

IE 400: Principles of Engineering Management

Study Set 2 Solutions

Summer 2020

Question 1.

•

x_1, x_2 basic,

x_3, x_4 nonbasic ($x_3 = 0, x_4 = 0$)

Then,

$$x_1 - x_2 = 15$$

$$x_1 + x_2 = 9$$

Solving this system, we obtain $x = (12, -3, 0, 0)$. This is basic but not feasible since $x_2 = -3 < 0$.

•

x_1, x_3 basic,

x_2, x_4 nonbasic ($x_2 = 0, x_4 = 0$)

Then,

$$x_1 + 3x_3 = 15$$

$$x_1 = 9$$

Solving this system, we obtain $x = (9, 0, 2, 0)$. This is a basic feasible solution.

•

x_1, x_4 basic,

x_2, x_3 nonbasic ($x_2 = 0, x_3 = 0$)

Then,

$$x_1 + 5x_4 = 15$$

$$x_1 + x_4 = 9$$

Solving this system, we obtain $x = (15/2, 0, 0, 3/2)$. This is a basic feasible solution.

•

x_2, x_3 basic,

x_1, x_4 nonbasic ($x_1 = 0, x_4 = 0$)

Then,

$$-x_2 + 3x_3 = 15$$

$$x_2 = 9$$

Solving this system, we obtain $x = (0, 9, 8, 0)$. This is a basic feasible solution.

•

x_2, x_4 basic,

x_1, x_3 nonbasic ($x_1 = 0, x_3 = 0$)

Then,

$$-x_2 + 5x_4 = 15$$

$$x_2 + x_4 = 9$$

Solving this system, we obtain $x = (0, 5, 0, 4)$. This is a basic feasible solution.

•

x_3, x_4 basic,

x_1, x_2 nonbasic ($x_1 = 0, x_2 = 0$)

Then,

$$3x_3 + 5x_4 = 15$$

$$x_4 = 9$$

Solving this system, we obtain $x = (0, 0, -10, 9)$. This is basic but not feasible since $x_3 = -10 < 0$.

Question 2.

First Tableau:

x_1	x_2	e_1	e_2	x_3	x_4	RHS
-2	-3	0	0	-M	-M	0
2	1	-1	0	1	0	4
1	-1	0	-1	0	1	1

x_3 and x_4 are basic variables but their row zero coefficients are not zero so we need to add the first and the second row to row zero after multiplying by M:

x_1	x_2	e_1	e_2	x_3	x_4	RHS
3M-2	-3	-M	-M	0	0	5M
2	1	-1	0	1	0	4
1	-1	0	-1	0	1	1

Then, x_1 enters x_4 leaves.

Second Tableau:

x_1	x_2	s_1	e_1	x_3	x_4	RHS
0	0	0	0	-1	-1	0
1	1	1	0	0	0	3
2	1	0	-1	1	0	4
1	1	0	0	0	1	3

x_3 and x_4 are basic variables but their row zero coefficients are not zero so we need to add the second and the third row to row zero:

x_1	x_2	s_1	e_1	x_3	x_4	RHS
3	2	0	-1	0	0	7
1	1	1	0	0	0	3
2	1	0	-1	1	0	4
1	1	0	0	0	1	3

Then, x_1 enters x_3 leaves.

Third Tableau:

x_1	x_2	x_3	s_1	s_2	e_1	RHS
-2	-1	-1	0	0	0	0
1/3	2/3	0	1	0	1/3	1
2	1	0	0	1	0	3
2/3	1/3	1	0	0	-1/3	1

x_3 is a basic variable but its row zero coefficient is not zero so we need to add the third row to row zero:

x_1	x_2	x_3	s_1	s_2	e_1	RHS
-4/3	-2/3	0	0	0	-1/3	1
1/3	2/3	0	1	0	1/3	1
2	1	0	0	1	0	3
2/3	1/3	1	0	0	-1/3	1

Then, x_1 enters and s_2 leaves (or x_3 leaves).

Question 3. First convert the LP into standard form:

$$\begin{aligned}
 \max \quad & 5x_1 + 3x_2 + x_3 \\
 \text{s.t.} \quad & x_1 + x_2 + x_3 + s_1 = 6 \\
 & 5x_1 + 3x_2 + 6x_3 + s_2 = 15 \\
 & x_i \geq 0 \quad i = 1, 2, 3 \\
 & s_1, s_2 \geq 0
 \end{aligned}$$

	z	x_1	x_2	x_3	s_1	s_2	RHS
1	1	-5	-3	-1	0	0	0
s_1	0	1	1	3	1	0	6
s_2	0	5	3	6	0	1	15

x_1 enters. Apply MRT: $\min\{6/1, 15/5\}=3$ Hence, s_2 leaves.

	z	x_1	x_2	x_3	s_1	s_2	RHS
1	1	0	0	5	0	1	15
s_1	0	0	2/5	9/5	1	-1/5	3
x_1	0	1	3/5	6/5	0	1/5	3

Since there is no negative row 0 coefficient, the solution is optimal: $x^* = (3, 0, 0)$ and $z^* = 15$.

Since there exists 0 in row 0, there may be alternative optimal solutions:
 Let x_2 enter the basis. Apply MRT: $\min\{\frac{3}{2/5}, \frac{3}{3/5}\} = 5$. Then x_1 leaves.

	z	x_1	x_2	x_3	s_1	s_2	RHS
1	1	0	0	5	0	1	15
s_1	0	-2/3	0	1	1	-1/3	1
x_2	0	5/3	1	2	0	1/3	5

$x^* = (0, 5, 0)$ and $z^*=15$.

(If x_1 enters again, you turn back to the previous tableau!)

Question 4. Introduce slack variables s_1 , s_2 and s_3 . Then the LP in the standard form becomes:

$$\begin{aligned}
 \max \quad & 2x_1 + x_2 + 6x_3 - 4x_4 \\
 \text{s.t.} \quad & x_1 + 2x_2 + 4x_3 - x_4 + s_1 = 6 \\
 & 2x_1 + 3x_2 - x_3 + x_4 + s_2 = 12 \\
 & x_1 + x_3 + x_4 + s_3 = 6 \\
 & x_i \geq 0 \quad i = 1, 2, 3, 4, 5, 6, 7
 \end{aligned}$$

Now construct the simplex tableau for initial basis s_1 , s_2 and s_3 . Now we will add the required variables to basis one by one. (Be careful! The following are not Simplex iterations to find the optimal solution!)

	z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
1	1	-2	-1	-6	4	0	0	0	0
s_1	0	1	2	4	-1	1	0	0	6
s_2	0	2	3	-1	1	0	1	0	12
s_3	0	1	0	1	1	0	0	1	6

Let x_1 enter and s_1 leave.

	z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
1	1	0	3	2	2	2	0	0	12
x_1	0	1	2	4	-1	1	0	0	6
s_2	0	0	-1	-9	3	-2	1	0	0
s_3	0	0	-2	-3	2	-1	0	1	0

Let x_2 enter and s_2 leave.

	z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
1	1	0	0	-25	11	-4	3	0	12
x_1	0	1	0	-14	5	-3	2	0	6
x_2	0	0	1	9	-3	2	-1	0	0
s_3	0	0	0	15	-4	3	-2	1	0

Let x_4 enter and s_3 leave.

	z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
1	1	0	0	65/4	0	17/4	-5/2	11/4	12
x_1	0	1	0	19/4	0	3/4	-1/2	5/4	6
x_2	0	0	1	-9/4	0	-1/4	1/2	-3/4	0
x_4	0	0	0	-15/4	1	-3/4	1/2	-1/4	0

The Simplex tableau is constructed for basis x_1 , x_2 and x_4 . This tableau is not optimal because there is a negative coefficient in row zero. Hence, s_2 enters.

To find the variable that leaves the basis, apply MRT: $\min\{0, 0\} = 0$. Hence, x_2 (or x_4) leaves.

Question 5.

- (a) The tableau is final and there exists a unique optimal solution.

$$A < 0, B = 0, C \text{ any}, D \geq 0, E = 1, F \text{ any}$$

(b) The simplex method determines an unbounded solution from this tableau.

$$A > 0, B = 0, C \leq 0, D \geq 0, E = 1, F \text{ any}$$

(c) The current bfs is degenerate (not necessarily optimal).

$$A \text{ any}, B = 0, C \text{ any}, D = 0, E = 1, F \text{ any}$$

(d) d) The current solution is optimal, there are alternative optimal solutions but no alternative optimal bfs.

$$A = 0, B = 0, C \leq 0, D \geq 0, E = 1, F \text{ any}$$

Question 6. Convert into standard form and add artificial variable.

$$\begin{aligned} \min \quad & -3x_1 + 4x_2 \\ \text{s.t.} \quad & x_1 + x_2 + s_1 = 4 \\ & 2x_1 + 3x_2 - e_1 + a_1 = 18 \\ & x_1, x_2, s_1, e_1, a_1 \geq 0 \end{aligned}$$

PHASE 1

	z	x_1	x_2	s_1	e_1	a_1	RHS
1	1	0	0	0	0	-1	0
s_1	0	1	1	1	0	0	4
a_1	0	2	3	0	-1	1	18

Add second row to row 0.

	z	x_1	x_2	s_1	e_1	a_1	RHS
1	1	2	3	0	-1	0	18
s_1	0	1	1	1	0	0	4
a_1	0	2	3	0	-1	1	18

x_2 enters. Apply MRT: $\min\{4/1, 18/2\} = 4$. Then, s_1 leaves the basis.

	z	x_1	x_2	s_1	e_1	a_1	RHS
1	1	-1	0	-3	-1	0	6
x_2	0	1	1	1	0	0	4
a_1	0	-1	0	-3	-1	1	6

The tableau is optimal, but a_1 is still in the basis. Hence, the Phase 1 problem is infeasible.

Question 7. Convert the following LP to standard form and solve with the Big-M method.

$$\begin{aligned}
& \max -x_1 + 3x_2 + x_3 \\
& \text{s.t.} \quad -x_1 + x_2 - x_3 \geq -4 \\
& \quad \quad 2x_1 + x_2 + x_3 \geq 10 \\
& \quad \quad x_1 \geq 0 \\
& \quad \quad x_2 \leq 0 \\
& \quad \quad x_3 \text{ free}
\end{aligned}$$

Let $x_3 = x'_3 - x''_3$ where $x'_3 \geq 0$ and $x''_3 \geq 0$. Let $-x_2 = x'_2$. Introduce slack variable s_1 and excess variable e_2 .

$$\begin{aligned}
& \max -x_1 - 3x'_2 + (x'_3 - x''_3) - Ma_2 \\
& \text{s.t.} \quad x_1 + x'_2 + (x'_3 - x''_3) + s_1 = 4 \\
& \quad \quad 2x_1 - x'_2 + (x'_3 - x''_3) - e_2 + a_2 = 10 \\
& \quad \quad x_1, x'_2, x'_3, x''_3, s_1, e_2 \geq 0
\end{aligned}$$

	z	x_1	x'_2	x'_3	x''_3	s_1	e_2	a_2	RHS
	1	1	3	-1	1	0	0	M	0
s_1	0	1	1	1	-1	1	0	0	4
a_1	0	2	-1	1	-1	0	-1	1	10

Multiply the second row with $-M$ and add it to row 0:

	z	x_1	x'_2	x'_3	x''_3	s_1	e_2	a_2	RHS
	1	$-2M+1$	$M+3$	$-M-1$	$M+1$	0	M	0	$-10M$
s_1	0	1	1	1	-1	1	0	0	4
a_1	0	2	-1	1	-1	0	-1	1	10

x_1 enters the basis. By minimum ratio test: $\min\{4/1\} = 4$, s_1 leaves:

	z	x_1	x'_2	x'_3	x''_3	s_1	e_2	a_2	RHS
	1	0	$3M+2$	$M-2$	$-M+2$	$2M-1$	M	0	$-2M-4$
x_1	0	1	1	1	-1	1	0	0	4
a_1	0	0	-3	-1	1	-2	-1	1	2

x''_3 enters the basis. Apply MRT: $\min\{4, 2\}$. Then, a_1 leaves:

	z	x_1	x'_2	x'_3	x''_3	s_1	e_2	a_2	RHS
	1	0	8	0	0	3	2	$M-2$	-8
x_1	0	1	-2	0	0	-1	-1	1	6
x''_3	0	0	-3	-1	1	-2	-1	1	2

No negative coefficient in row 0. The tableau is final. (a_2 is not in the basis so remove the column a_2 from the tableau)

Then,

$$x^* = (x_1, x'_2, x'_3, x''_3, s_1, e_2) = (6, 0, 0, 2, 0, 0)$$

and $z^* = -8$. Note that $x_2 = -x'_2 = 0$, $x_3 = x'_3 - x''_3 = 0 - 2 = -2$.

Question 8.

Two-Phase Method:

The phase-I problem is the following:

$$\begin{aligned} \min \quad & a_1 + a_2 \\ \text{s.t.} \quad & 2x_1 + x_2 + 3x_3 + a_1 = 60 \\ & 3x_1 + 3x_2 + 5x_3 - e_1 + a_2 = 120 \\ & x_1, x_2, x_3, e_1, a_1, a_2 \geq 0 \end{aligned}$$

Now we need to solve phase-I LP:

Tableau 0:

z	x ₁	x ₂	x ₃	e ₁	a ₁	a ₂	RHS
1	0	0	0	0	-1	-1	0
0	2	1	3	0	1	0	60
0	3	3	5	-1	0	1	120

The simplex tableau above is not in proper format since the row zero coefficients of basic variables a_1 and a_2 are not zero. We add the first and second row to row zero to make these coefficients zero:

Tableau 0:

z	x ₁	x ₂	x ₃	e ₁	a ₁	a ₂	RHS
1	5	4	8	-1	0	0	180
0	2	1	3	0	1	0	60
0	3	3	5	-1	0	1	120

 60/3
 120/5

Now we can start simplex iterations: since it is a minimization problem we are looking for positive coefficients to decide on entering variables.

x_3 enters and a_1 leaves:

Tableau 1:

z	x ₁	x ₂	x ₃	e ₁	a ₁	a ₂	RHS
1	-1/3	4/3	0	-1	-8/3	0	20
0	2/3	1/3	1	0	1/3	0	20
0	-1/3	4/3	0	-1	-5/3	1	20

 20/(1/3)
 20/(4/3)

x_2 enters and a_2 leaves:

Tableau 2:

z	x ₁	x ₂	x ₃	e ₁	a ₁	a ₂	RHS
1	0	0	0	0	-1	-1	0
0	3/4	0	1	1/4	3/4	-1/4	15
0	-1/4	1	0	-3/4	-5/4	3/4	15

Both a₁ and a₂ left the basis so we can delete their columns from the tableau and move onto phase-II. We write the original objective function coefficients to the tableau:

Tableau 0:

z	x ₁	x ₂	x ₃	e ₁	RHS
1	-3	-2	-4	0	0
0	3/4	0	1	1/4	15
0	-1/4	1	0	-3/4	15

Again, notice that this tableau is not in proper format, we need to make row zero coefficients of basic variables x₂ and x₃ zero:

Tableau 0:

z	x ₁	x ₂	x ₃	e ₁	RHS
1	-1/2	0	0	-1/2	90
0	3/4	0	1	1/4	15
0	-1/4	1	0	-3/4	15

Stop. The tableau is optimal since there is no positive entry in row zero.

The optimal solution is x₁^{*}=0, x₂^{*}=x₃^{*}=15 with objective value 90.

Now let's solve the same problem with Big-M method.

Big-M method:

The LP is the following:

$$\begin{aligned}
 \min \quad & 3x_1 + 2x_2 + 4x_3 + Ma_1 + Ma_2 \\
 \text{s.t.} \quad & 2x_1 + x_2 + 3x_3 + a_1 = 60 \\
 & 3x_1 + 3x_2 + 5x_3 - e_1 + a_2 = 120 \\
 & x_1, x_2, x_3, e_1, a_1, a_2 \geq 0
 \end{aligned}$$

Tableau 0:

z	x_1	x_2	x_3	e_1	a_1	a_2	RHS
1	-3	-2	-4	0	-M	-M	0
0	2	1	3	0	1	0	60
0	3	3	5	-1	0	1	120

Let's put the tableau in the proper format first:

Tableau 0:

z	x_1	x_2	x_3	e_1	a_1	a_2	RHS
1	$5M-3$	$4M-2$	$8M-4$	-M	0	0	$180M$
0	2	1	3	0	1	0	60
0	3	3	5	-1	0	1	120

→ $60/3$
 $120/5$

x_3 enters and a_1 leaves:

Tableau 1:

z	x_1	x_2	x_3	e_1	a_1	a_2	RHS
1	$-(M+1)/3$	$(4M-2)/3$	0	-M	$(4-8M)/3$	0	$80+20M$
0	$2/3$	$1/3$	1	0	$1/3$	0	20
0	$-1/3$	$4/3$	0	-1	$-5/3$	1	20

→ $20/(1/3)$
 $20/(4/3)$

x_2 enters and a_2 leaves:

Tableau 2:

z	x_1	x_2	x_3	e_1	a_1	a_2	RHS
1	$-1/2$	0	0	$-1/2$	$1/2-M$	$1/2-M$	90
0	$3/4$	0	1	$1/4$	$3/4$	$-1/4$	15
0	$-1/4$	1	0	$-3/4$	$-5/4$	$3/4$	15

Stop. The tableau is optimal since there is no positive entry in row zero.

The optimal solution is $x_1^*=0$, $x_2^*=x_3^*=15$ with objective value 90.