IE 400: Principles of Engineering Management
Study Set 5 Solutions

Summer 2020

Question 1. (a) First convert the LP into standard form:

$$\begin{aligned} \max\ & 5x_1 + 3x_2 \\ \text{s.t.}\ & 4x_1 + 2x_2 + s_1 = 12 \\ & 4x_1 + x_2 + s_2 = 10 \\ & x_1 + x_2 + s_3 = 4 \\ & x_1, x_2, s_1, s_2, s_3 \geq 0 \end{aligned}$$

For $x_1 = 0$ and $x_2 = 0$, the BFS is:

$$\text{x}=\begin{pmatrix} 0 \\ 0 \\ 12 \\ 10 \\ 4 \end{pmatrix}$$

(b)

	z	x_1	x_2	s_1	s_2	s_3	**RHS**
	1	-5	-3	0	0	0	0
s_1	0	4	2	1	0	0	12
s_2	0	4	1	0	1	0	10
s_3	0	1	1	0	0	1	4

x_1 enters. MRT=min$\left\{\frac{12}{4}, \frac{10}{4}, \frac{4}{1}\right\} = \frac{10}{4}$. Hence, s_2 leaves.

	z	x_1	x_2	s_1	s_2	s_3	**RHS**
	1	0	-7/4	0	5/4	0	25/2
s_1	0	0	1	1	-1	0	2
x_1	0	1	1/4	0	1/4	0	5/2
s_3	0	0	3/4	0	-1/4	1	3/2

MRT=min $\left\{\frac{2}{1}, \frac{5/2}{1/4}, \frac{3/2}{3/4}\right\} = 2$

There is a tie in MRT. Choose choose one of s_1 and s_3 arbitrarily. x_2 enters, s_1 leaves.

	z	x_1	x_2	s_1	s_2	s_3	**RHS**
	1	0	0	1	0	1	16
x_2	0	0	1	1	-1	0	2
x_1	0	1	0	-1/4	1/2	0	2
s_3	0	0	0	-3/4	1/2	1	0

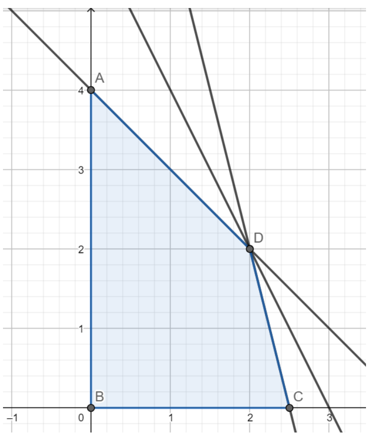
There is no negative value in row 0.

$z^* = 16$ and

$$x^* = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Since RHS of s_3 is 0, x^* is a degenerate basic feasible solution.

(c)



$B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $C = \begin{pmatrix} 5/2 \\ 0 \end{pmatrix}$ $D = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

The path followed by Simplex algorithm is B-C-D.

Question 2.

Decision Variables:

x_i : Number of workers employed on line i $\; i = 1, 2$

$$y_i = \begin{cases} 1 & \text{if line i is used } \; i = 1, 2 \\ 0 & \text{o.w} \end{cases}$$

Model:

$$\begin{aligned} \min \;& 1000y_1 + 2000y_2 + 500x_1 + 900x_2 \\ \text{s.t.} \;& 20x_1 + 50x_2 \geq 120 \\ & 30x_1 + 35x_2 \geq 150 \\ & 40x_1 + 45x_2 \geq 200 \\ & x_1 \leq 7y_1 \\ & x_2 \leq 7y_2 \\ & x_1, x_2 \geq 0 \\ & y_1, y_2 \in \{0, 1\} \end{aligned}$$

Question 3.

Let c_{ij} be the total net cost incurred from year i to j.

c_{ij}= maintenance cost incurred during years$i, i+1, \cdots, j-1$ +cost of purchasing a car at the beginning of year i- trade-in value received at the beginning of year j.

$c_{12} = 300 + 12000 - 7000 = 5300$
$c_{13} = 300 + 500 + 12000 - 6000 = 6800$
$c_{14} = 300 + 500 + 800 + 12000 - 4000 = 9600$
$c_{15} = 300 + 500 + 800 + 1200 + 12000 - 3000 = 11800$
$c_{16} = 300 + 500 + 800 + 1200 + 1600 + 12000 - 2000 = 13500$
$c_{17} = 300 + 500 + 800 + 1200 + 1600 + 2200 + 12000 - 1000 = 16700$

$c_{23} = 300 + 12000 - 7000 = 5300$

$c_{24} = 300 + 500 + 12000 - 6000 = 6800$

$c_{25} = 300 + 500 + 800 + 12000 - 4000 = 9600$

$c_{26} = 300 + 500 + 800 + 1200 + 12000 - 3000 = 11800$

$c_{27} = 300 + 500 + 800 + 1200 + 1600 + 12000 - 2000 = 13500$

$c_{34} = 300 + 12000 - 7000 = 5300$

$c_{35} = 300 + 500 + 12000 - 6000 = 6800$

$c_{36} = 300 + 500 + 800 + 12000 - 4000 = 9600$

$c_{37} = 300 + 500 + 800 + 1200^{+}12000 - 3000 = 11800$

$c_{45} = 300 + 12000 - 7000 = 5300$

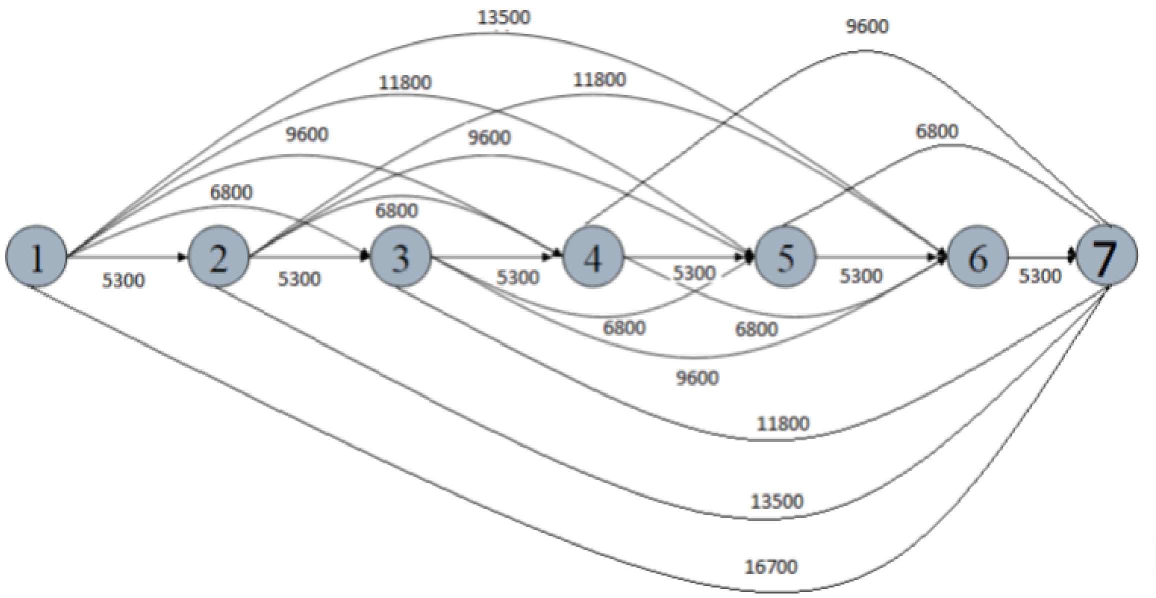
$c_{46} = 300 + 500 + 12000 - 6000 = 6800$

$c_{47} = 300 + 500 + 800 + 12000 - 4000 = 9600$

$c_{56} = 300 + 12000 - 7000 = 5300$

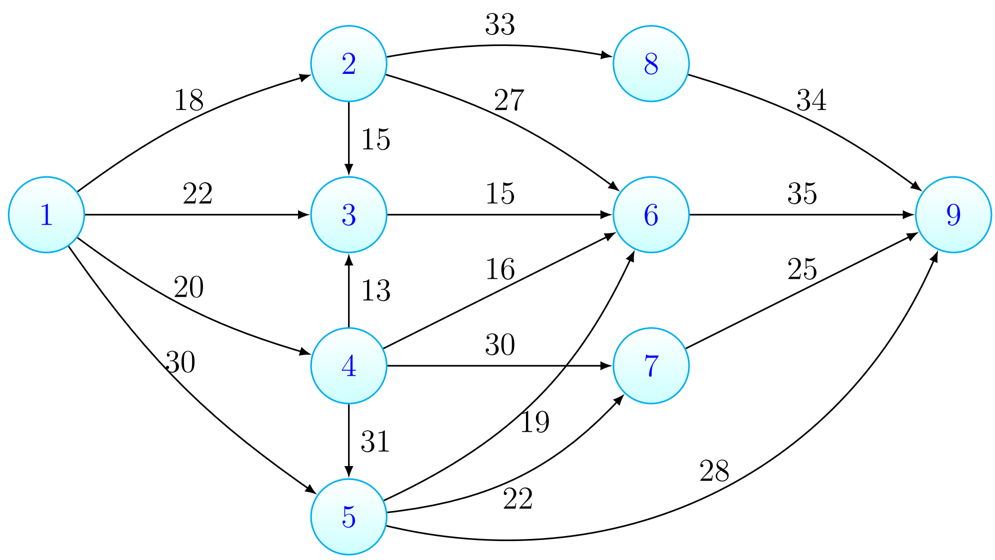
$c_{57} = 300 + 500 + 12000 - 6000 = 6800$

$c_{67} = 300 + 12000 - 7000 = 5300$



Replacing the car at years 2,4 and 6 yields the least costly solution.

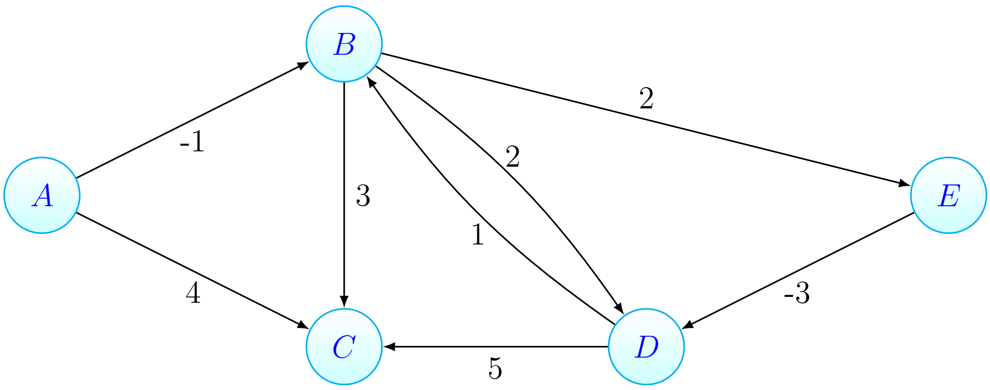
Question 4. Find shortest paths from 1 to all other nodes using Dijkstra's algorithm.



p	**v[1]**	**v[2]**	**v[3]**	**v[4]**	**v[5]**	**v[6]**	**v[7]**	**v[8]**	**v[9]**
(init)	0	∞	∞	∞	∞	∞	∞	∞	∞
1	(perm)	18	22	20	30				
2		(perm)				45		51	
4				(perm)		36	50		
3			(perm)						
5					(perm)				58
6						(perm)			
7							(perm)		
8								(perm)	
9									(perm)
(final)	0	18	22	20	30	36	50	51	58

p	**d[1]**	**d[2]**	**d[3]**	**d[4]**	**d[5]**	**d[6]**	**d[7]**	**d[8]**	**d[9]**
1	(perm)	1	1	1	1				
2		(perm)				2		2	
4				(perm)		4	4		
3			(perm)						
5					(perm)				5
6						(perm)			
7							(perm)		
8								(perm)	
9									(perm)
(final)	-	1	1	1	1	4	4	2	5

Question 5. Use Bellman Ford algorithm to find the shortest path of the following graph where A is the source node.

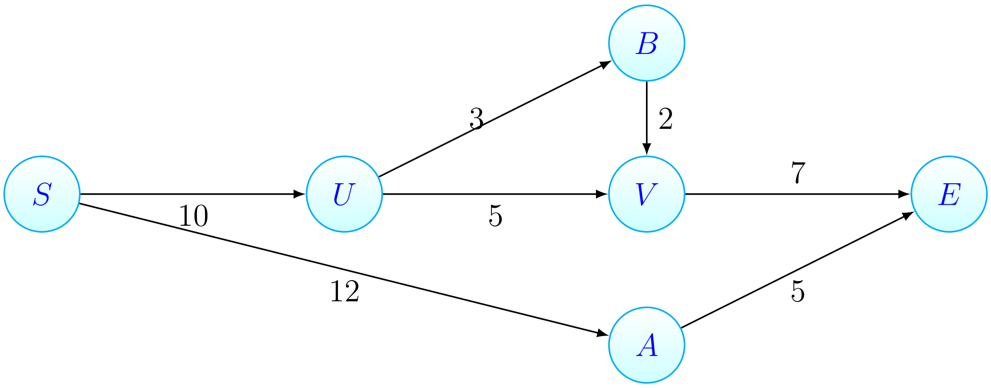


$$V_{[k]}^{(t)} = \min\{V_{[i]}^{t-1} + c_{ik}\}$$

t	$V_{(A)}^t$	$V_{(B)}^t$	$V_{(C)}^t$	$V_{(D)}^t$	$V_{(E)}^t$	d(A)	d(B)	d(C)	d(D)	d(E)
0	0	$+\infty$	$+\infty$	$+\infty$	$+\infty$					
1	0	-1	4	$+\infty$	$+\infty$		A	A		
2	0	-1	2	1	1			B	B	B
3	0	-1	2	-2	1				E	

Question 6. (a)

The principal of optimality does not apply. See the following **counter example**:



Since we are trying to minimize the highest height encountered: Shortest path from S to E is S-U-V-E=10 (OR S-U-B-V-E =10) **but** shortest path from U to V is U-B-V=3, not U-V=5.

(b)

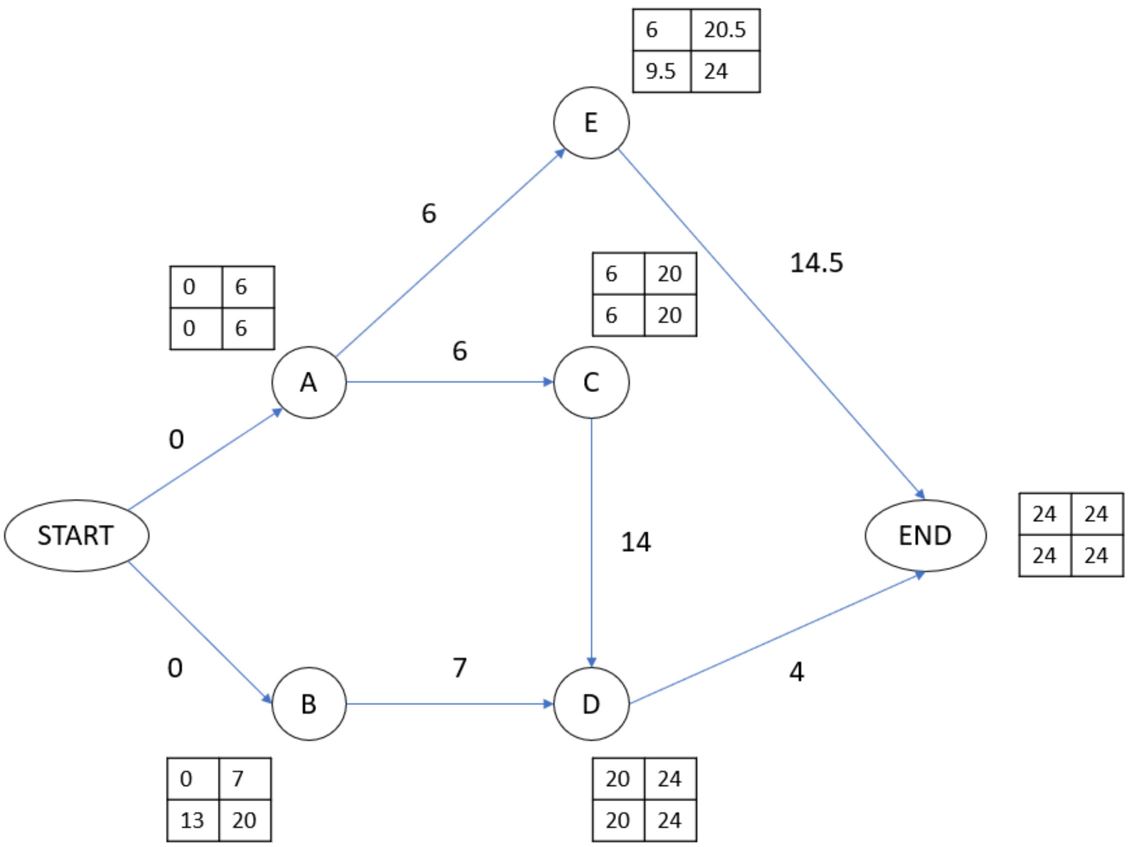
$$V[S] = 0$$

$$V[i] = \min_{(i,k)\in A} \{\max\{V[k], c_{ik}\}\}$$

(c) You can adapt Dijkstra's algorithm with the functional equations in part (b).

Question 7. (a)

Activity	**Mean**	**Variance**
A	6	4/36
B	7	196/36
C	14	256/36
D	4	4/36
E	14.5	169/36



The critical path is A-C-D.

(b) Note that

$$\mathbb{P}(18 \leq t \leq 26) = \mathbb{P}(t \leq 26) - \mathbb{P}(t \leq 18)$$

We have $\sigma_p^2 = \frac{4}{36} + \frac{196}{36} + \frac{256}{36} + \frac{4}{36} + \frac{169}{36} = 17.4722$

Then, $\sigma_p = 4.18$

Note that $Z_{18} = \frac{18-24}{4.18} = -1.4354$

Then, $\mathbb{P}(t \leq 18) = \mathbb{P}(Z \leq -1.4354) = 0.0757$

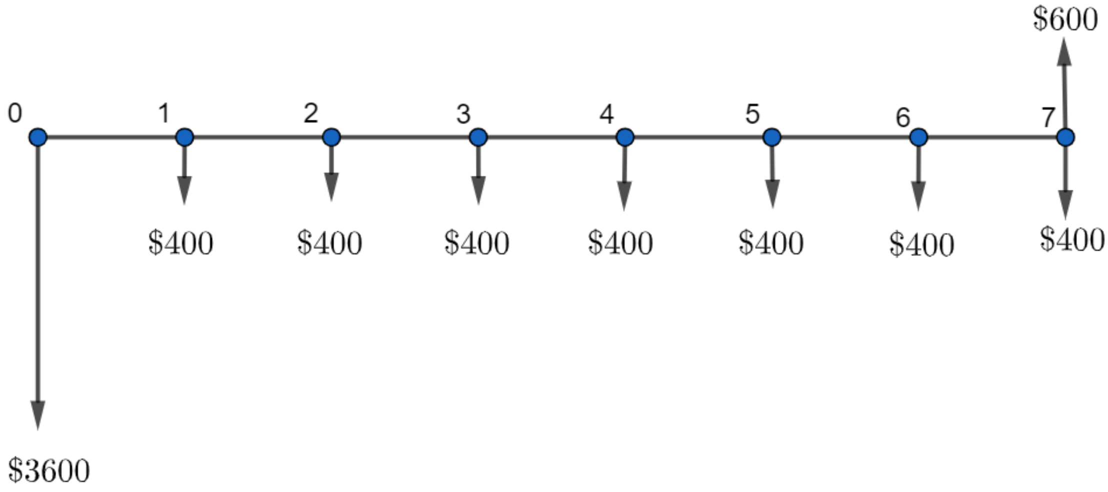
$Z_{26} = \frac{26-24}{4.18} = 0.4785$

$\mathbb{P}(t \leq 26) = \mathbb{P}(Z \leq 0.4785) = 0.6844$

Hence,

$$\mathbb{P}(18 \leq t \leq 26) = \mathbb{P}(t \leq 26) - \mathbb{P}(t \leq 18) = 0.6087$$

Question 8.



The present value is:

$$-\$3600 - 400\underbrace{(P/A, 10\%, 7)}_{4.868} + 600\underbrace{(P/F, 10\%, 7)}_{0.5132} = -5239.28$$

Question 9. The model is in the following form:

$$\begin{aligned}
\max \sum_{j=1}^{n} c_j x_j & \\
\text{s.t.} \sum_{j=1}^{n} \alpha_j x_j \le b_1 & \\
\sum_{j=1}^{n} d_j x_j \le b_2 & \\
x_j \ge 0 & \qquad j = 1, \ldots, n
\end{aligned}$$

x_j : Decision at stage j, $j = 1, \ldots, n$ (amount of type i items taken)
States: R_1, R_2 (Amount of resources available)
$f_j(R_1, R_2)$: Maximum value of the objective function considering stages (items) $j, j+1, \ldots, n$ with available resources R_1 and R_2.

So the **recursion function** is as follows:

$$f_j(R_1, R_2) = \max_{\substack{\alpha_j x_j \leq R_1 \\ d_j x_j \leq R_2 \\ x_j \geq 0}} \{c_j x_j + f_{j+1}(R_1 - a_j x_j, R_2 - d_j x_j)\} j = 1, \ldots, n$$

Base Case: $f_{n+1}(k_1, k_2) = 0$ for all k_1, k_2
Optimal Solution Value: $f_1(b_1, b_2)$

Then, for the LP given in the question;
Base Case: $f_3(430 - 2x_1 - x_2, 230 - x_2) = 0$
Optimal Solution Value: $f_1(430, 230)$

$$\begin{aligned} f_2(430 - 2x_1, 230) = & \max_{\substack{x_2 \leq 430 - 2x_1 \\ x_2 \leq 230 \\ x_2 \geq 0}} \{5x_2 + \underbrace{f_3(430 - 2x_1 - x_2, 230 - x_2)}_{0}\} \\ = & \max_{\substack{x_2 \leq 430 - 2x_1 \\ x_2 \leq 230 \\ x_2 \geq 0}} \{5x_2\} \\ = & \min\{5(430 - 2x_1), 5 \cdot 230\} \\ = & \begin{cases} -10x_1 + 2150 & \text{if } 100 \leq x_1 \leq 215 \\ 1150 & \text{if } 0 \leq x_1 \leq 100 \end{cases} \end{aligned}$$

The minimization in the third equality follows as x_2 cannot be greater than the minimum of $430 - 2x_1$ and 230.

$$\begin{aligned} f_1(430, 230) = & \max_{\substack{2x_1 \leq 430 \\ x_1 \geq 0}} \{2x_1 + f_2(430 - 2x_1, 230)\} \\ = & \max_{x_1} \begin{cases} 2x_1 + 1150 & \text{if } 0 \leq x_1 \leq 100 \\ -8x_1 + 2150 & \text{if } 100 \leq x_1 \leq 215 \end{cases} \\ = & 1350 \end{aligned}$$

(You can also graphically verify that $x_1^* = 100$ and $x_2^* = 230$. Therefore, $f_1(430, 230) = 2x_1^* + 5x_2^*$.)

Question 10.
State: s (Capacity available in the beginning of a year)
Stage: t (Years $1, \ldots, T$)
Decision: x (Change in the capacity)

$f_t(s)$: the minimum cost of meeting power requirements from year t to T having capacity s in the beginning of year t.

Base Case: $f_{T+1}(s) = 0$ for all $s \geq 0$.
Optimal Solution Value: $f_1(100)$
Recursion Function: $f_t(s) = \min_{s+x \geq r_t}\{c_t(x) + m_t(s+x) + f_{t+1}(0.9(s+x))\}$

Question 11.
b_i: the minimum number of workforce needed for year i
x_i: the number of workers in year i
Salaries: $\$30000x_i$
Hiring cost: $\$10000(x_i - \lfloor 0.9x_{i-1} \rfloor)^+$
Firing cost: $\$20000(\lfloor 0.9x_{i-1} \rfloor - x_i)^+$
Stages: Years $i = 1, \ldots, 5$
States: Number of workers in the previous year
$f_i(x_{i-1})$: minimal cost of maintaining a workforce from year i to 5 with workforce (x_{i-1}) in the previous year

Base Case: $f_6(x_5) = 0$
Optimal Solution Value: $f_1(20)$ (Since we started with 20 workers)
Recursion Function:

$$f_i(x_{i-1}) = \min_{b_i \leq x_i \leq \max\{b_1, \ldots, b_n\}} \{f_{i+1}(x_i) + 30000x_i + \$10000(x_i - \lfloor 0.9x_{i-1} \rfloor)^+ + \$20000(\lfloor 0.9x_{i-1} \rfloor - x_i)^+\}$$

for $i = 1, \ldots, 5$