1 \quad \forall i=1..4 \quad (\text{each ast. should be assigned to a single post})$$

$$x_{ij} \in \{0, 1\} \quad \forall i=1..4, \forall j=1..4$$

Q6)

Dec. Var : given

$$\text{Model: } \min \sum_{i=1}^M x_i$$

$$\text{s.t} \quad \sum_{i=1}^M y_{ik} = d_k \quad \forall k=1..n$$

$$\sum_{k=1}^n y_{ik} \cdot p_k \leq p \cdot x_i \quad \forall i=1..M$$

$$x_i \in \{0, 1\} \quad \forall i=1..M$$

$$y_{ik} \geq 0 \text{ and integer} \quad \forall i=1..M, k=1..n$$

Q7)

④ Convert into standard form and add artificial variable.

$$\begin{aligned} \min \quad & -3x_1 + 4x_2 \\ \text{s.t.} \quad & x_1 + x_2 + s_1 = 4 \\ & 2x_1 + 3x_2 - e_1 + a_1 = 18 \\ & x_1, x_2, s_1, e_1, a_1 \geq 0 \end{aligned}$$

Phase I: change obj. function with $\min a_1$ s.t. (some const.)

	x_1	x_2	s_1	e_1	a_1	RHS
1	0	0	0	0	-1	0
1	2	3	0	-1	0	18
s_1	0	1	1	0	0	4
a_1	0	2	3	0	-1	18

1	-1	0	-3	-1	0	6
x_2	0	1	1	0	0	4
a_1	0	-1	0	-1	1	6

There does not exist any positive value in row zero, so the tableau is optimal for Phase I.

Since a_1 is still basic, the original problem is infeasible.

Q8)

a) $A < 0$ $B = 0$ C any $D \geq 0$ $E = 1$ F any

b) $A > 0$ $B = 0$ $C \leq 0$ $D \geq 0$ $E = 1$ F any

c) A any $B = 0$ C any $D = 0$ $E = 1$ F any

d) $A = 0$ $B = 0$ $C \leq 0$ $D \geq 0$ $E = 1$ F any

Q9)

$$1) \quad x_1 - x_2 + 3x_3 + 5x_4 = 15$$

$$x_1 + x_2 + x_4 = 9$$

$$x_i \geq 0 \quad i=1,2,3,4$$

$$n=4 \quad m=2 \quad \binom{n}{m} = \binom{4}{2} = 6 \quad \# \text{ of basic solutions}$$

• x_1, x_2 : basic

$$\begin{aligned} x_3, x_4 &: \text{nonbasic} \Rightarrow \begin{aligned} x_1 - x_2 &= 15 \\ x_1 + x_2 &= 9 \end{aligned} \Rightarrow x = \begin{pmatrix} 12 \\ -3 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

$$(x_3=0, x_4=0)$$

This is a basic solution but not feasible since $x_2 = -3 < 0$

• x_1, x_3 : basic

$$\begin{aligned} x_2, x_4 &: \text{nonbasic} \Rightarrow \begin{aligned} x_1 + 3x_3 &= 15 \\ x_1 &= 9 \end{aligned} \Rightarrow x = \begin{pmatrix} 9 \\ 0 \\ 2 \\ 0 \end{pmatrix} \text{ This's a bfs.} \end{aligned}$$

$$(x_2=0, x_4=0)$$

• x_1, x_4 : basic

$$\begin{aligned} x_2, x_3 &: \text{nonbasic} \Rightarrow \begin{aligned} x_1 + 5x_4 &= 15 \\ x_1 + x_4 &= 9 \end{aligned} \Rightarrow x = \begin{pmatrix} 15/2 \\ 0 \\ 0 \\ 3/2 \end{pmatrix} \text{ It's a bfs.} \end{aligned}$$

$$(x_2=0, x_3=0)$$

• x_2, x_3 : basic

$$\begin{aligned} x_1, x_4 &: \text{nonbasic} \Rightarrow \begin{aligned} -x_2 + 3x_3 &= 15 \\ x_2 &= 9 \end{aligned} \Rightarrow x = \begin{pmatrix} 0 \\ 9 \\ 8 \\ 0 \end{pmatrix} \text{ It's a bfs.} \end{aligned}$$

$$(x_1=0, x_4=0)$$

• x_2, x_4 : basic

$$\begin{aligned} x_1, x_3 &: \text{nonbasic} \Rightarrow \begin{aligned} -x_2 + 5x_4 &= 15 \\ x_2 + x_4 &= 9 \end{aligned} \Rightarrow x = \begin{pmatrix} 0 \\ 5 \\ 0 \\ 4 \end{pmatrix} \text{ It's a bfs.} \end{aligned}$$

$$(x_1=0, x_3=0)$$

• x_3, x_4 : basic

$$\begin{aligned} x_1, x_2 &: \text{nonbasic} \Rightarrow \begin{aligned} 3x_3 + 5x_4 &= 15 \\ x_4 &= 9 \end{aligned} \Rightarrow x = \begin{pmatrix} 0 \\ 0 \\ -10 \\ 9 \end{pmatrix} \end{aligned}$$

$$(x_1=0, x_2=0)$$

This is basic but not feasible since $x_3 < 0$

Q10)

④ First convert to standard form

$$\begin{aligned} \max \quad & -x_1 - 3x_2' + (x_3' - x_3'') - M a_2 \\ \text{s.t.} \quad & x_1 + x_2' + (x_3' - x_3'') + s_1 = 4 \\ & 2x_1 - x_2' + (x_3' - x_3'') - e_2 + a_2 = 10 \\ & x_1, x_2', x_3', x_3'', s_1, e_2 \geq 0 \end{aligned}$$

$$\begin{aligned} x_2 &= -x_2' \\ x_3 &= x_3' - x_3'' \end{aligned}$$

	x_1	x_2'	x_3'	x_3''	s_1	e_2	a_2	RHS
s_1	1	1	1	-1	1	0	0	4
a_2	2	-1	1	-1	0	-1	1	10
	1	1	1	-1	0	0	M	0
	1	2M+1	M+3	-M-1	M+1	0	M	-10M

$x_1 \leftarrow M$ $\left(\begin{array}{c} 4 \\ 1 \\ 10/2 \end{array} \right)$

	x_1	x_2'	x_3'	x_3''	s_1	e_2	a_2	RHS
s_1	0	3M+2	M-2	-M+2	2M-1	M	0	-2M-4
x_1	0	1	1	-1	1	0	0	4
a_2	0	-3	-1	1	-2	-1	1	2

$\left(\begin{array}{c} X \\ 2/1 \end{array} \right)$

	x_1	x_2'	x_3'	x_3''	s_1	e_2	a_2	RHS
s_1	0	8	0	0	3	2	M-2	-8
x_1	0	1	-2	0	-1	-1	1	6
x_3''	0	0	-3	-1	1	-2	1	2

$x^* = \begin{pmatrix} x_1 \\ x_2' \\ x_3' \\ x_3'' \\ s_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} \quad z^* = -8$

3 no neg row 0 value. So the tableau is final. Since a_2 is not in the basis, remove the column a_2 . the remaining tableau is for optimal soln.

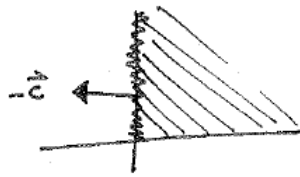
$$x_2 = -x_2' = 0$$

$$x_3 = x_3' - x_3'' = 0 - 2 = -2$$

Q11)

- ② a) Every LP with an unbounded feasible region is unbounded.

FALSE,



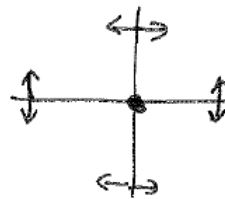
For the following
min x_1

$$z^* = 0$$

- ② b) A linear programming problem cannot have exactly one feasible solution.

FALSE

$$\begin{aligned} \min & 0 \\ \text{s.t.} & x_1 \leq 0 \\ & x_1 \geq 0 \\ & x_2 \leq 0 \\ & x_2 \geq 0 \end{aligned}$$



(0,0) is the
feasible region
for this LP

c) **TRUE:** Since the objective value of phase-I (w) is bounded, phase-I will always yield an optimal tableau.

d) **FALSE:** Since Big-M and 2-Phase methods make the same sequence of pivots, if Big-M stops with infeasibility (at least one artificial variable is positive and there is no entering variable) then phase-I will also stop with infeasibility and won't move to phase-II.

e) f) **FALSE:** counter example:

A Degenerate LP

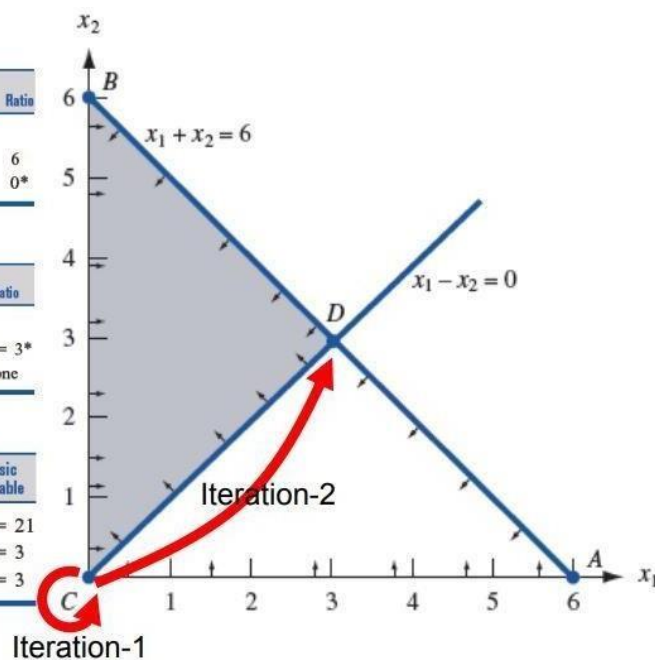
z	x_1	x_2	s_1	s_2	rhs	Basic Variable	Ratio
1	-5	-2	0	0	0	$z = 0$	
0	1	1	1	0	6	$s_1 = 6$	6
0	①	-1	0	1	0	$s_2 = 0$	0*

First Tableau for (16)

z	x_1	x_2	s_1	s_2	rhs	Basic Variable	Ratio
1	0	-7	0	5	0	$z = 0$	
0	0	②	1	-1	6	$s_1 = 6$	$\frac{6}{2} = 3^*$
0	1	-1	0	1	0	$x_1 = 0$	None

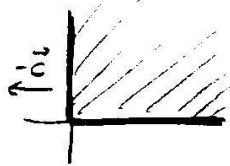
Optimal Tableau for (16)

z	x_1	x_2	s_1	s_2	rhs	Basic Variable
1	0	0	3.5	1.5	21	$z = 21$
0	0	1	0.5	-0.5	3	$x_2 = 3$
0	1	0	0.5	0.5	3	$x_1 = 3$



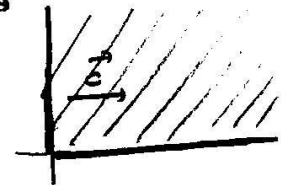
g) TRUE: By definition, feasible region of an LP is convex. Again by definition convex combination of any two points in a convex set is also convex.

h) TRUE: Consider the feasible region below for a minimization problem.



$$\begin{aligned} \min \quad & cx \\ \text{s.t.} \quad & Ax \geq b \end{aligned}$$

by multiplying c with -1



unbounded

Q12)

(4) First convert into standard form

$$\begin{aligned}
 \max \quad & 5x_1 + 3x_2 + x_3 \\
 \text{s.t.} \quad & x_1 + x_2 + 3x_3 + s_1 = 6 \\
 & 5x_1 + 3x_2 + 6x_3 + s_2 = 15 \\
 & x_i \geq 0 \quad (i=1,2,3) \quad s_i \geq 0 \quad (i=1,2)
 \end{aligned}$$

	θ	x_1	x_2	x_3	s_1	s_2	RHS
	1	-5	-3	-1	0	0	0
s_1	0	1	1	3	1	0	6
s_2	0	(5)	3	6	0	1	15

x_2 enters (since $-5 < 0$)
(problem is max)

Apply MMT and $\min\{\frac{6}{1}, \frac{15}{3}\}$
so s_2 leaves

	θ	x_1	x_2	x_3	s_1	s_2	RHS
	1	0	0	5	0	1	15
s_1	0	0	$\frac{2}{5}$	$\frac{1}{5}$	1	$-\frac{1}{5}$	3
s_2	0	1	($\frac{3}{5}$)	$\frac{6}{5}$	0	$\frac{1}{5}$	3

Since there is no negative row 0 coefficient, the solution is optimal

$x^* = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ with opt. soln. value = 15

If there exists '0' in the row 0, there may be alternative optimal soln. (0's for nonbasic variables)

So x_2 enters, apply MMT $\min\{\frac{3}{2/5}, \frac{3}{3/5}\}$, s_2 leaves

	θ	x_1	x_2	x_3	s_1	s_2	RHS
	1	0	0	5	0	1	15
s_1	0	$\frac{2}{5}$	0	1	1	$-\frac{1}{5}$	3
s_2	0	$\frac{3}{5}$	1	2	0	$\frac{1}{5}$	3

→ optimal soln

$x^* = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}$ with opt. soln. value = 15

If you again enter x_1 , you will turn back to previous tableau!

Q13)

- (a) FALSE; this cannot be true for a maximization problem because a newly added constraint either does not change the feasible region or it makes it narrower. Hence the objective function may stay the same or decrease but there cannot be a constraint to improve the optimal solution.
- (b) TRUE; $\binom{m+1}{m} = m+1$
- (c) FALSE; it can have "at most" m positive variables. If it is less than m variables then that is called degeneracy.
- (d) FALSE; because the set P is a convex set and the points on the convex combination of two optimal solutions also become optimal.
- (e) FALSE; due to degeneracy, an extreme point of P may correspond to more than one basis.

Q14)

Converting the solution into standard form:

$$\begin{array}{llllllllll}
 \text{max} & 2x_1 & +x_2 & +6x_3 & -4x_4 & & & & & \\
 \text{st.} & x_1 & +2x_2 & +4x_3 & -x_4 & +x_5 & & & & =6 \\
 & 2x_1 & +3x_2 & -x_3 & +x_4 & & +x_6 & & & =12 \\
 & x_1 & & +x_3 & +x_4 & & & +x_7 & & =6 \\
 & x_1, & x_2, & x_3, & x_4, & x_5, & x_6, & x_7 & & \geq 0
 \end{array}$$

$$A = \begin{bmatrix} 1 & 2 & 4 & -1 \\ 2 & 3 & -1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

For basic variables x_1, x_2, x_4 the basis $B = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$$B^{-1} = \begin{bmatrix} 3/4 & -1/2 & 5/4 \\ -1/4 & 1/2 & -3/4 \\ -3/4 & 1/2 & -1/4 \end{bmatrix}$$

$$c_B = [2 \ 1 \ -4] \quad b = \begin{bmatrix} 6 \\ 12 \\ 6 \end{bmatrix} \quad c = [2 \ 1 \ 6 \ -4]$$

Remember the matrix notation

	z	x	x ₅	
z	1	$c_B B^{-1} A - c$	$c_B B^{-1}$	$c_B B^{-1} b$
x _B	0	$B^{-1} A$	B^{-1}	$B^{-1} b$

$$B^{-1} A = \begin{bmatrix} 1 & 0 & 19/4 & 0 \\ 0 & 1 & -9/4 & 0 \\ 0 & 0 & -15/4 & 1 \end{bmatrix}$$

$$c_B B^{-1} A = [2 \ 1 \ 89/4 \ -4]$$

$$c_B B^{-1} A - c = [0 \ 0 \ 65/4 \ 0]$$

$$c_B B^{-1} = [17/4 \ -10/4 \ 11/4] \quad c_B B^{-1} b = [12] \quad B^{-1} b = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

Putting these values into the matrix:

	z	x ₁	x ₂	x ₃	x ₄	x ₅	x ₆	x ₇	RHS
z	1	0	0	65/4	0	17/4	-10/4	11/4	12
x ₁	0	1	0	19/4	0	3/4	-2/4	5/4	6
x ₂	0	0	1	-9/4	0	-1/4	2/4	-3/4	0
x ₄	0	0	0	-15/4	1	-3/4	2/4	-1/4	0

Since we have negative entry in Row 0, this tableau is not optimal.

x₆ enters and x₂ leaves the basis. New tableau: