IE 400: Principles of Engineering Management
Study Set 1 Solutions

Summer 2020

Question 1.
Decision Variables:

x_{ij} =amount of land allocated by kibbutz j for crop i,
$i = 1, 2, 3$ (1:sugar beets, 2:cotton, 3:sorghum), $j = 1, 2, 3$

Model:

$$
\begin{aligned}
\max\ & 1000(x_{11} + x_{12} + x_{13}) + 750(x_{21} + x_{22} + x_{23}) + 250(x_{31} + x_{32} + x_{33}) \\
\text{s.t. } & x_{11} + x_{12} + x_{13} \leq 600 && (1)\\
& x_{21} + x_{22} + x_{23} \leq 500 && (2)\\
& x_{31} + x_{32} + x_{33} \leq 325 && (3)\\
\\
& x_{11} + x_{21} + x_{31} \leq 600 && (4)\\
& x_{12} + x_{22} + x_{32} \leq 800 && (5)\\
& x_{13} + x_{23} + x_{33} \leq 375 && (6)\\
\\
& x_{11} + x_{21} + x_{31} \leq 400 && (7)\\
& x_{12} + x_{22} + x_{32} \leq 600 && (8)\\
& x_{13} + x_{23} + x_{33} \leq 300 && (9)\\
\\
& (x_{12} + x_{22} + x_{32}) - 2(x_{13} + x_{23} + x_{33}) = 0 && (10)\\
& 3(x_{11} + x_{21} + x_{31}) - 2(x_{12} + x_{22} + x_{32}) = 0 && (11)\\
& 3(x_{11} + x_{21} + x_{31}) - 4(x_{13} + x_{23} + x_{33}) = 0 && (12)\\
\\
& x_{ij} \geq 0 \quad i = 1, 2, 3 \quad j = 1, 2, 3
\end{aligned}
$$

(1), (2), (3): Constraints for maximum quota limitations
(4), (5), (6): Constraints for water consumption limitations
(7), (8), (9): Constraints for the usable land

Note: Constraints (10), (11), (12) are obtained by linearizing the following equations that ensures each kibbutz will plant the same proportion of its available irrigable land:

$$\frac{x_{11} + x_{21} + x_{31}}{400} = \frac{x_{12} + x_{22} + x_{32}}{600} = \frac{x_{13} + x_{23} + x_{33}}{300}$$

Question 2.

Decision Variables:

$$\begin{aligned} x_1 &= \text{Total number of process 1} \\ x_2 &= \text{Total number of process 2} \\ a &= \text{Amount of A that is produced} \\ b_1 &= \text{Amount of B that is sold} \\ b_2 &= \text{Amount of B that is disposed} \end{aligned}$$

(Note that $b_1 + b_2$ gives the amount of B produced.)
Model:

$$\begin{aligned} \max\ & 16a + 14b_1 - 2b_2 \\ \text{s.t. } & a = 2x_1 + 3x_2 \\ & b_1 + b_2 = x_1 + 2x_2 \\ & 2x_1 + 3x_2 \leq 60 \\ & x_1 + 2x_2 \leq 40 \\ & b_1 \leq 20 \\ & x_1, x_2, a, b_1, b_2 \geq 0 \end{aligned}$$

Note: One can remove the variables a and b_2 from the model by letting $a = 2x_1 + 3x_2$ and $b_2 = x_1 + 2x_2 - b_1$ (constraints 1 and 2). Then the modified model becomes:

$$
\begin{aligned}
\max\ & 16(2x_1+3x_2)+14b_1-2(x_1+2x_2-b_1) \\
\text{s.t.}\ & 2x_1+3x_2 \le 60 \\
& x_1+2x_2 \le 40 \\
& b_1 \le 20 \\
& x_1, x_2, b_1 \ge 0
\end{aligned}
$$

Question 3.
Decision Variables:

c_i = Cash on hand at the end of month i, i=1,2,3,4
x_i = Amount of tons purchased at the beginning of month i, i=1,2,3,4
y_i = Amount of tons sold at the end of month i, i=1,2,3,4
I_i = Amount of tons on hand after purchase at month i, i=1,2,3,4

Parameters:

p_i = Price of buying at month i, i=1,2,3,4
s_i = Price of selling at month i, i=1,2,3,4

Model:

$$
\begin{aligned}
\max\ & c_4 & & \\
\text{s.t.}\ & I_i = I_{i-1}+x_i-y_{i-1} & i=1,\cdots,4 & \quad \text{(Inventory Balance Equations)} \\
& c_i = c_{i-1}-p_ix_i+s_iy_i & i=1,\cdots,4 & \quad \text{(Cash Balance Equations)} \\
& I_i \le 100 & i=1,\cdots,4 & \quad \text{(Capacity Constraint)} \\
& y_i \le I_i & i=1,\cdots,4 & \\
& p_ix_i \le c_{i-1} & i=1,\cdots,4 & \\
& I_0 = 50 & & \\
& c_0 = 1000 & & \\
& y_0 = 0 & & \\
& c_i, x_i, y_i, I_i \ge 0 & i=1,\cdots,4 &
\end{aligned}
$$

Question 4.

(a) **Decision Variables:**
$x_i =$ Number of hours process i runs, i=1,2,3

$$\begin{aligned}
\max\ & 9(2x_1 + 2x_3) + 10(x_1 + 3x_2) + 24x_3 - (5x_1 + 4x_2 + x_3) - 2(2x_1 + x_2 + 3x_3) \\
& -3(3x_1 + 3x_2 + 2x_3) \\
\text{s.t. } & 2x_1 + x_2 + 3x_3 \leq 200 && (1) \\
& 3x_1 + 3x_2 + 2x_3 \leq 300 && (2) \\
& x_1 + x_2 + x_3 \leq 100 && (3) \\
& x_1, x_2, x_3 \geq 0 && (4)
\end{aligned}$$

(b) Remove constraints (1) and (2) and add the following constraint to the model in part (a)

$$(2x_1 + x_2 + 3x_3) + (3x_1 + 3x_2 + 2x_3) \leq 500 \implies 5x_1 + 4x_2 + 5x_3 \leq 500$$

(c)

Additional Decision Variable:
y : Amount of Wonka Boxes sold

The following constraint is required:

$$y = \min\left\{2x_1 + 2x_3, \frac{x_1 + 3x_2}{2}, x_3\right\}$$

Observe that this is not a linear constraint. It **must** be linearized before it is added to the model. The equality above implies the following:

$$y \leq 2x_1 + 2x_3, \quad y \leq \frac{x_1 + 3x_2}{2}, \quad y \leq x_3$$

Then the model becomes:

$$\begin{aligned}
\max\ & 54y - (5x_1 + 4x_2 + x_3) - 2(2x_1 + x_2 + 3x_3) - 3(3x_1 + 3x_2 + 2x_3) \\
\text{s.t. } & (1) - (4) \\
& y \leq 2x_1 + 2x_3 \\
& 2y \leq x_1 + 3x_2 \\
& y \leq x_3 \\
& y \geq 0
\end{aligned}$$

Question 5.
Decision Variables:

x_{ij} = amount of product i produced at period j, i=1,2 j=1,...,12
I_{ij} = amount of product i stored at the end of period j, i=1,2 j=1,...,12

Parameters:

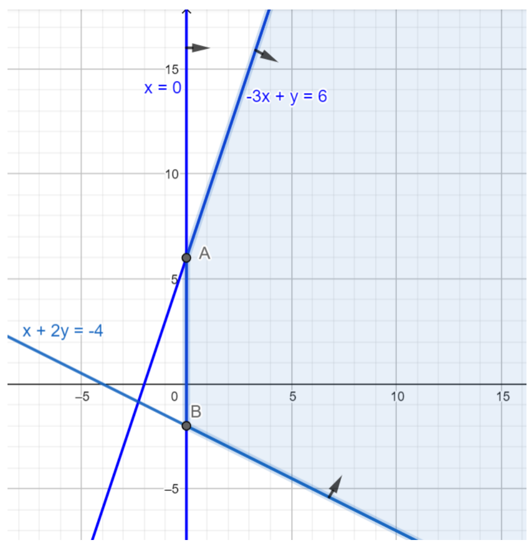
D_{ij} = demand of product i at period j, i=1,2 j=1,...,12

Model:

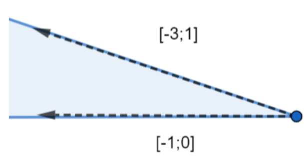
$$
\begin{aligned}
\min\ & \sum_{j=1}^{5}(5x_{1j}+8x_{2j})+\sum_{j=6}^{12}(4.5x_{1j}+7x_{2j})+\sum_{j=1}^{12}(0.2I_{1j}+0.4I_{2j}) & & \\
\text{s.t. } & x_{1j}+x_{2j}\leq 120000 & & j=1,\cdots,9 \\
& x_{1j}+x_{2j}\leq 150000 & & j=10,11,12 \\
& 2I_{1j}+4I_{2j}\leq 150000 & & j=1,\cdots,12 \\
& x_{ij}+I_{i(j-1)}=D_{ij}+I_{ij} & i=1,2 & \quad j=1,\cdots,12 \\
& I_{i0}=0 & i=1,2 & \\
& x_{ij},I_{ij}\geq 0 & i=1,2 & \quad j=1,\cdots,12
\end{aligned}
$$

Question 6.

(a)

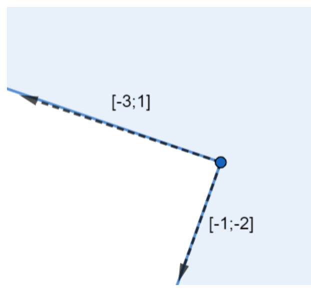


(b) Let's suppose that $A = [0; 6]$ is the unique optimal solution. Then, $-c = (-a, -b)$ should lie in the following region:



An example of such vector is $-c = (-4, 1)$. Then, we obtain $a = 4$ and $b = -1$. Then, the unique optimal solution is $A = [0; 6]$. Hence, optimal value becomes: $4 \times 0 + (-1) \times 6 = -6$

(c) $-c$ should lie in the following shaded region for the LP to be unbounded:

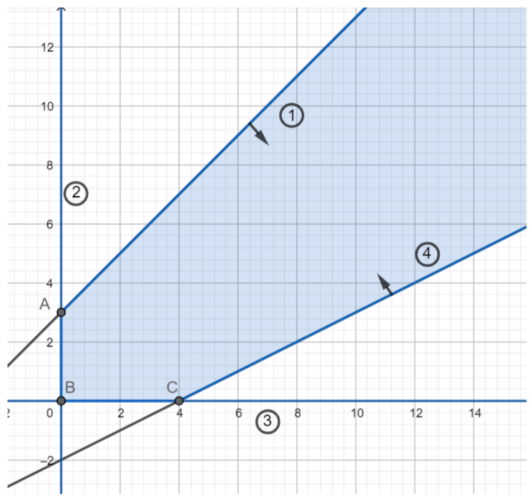


For instance; $-c = (1, 1)$ Then, $a = -1, b = -1$.

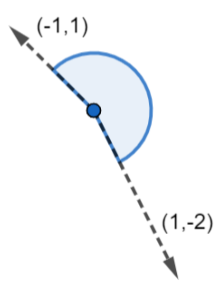
(d) There are two possible cases to obtain alternate solutions with only one extreme point optimal solution :

- **Case1**: -c=(-3,1). Then, point A is the only extreme point.
- **Case2**: -c=(-1,-2). Then, point B is the only extreme point.

Question 7.

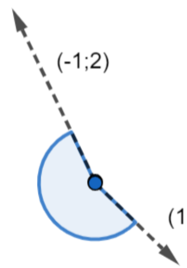


(a) First consider **LP1** (minimization problem):
$-c = (-c_1, -c_2)$ should be in the following shaded region for LP1 to be unbounded:



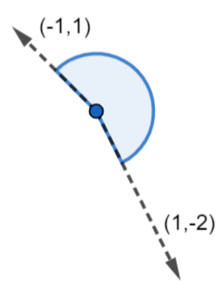
Then c lies in the following shaded region (each vector above is mutliplied by -1 for transforming $-c \to c$):

Figure 1:

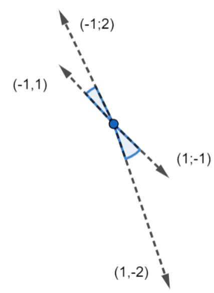


Now consider **LP2** (Maximization problem)
$c = (c_1, c_2)$ should be in the following shaded region for LP2 to be unbounded:

Figure 2:



Then, LP1 and LP2 is unbounded if c is in the following shaded region (intersection of the regions in Figure 1 (the vectors c that makes LP1 unbounded) and Figure 2 (the vectors c that makes LP2 unbounded)):



As an example, we can take $c = (1, -3/2)$ that makes both LP1 and LP2 unbounded (You can also choose any other c that is part of the shaded region of the above figure).

(b) LP1 has alternate optimal solutions for the following values of $-c$:

$$(-1, 1), (-1, 0), (0, -1), (1, -2)$$

which are perpendicular to lines 1, 2, 3 and 4 respectively.

Then c takes the following values:

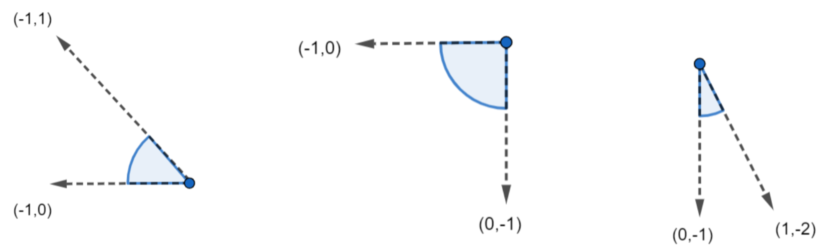
$$(1, -1), (1, 0), (0, 1), (-1, 2)$$

From part (a), it is known that c should be as in Figure 2 for LP2 to be unbounded:

Hence, c can take all the values above for LP1 to have alternate optimal solutions and LP2 to be unbounded. Hence, c can take:

$$(1, -1), (1, 0), (0, 1), (-1, 2)$$

(c) LP1 has unique optimal solution, if $-c$ is in one of the following three regions that make points A,B and C unique optimal solution respectively:



- Take $c = (1, -1/2)$. Then, $-c = (-1, 1/2)$. Then A becomes the unique solution for LP1 where LP2 is unbounded (see Figure 2)
- Take $c = (1, 1)$. Then, $-c = (-1, -1)$. This makes B the unique solution for LP1, and LP2 unbounded (see Figure 2)
- Take $c = (-1, 3)$. Then, $-c = (1, -3)$. Then C becomes the unique solution for LP1 where LP2 is unbounded (see Figure 2)

Question 8.

(a) **False**. Consider $f(x) = \sin(x) + 9.5$ where clearly f is continuous. Let $x_1 = \frac{\pi}{6}$ and $x_2 = \frac{5\pi}{6}$, then as $f(x_1) = 0.5 + 9.5 \leq 10$ and $f(x_2) = 0.5 + 9.5 \leq 10$ we have $x_1, x_2 \in S$. Now let $\lambda = \frac{1}{2}$ and $z = \lambda x_1 + (1-\lambda)x_2$, hence we have $z = \frac{\pi}{2}$ and $f(z) = 1 + 9.5 > 10$. Thus $z \notin S$, so S does not contain a convex combination of two of its elements hence S is not a convex set.

(b) **True**. Let $x, y \in S$. We need to show that any convex combination of x and y is also an element of S. To that end let $z = \lambda x + (1-\lambda)y$ for some $\lambda \in [0, 1]$. As $x, y \in S$ we have that $Ax \leq b$ and $Ay \leq b$.

$$\begin{aligned} Ax \leq b &\implies \lambda Ax \leq \lambda b \\ Ay \leq b &\implies (1-\lambda)Ay \leq (1-\lambda)b \end{aligned} \tag{1}$$

Sum these constraints up:

$$\lambda Ax + (1-\lambda)Ay \leq \lambda b + (1-\lambda)b \implies A(\lambda x + (1-\lambda)y) \leq b$$

Thus $\lambda x + (1-\lambda)y \in S$ and S is a convex set.

(c) **True**. From part (b) we know that both S_1 and S_2 are convex sets. Let $S = S_1 \cap S_2$ and $x, y \in S$. Then, $x, y \in S_1$ and $x, y \in S_2$. Further let $z = \lambda x + (1-\lambda)y$. As $x, y \in S_1$ and S_1 is a convex set we have $z \in S_1$. Similarly as S_2 is a convex set we also have $z \in S_2$. Thus $z \in S_1 \cap S_2 = S$ hence S is a convex set.