IE 400: Principles of Engineering Management
Study Set 3 Solutions

Summer 2020

Question 1.

(a) **Parameters:**

f_i= fixed cost of plant i, $i = 1, \ldots, 4$

c_{ij}= cost of producing car j at plant i $i = 1, \ldots, 4$, $j = 1, 2, 3$

Decision Variables:

$$y_i = \begin{cases} 1 & \text{if plant i is used} \quad i = 1, \ldots, 4 \\ 0 & \text{o.w} \end{cases}$$

$$x_{ij} = \begin{cases} 1 & \text{if car j is produced at plant i} \quad i = 1, \ldots, 4, \quad j = 1, 2, 3 \\ 0 & \text{o.w} \end{cases}$$

Model:

$$\begin{aligned}
\min \ & \sum_{i=1}^{4} f_i y_i + \sum_{i=1}^{4} \sum_{j=1}^{3} c_{ij} x_{ij} \\
\text{s.t.} \ & \sum_{j=1}^{3} x_{ij} \leq 1 && i = 1, 2, 3, 4 && (1) \\
& \sum_{i=1}^{4} x_{ij} = 1 && j = 1, 2, 3 && (2) \\
& y_3 + y_4 \leq 1 + y_1 && && (3) \\
& x_{ij} \leq y_i && i = 1, 2, 3, 4 \quad j = 1, 2, 3 \\
& x_{ij} \in \{0, 1\} && i = 1, 2, 3, 4 \quad j = 1, 2, 3 \\
& y_j \in \{0, 1\} && j = 1, 2, 3
\end{aligned}$$

Constraint (1): Each plant can produce one type of car.
Constraint (2): The total production of each type of car must be at a single plant; that is, for example, if any Tauruses are made at plant 1, then all Tauruses must be made there.
Constraint (3): If plants 3 and 4 are used, then plant 1 must also be used.

(b)

either	$y_1 + y_2 \leq 1$	(*)
or	$y_3 + y_4 \geq 1$	(**)

In order to linearize the constraint, add the following decision variable:

$$w = \begin{cases} 1 & \text{if constraint (*) holds} \\ 0 & \text{if constraint (**) holds} \end{cases}$$

Then add the following constraints to the model:

$$y_1 + y_2 \leq 1w - (1 - w)M$$
$$y_3 + y_4 \geq 1(1 - w) + Mw$$
$$w \in \{0, 1\}$$

where M is a sufficiently large number.

(For $w = 1$, the constraint (*) holds and (**) becomes redundant. For $w = 0$, the constraint (**) holds and (*) becomes redundant.)

Question 2.

Decision Variables:

$$x_{ij} = \begin{cases} 1 & \text{if astronaut i is assigned to post j } i = 1,2,3,4,\ j = 1,2,3,4 \\ 0 & \text{o.w} \end{cases}$$

Model:

$$\begin{aligned} \max \ & \sum_{i=1}^{4} \sum_{j=1}^{4} p_{ij} x_{ij} \\ \text{s.t.} \ & \sum_{i=1}^{4} x_{ij} = 1 && j = 1,2,3,4 && (1) \\ & \sum_{j=1}^{4} x_{ij} = 1 && i = 1,2,3,4 && (2) \\ & x_{ij} \in \{0,1\} && i = 1,2,3,4 \quad j = 1,2,3,4 \end{aligned}$$

- Constraint (1) indicates that each post should have a single astronaut.
- Constraint (2) indicates that each astronaut should be assigned to a single post.

Question 3.

Parameters: As given in the question

Decision Variables: As given in the question

Model:

$$
\begin{aligned}
\min \ & \sum_{i=1}^{M} x_i \\
\text{s.t.} \ & \sum_{i=1}^{M} y_{ik} = d_k & & k = 1, \cdots, n \\
& \sum_{k=1}^{n} y_{ik} \cdot l_k \leq l \cdot x_i & & i = 1, \cdots, M \\
& x_i \in \{0, 1\} & & i = 1, \cdots, M \\
& y_{ik} \geq 0 & & i = 1, \cdots, M \qquad k = 1, \cdots, n \\
& y_{ik} \text{ integer} & & i = 1, \cdots, M \qquad k = 1, \cdots, n
\end{aligned}
$$

Question 4.

(a) **Parameters:**

w_j= weight capacity of compartment j (in tons)

s_j= space capacity of compartment j (in cubic meters)

wt_i= weight of cargo i (in tons)

v_i= weight of cargo i (in cubic meters/ton)

p_i= profit gained from cargo i (TL/ton)

Decision Variables:

x_{ij}= amount of cargo i distributed to compartment j in tons

$$
\begin{aligned}
\max\ & \sum_{i=1}^{4}\sum_{j=1}^{3} p_i x_{ij} \\
\text{s.t.}\ & \sum_{j=1}^{3} x_{ij} \le wt_i & i = 1,2,3,4 \qquad (1)\\
& \sum_{i=1}^{4} v_i x_{ij} \le s_j & j = 1,2,3 \qquad (2)\\
& \sum_{i=1}^{4} x_{ij} \le w_j & j = 1,2,3 \qquad (3)\\
& \frac{\sum_{i=1}^{4} x_{i1}}{w_1} = \frac{\sum_{i=1}^{4} x_{i2}}{w_2} = \frac{\sum_{i=1}^{4} x_{i3}}{w_3} & (4)\\
& x_{ij} \ge 0 & i = 1,2,3,4 \quad j = 1,2,3
\end{aligned}
$$

(b) Replace constraint (4) with

$$
\left|\frac{\sum_{i=1}^{4} x_{i1}}{w_1} - \frac{\sum_{i=1}^{4} x_{i2}}{w_2}\right| \le 0.1
$$

$$
\left|\frac{\sum_{i=1}^{4} x_{i1}}{w_1} - \frac{\sum_{i=1}^{4} x_{i3}}{w_3}\right| \le 0.1
$$

$$
\left|\frac{\sum_{i=1}^{4} x_{i2}}{w_2} - \frac{\sum_{i=1}^{4} x_{i3}}{w_3}\right| \le 0.1
$$

You need to linearize these constraints before adding to the model:

$$
\begin{aligned}
-0.1 \le \frac{\sum_{i=1}^{4} x_{i1}}{w_1} - \frac{\sum_{i=1}^{4} x_{i2}}{w_2} \le 0.1\\
-0.1 \le \frac{\sum_{i=1}^{4} x_{i1}}{w_1} - \frac{\sum_{i=1}^{4} x_{i3}}{w_3} \le 0.1\\
-0.1 \le \frac{\sum_{i=1}^{4} x_{i2}}{w_2} - \frac{\sum_{i=1}^{4} x_{i3}}{w_3} \le 0.1
\end{aligned}
$$

(c) (b) is more likely to have a larger optimal value since the feasible region in (b) includes the feasible region in (a).

(d) **Decision Variables:**

x_{ij}= amount of cargo i distributed to compartment j in tons

$$y_{ij} = \begin{cases} 1 & \text{if cargo i is shipped in compartment j} \\ 0 & \text{o.w} \end{cases}$$

Model: All constraints in (b) +

$$\begin{aligned} & x_{ij} \leq w_j y_{ij} && i = 1, 2, 3, 4 \quad j = 1, 2, 3 \\ & \sum_{j=1}^{3} y_{ij} \leq 1 && i = 1, 2, 3, 4 \\ & y_{ij} \in \{0, 1\} && i = 1, 2, 3, 4 \quad j = 1, 2, 3 \\ & x_{ij} \text{ integer} && i = 1, 2, 3, 4 \quad j = 1, 2, 3 \end{aligned}$$

Question 5.

Decision Variables:

$$y_i = \begin{cases} 1 & \text{if constraint i is satisfied} \\ 0 & \text{o.w} \end{cases}$$

$$z_1 = \begin{cases} 1 & \text{if at least one of 7,8,9 is satisfied} \\ 0 & \text{o.w} \end{cases}$$

$$z_2 = \begin{cases} 1 & \text{if at least two of 10,11,12,13 are satisfied} \\ 0 & \text{o.w} \end{cases}$$

Model:

$$
\begin{aligned}
&\min 0 && && \\
&\text{s.t.} \sum_{j=1}^{n} a_{ij}x_j \le b_i + M(1-y_i) && i = 1,\dots,20 && (1)\\
&\sum_{j=1}^{n} a_{ij}x_j \ge b_i - My_i && i = 1,\dots,20 && (2)\\
&\sum_{i=1}^{20} y_i \ge 10 && && (3)\\
&y_1 \le y_4 && && (4)\\
&y_2 + y_3 \le 1 && && (5)\\
&y_5 = y_6 && && (6)\\
&y_7 + y_8 + y_9 \ge z_1 && && (7)\\
&y_{10} + y_{11} + y_{12} + y_{13} \ge 2z_2 && && (8)\\
&z_1 + z_2 \ge 1 && && (9)\\
&z_1, z_2 \in \{0,1\} && && (10)\\
&y_i \in \{0,1\} && i = 1,\dots,20 && (11)
\end{aligned}
$$

where M is a sufficiently large number. Observe that we minimize zero, which means that the model is a feasibility problem. If a feasible solution is found for the above model, the objective will take value 0.

Let's take a closer look at constraints (1) and (2):

- If $y_i = 1$:

  Constraint (1) becomes $\sum_{j=1}^{n} a_{ij}x_j \le b_i$ which means that the constraint i is satisfies.
  Constraint (2) becomes $\sum_{j=1}^{n} a_{ij}x_j \ge b_i - M$. This makes the constraint redundant as $-M$ is a really small number.

- If $y_i = 0$:

  Constraint (1) becomes $\sum_{j=1}^{n} a_{ij}x_j \le b_i + M$ which makes the constraint redundant as M is a really large number.

Constraint (2) becomes $\sum_{j=1}^{n} a_{ij}x_j \geq b_i$. This means constraint is not satisfied.

(3): Point x should satisfy at least 10 of the 20 constraints.
(4): If point x satisfies constraint 1, then it should also satisfy constraint 4.
(5): If point x satisfies constraint 2, then it should not satisfy constraint 3.
(6): If point x satisfies either both of the constraints 5 and 6 or neither of them.
(7)-(8)-(9): Point x should satisfy either at least one of the constraints 7,8,9 or at least two of constraints 10,11,12,13.

Question 6.

First calculate the ratios:

x_1 : 9/6

x_2 : 13/3

x_3 : 10/2

x_4 : 8/4

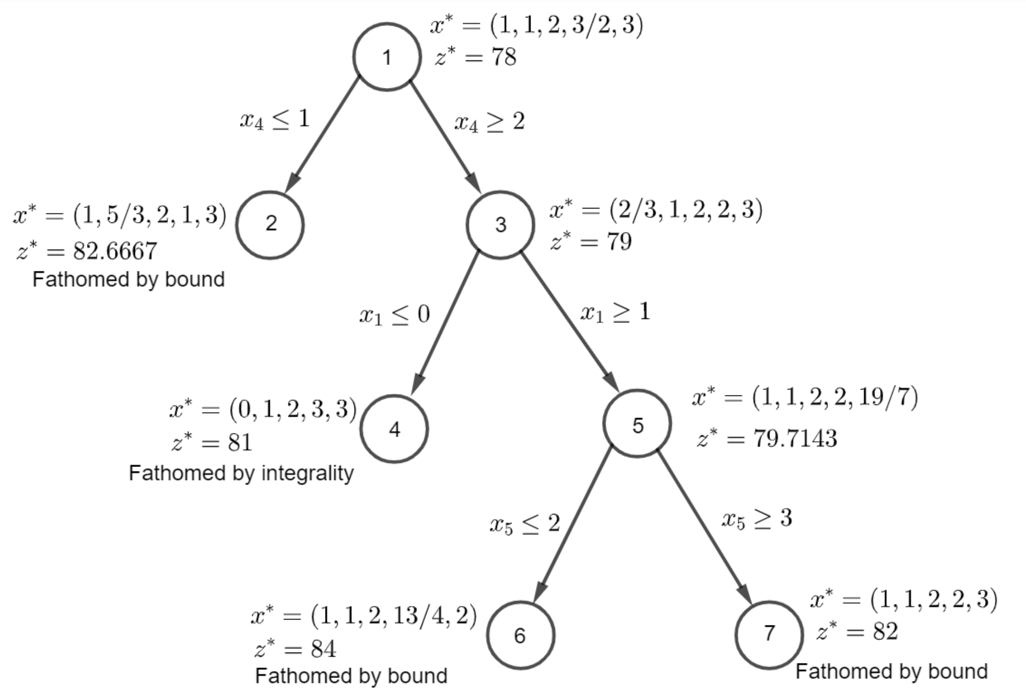
x_5 : 8/7

For the **minimization problem**, sort the ratios from smallest to largest:

$$x_5 < x_1 < x_4 < x_2 < x_3$$

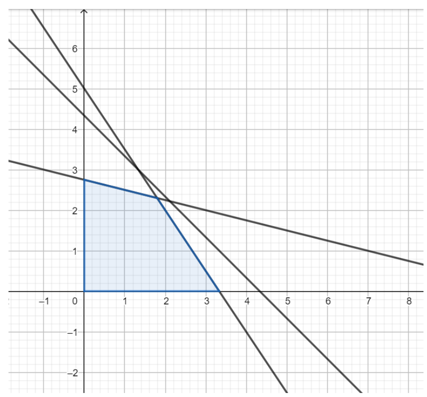
Solve the LP relaxation of the knapsack model according to the above priority list (x_5 is the most preferable and x_2 is the least preferable one). The solution to the LP relaxation of the given model is displayed at node 1. Since this is not an integer solution, we branch from node 1, and add new constraints as given in the Branch and Bound tree below. Note that at each node we solve the LP relaxation of the IP model along with the newly added constraints.
Note that node 2 is fathomed by bound, because we have found an integer solution at node 4 that has a better objective value than the one at node 2.

(If we branch node 2, then the resulting nodes would have an objective value ≥ 82.6667 which will always result in a solution worse than node 4)

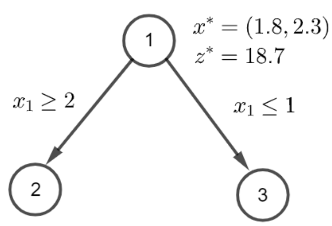


The optimal solution is $x^* = (0, 1, 2, 3, 3)$.

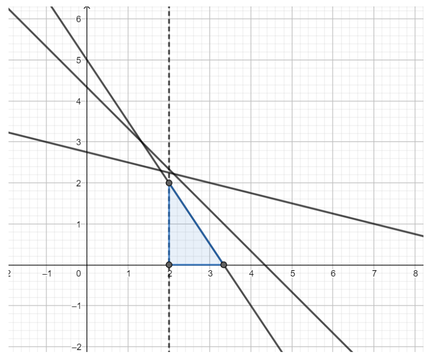
Question 7.
Solve the LP relaxation of the original model using graphical method:



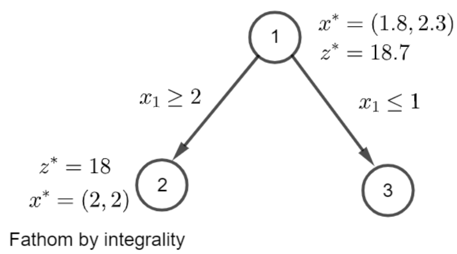
For c=(4,5), we obtain $x^* = (1.8, 2.3)$. This is not an integer solution, so start branching:



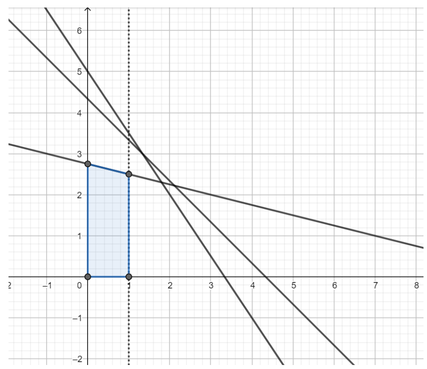
Now move to node 2. Here the constraint $x_1 \geq 2$ is added to the model:



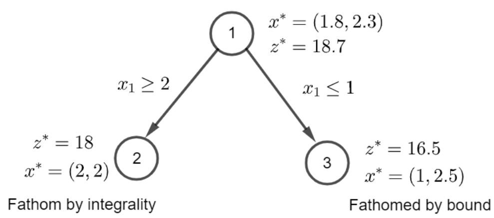
Solve the new model with graphical method and update the branch and bound tree:



We obtained an integer solution at node 2. Hence, it is fathomed by integrality. Now move to the third node. Add the constraint $x_1 \leq 1$ to the original model:



Solving via the graphical method we obtain:



Node 3 is fathomed by bound, because we have already obtained an integer solution at node 2 that has a better objective value (for maximization) than the model at node 3 . If we branch the third node, the objective value will be ≤ 16.5 !

We cannot make further branches. Then, the optimal solution is $x^* = (2, 2)$ and $z^* = 18$.