

IE 400: Principles of Engineering Management

Study Set 2 Solutions

Fall 2020

Question 1.

•

x_1, x_2 basic,

x_3, x_4 nonbasic ($x_3 = 0, x_4 = 0$)

Then,

$$x_1 - x_2 = 15$$

$$x_1 + x_2 = 9$$

Solving this system, we obtain $x = (12, -3, 0, 0)$. This is basic but not feasible since $x_2 = -3 < 0$.

•

x_1, x_3 basic,

x_2, x_4 nonbasic ($x_2 = 0, x_4 = 0$)

Then,

$$x_1 + 3x_3 = 15$$

$$x_1 = 9$$

Solving this system, we obtain $x = (9, 0, 2, 0)$. This is a basic feasible solution.

•

x_1, x_4 basic,

x_2, x_3 nonbasic ($x_2 = 0, x_3 = 0$)

Then,

$$x_1 + 5x_4 = 15$$

$$x_1 + x_4 = 9$$

Solving this system, we obtain $x = (15/2, 0, 0, 3/2)$. This is a basic feasible solution.

•

x_2, x_3 basic,

x_1, x_4 nonbasic ($x_1 = 0, x_4 = 0$)

Then,

$$-x_2 + 3x_3 = 15$$

$$x_2 = 9$$

Solving this system, we obtain $x = (0, 9, 8, 0)$. This is a basic feasible solution.

•

x_2, x_4 basic,

x_1, x_3 nonbasic ($x_1 = 0, x_3 = 0$)

Then,

$$-x_2 + 5x_4 = 15$$

$$x_2 + x_4 = 9$$

Solving this system, we obtain $x = (0, 5, 0, 4)$. This is a basic feasible solution.

•

x_3, x_4 basic,

x_1, x_2 nonbasic ($x_1 = 0, x_2 = 0$)

Then,

$$3x_3 + 5x_4 = 15$$

$$x_4 = 9$$

Solving this system, we obtain $x = (0, 0, -10, 9)$. This is basic but not feasible since $x_3 = -10 < 0$.

Question 2.

First Tableau:

x_1	x_2	e_1	e_2	x_3	x_4	RHS
-2	-3	0	0	-M	-M	0
2	1	-1	0	1	0	4
1	-1	0	-1	0	1	1

x_3 and x_4 are basic variables but their row zero coefficients are not zero so we need to add the first and the second row to row zero after multiplying by M:

x_1	x_2	e_1	e_2	x_3	x_4	RHS
$3M-2$	-3	-M	-M	0	0	$5M$
2	1	-1	0	1	0	4
1	-1	0	-1	0	1	1

Then, x_1 enters x_4 leaves.

Second Tableau:

x_1	x_2	s_1	e_1	x_3	x_4	RHS
0	0	0	0	-1	-1	0
1	1	1	0	0	0	3
2	1	0	-1	1	0	4
1	1	0	0	0	1	3

x_3 and x_4 are basic variables but their row zero coefficients are not zero so we need to add the second and the third row to row zero:

x_1	x_2	s_1	e_1	x_3	x_4	RHS
3	2	0	-1	0	0	7
1	1	1	0	0	0	3
2	1	0	-1	1	0	4
1	1	0	0	0	1	3

Then, x_1 enters x_3 leaves.

Third Tableau:

x_1	x_2	x_3	s_1	s_2	e_1	RHS
-2	-1	-1	0	0	0	0
$1/3$	$2/3$	0	1	0	$1/3$	1
2	1	0	0	1	0	3
$2/3$	$1/3$	1	0	0	$-1/3$	1

x_3 is a basic variable but its row zero coefficient is not zero so we need to add the third row to row zero:

x_1	x_2	x_3	s_1	s_2	e_1	RHS
-4/3	-2/3	0	0	0	-1/3	1
1/3	2/3	0	1	0	1/3	1
2	1	0	0	1	0	3
2/3	1/3	1	0	0	-1/3	1

Then, x_1 enters and s_2 leaves (or x_3 leaves).

Question 3. First convert the LP into standard form:

$$\begin{aligned}
 \max \quad & 5x_1 + 3x_2 + x_3 \\
 \text{s.t.} \quad & x_1 + x_2 + x_3 + s_1 = 6 \\
 & 5x_1 + 3x_2 + 6x_3 + s_2 = 15 \\
 & x_i \geq 0 \quad i = 1, 2, 3 \\
 & s_1, s_2 \geq 0
 \end{aligned}$$

	z	x_1	x_2	x_3	s_1	s_2	RHS
1	1	-5	-3	-1	0	0	0
s_1	0	1	1	3	1	0	6
s_2	0	5	3	6	0	1	15

x_1 enters. Apply MRT: $\min\{6/1, 15/5\}=3$ Hence, s_2 leaves.

	z	x_1	x_2	x_3	s_1	s_2	RHS
1	1	0	0	5	0	1	15
s_1	0	0	2/5	9/5	1	-1/5	3
x_1	0	1	3/5	6/5	0	1/5	3

Since there is no negative row 0 coefficient, the solution is optimal: $x^* = (3, 0, 0)$ and $z^* = 15$.

Since there exists 0 in row 0, there may be alternative optimal solutions:
 Let x_2 enter the basis. Apply MRT: $\min\{\frac{3}{2/5}, \frac{3}{3/5}\} = 5$. Then x_1 leaves.

	z	x_1	x_2	x_3	s_1	s_2	RHS
1	1	0	0	5	0	1	15
s_1	0	-2/3	0	1	1	-1/3	1
x_2	0	5/3	1	2	0	1/3	5

$x^* = (0, 5, 0)$ and $z^*=15$.

(If x_1 enters again, you turn back to the previous tableau!)

Question 4. Introduce slack variables s_1 , s_2 and s_3 . Then the LP in the standard form becomes:

$$\begin{aligned}
 \max \quad & 2x_1 + x_2 + 6x_3 - 4x_4 \\
 \text{s.t.} \quad & x_1 + 2x_2 + 4x_3 - x_4 + s_1 = 6 \\
 & 2x_1 + 3x_2 - x_3 + x_4 + s_2 = 12 \\
 & x_1 + x_3 + x_4 + s_3 = 6 \\
 & x_i \geq 0 \quad i = 1, 2, 3, 4, 5, 6, 7
 \end{aligned}$$

Now construct the simplex tableau for initial basis s_1 , s_2 and s_3 . Now we will add the required variables to basis one by one. (Be careful! The following are not Simplex iterations to find the optimal solution!)

	z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
1	1	-2	-1	-6	4	0	0	0	0
s_1	0	1	2	4	-1	1	0	0	6
s_2	0	2	3	-1	1	0	1	0	12
s_3	0	1	0	1	1	0	0	1	6

Let x_1 enter and s_1 leave.

	z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
1	1	0	3	2	2	2	0	0	12
x_1	0	1	2	4	-1	1	0	0	6
s_2	0	0	-1	-9	3	-2	1	0	0
s_3	0	0	-2	-3	2	-1	0	1	0

Let x_2 enter and s_2 leave.

	z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
1	1	0	0	-25	11	-4	3	0	12
x_1	0	1	0	-14	5	-3	2	0	6
x_2	0	0	1	9	-3	2	-1	0	0
s_3	0	0	0	15	-4	3	-2	1	0

Let x_4 enter and s_3 leave.

	z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
1	1	0	0	65/4	0	17/4	-5/2	11/4	12
x_1	0	1	0	19/4	0	3/4	-1/2	5/4	6
x_2	0	0	1	-9/4	0	-1/4	1/2	-3/4	0
x_4	0	0	0	-15/4	1	-3/4	1/2	-1/4	0

The Simplex tableau is constructed for basis x_1 , x_2 and x_4 . This tableau is not optimal because there is a negative coefficient in row zero. Hence, s_2 enters.

To find the variable that leaves the basis, apply MRT: $\min\{0, 0\} = 0$. Hence, x_2 (or x_4) leaves.

Question 5.

- (a) The tableau is final and there exists a unique optimal solution.

$$A < 0, B = 0, C \text{ any}, D \geq 0, E = 1, F \text{ any}$$

(b) The simplex method determines an unbounded solution from this tableau.

$$A > 0, B = 0, C \leq 0, D \geq 0, E = 1, F \text{ any}$$

(c) The current bfs is degenerate (not necessarily optimal).

$$A \text{ any}, B = 0, C \text{ any}, D = 0, E = 1, F \text{ any}$$

(d) d) The current solution is optimal, there are alternative optimal solutions but no alternative optimal bfs.

$$A = 0, B = 0, C \leq 0, D \geq 0, E = 1, F \text{ any}$$

Question 6. Convert into standard form and add artificial variable.

$$\begin{aligned} \min & -3x_1 + 4x_2 \\ \text{s.t. } & x_1 + x_2 + s_1 = 4 \\ & 2x_1 + 3x_2 - e_1 + a_1 = 18 \\ & x_1, x_2, s_1, e_1, a_1 \geq 0 \end{aligned}$$

PHASE 1

	z	x_1	x_2	s_1	e_1	a_1	RHS
1	1	0	0	0	0	-1	0
s_1	0	1	1	1	0	0	4
a_1	0	2	3	0	-1	1	18

Add second row to row 0.

	z	x_1	x_2	s_1	e_1	a_1	RHS
1	1	2	3	0	-1	0	18
s_1	0	1	1	1	0	0	4
a_1	0	2	3	0	-1	1	18

x_2 enters. Apply MRT: $\min\{4/1, 18/2\} = 4$. Then, s_1 leaves the basis.

	z	x_1	x_2	s_1	e_1	a_1	RHS
1	1	-1	0	-3	-1	0	6
x_2	0	1	1	1	0	0	4
a_1	0	-1	0	-3	-1	1	6

The tableau is optimal, but a_1 is still in the basis. Hence, the Phase 1 problem is infeasible.

Question 7. Convert the following LP to standard form and solve with the Big-M method.

$$\begin{aligned}
&\max -x_1 + 3x_2 + x_3 \\
&\text{s.t.} \quad -x_1 + x_2 - x_3 \geq -4 \\
&\quad \quad 2x_1 + x_2 + x_3 \geq 10 \\
&\quad \quad x_1 \geq 0 \\
&\quad \quad x_2 \leq 0 \\
&\quad \quad x_3 \text{ free}
\end{aligned}$$

Let $x_3 = x'_3 - x''_3$ where $x'_3 \geq 0$ and $x''_3 \geq 0$. Let $-x_2 = x'_2$. Introduce slack variable s_1 and excess variable e_2 .

$$\begin{aligned}
&\max -x_1 - 3x'_2 + (x'_3 - x''_3) - Ma_2 \\
&\text{s.t.} \quad x_1 + x'_2 + (x'_3 - x''_3) + s_1 = 4 \\
&\quad \quad 2x_1 - x'_2 + (x'_3 - x''_3) - e_2 + a_2 = 10 \\
&\quad \quad x_1, x'_2, x'_3, x''_3, s_1, e_2 \geq 0
\end{aligned}$$

	z	x_1	x'_2	x'_3	x''_3	s_1	e_2	a_2	RHS
	1	1	3	-1	1	0	0	M	0
s_1	0	1	1	1	-1	1	0	0	4
a_1	0	2	-1	1	-1	0	-1	1	10

Multiply the second row with $-M$ and add it to row 0:

	z	x_1	x'_2	x'_3	x''_3	s_1	e_2	a_2	RHS
	1	$-2M+1$	$M+3$	$-M-1$	$M+1$	0	M	0	$-10M$
s_1	0	1	1	1	-1	1	0	0	4
a_1	0	2	-1	1	-1	0	-1	1	10

x_1 enters the basis. By minimum ratio test: $\min\{4/1, 10/2\} = 4$, s_1 leaves:

	z	x_1	x'_2	x'_3	x''_3	s_1	e_2	a_2	RHS
	1	0	$3M+2$	$M-2$	$-M+2$	$2M-1$	M	0	$-2M-4$
x_1	0	1	1	1	-1	1	0	0	4
a_1	0	0	-3	-1	1	-2	-1	1	2

x''_3 enters the basis. Apply MRT: $\min\{2/1\}$. Then, a_1 leaves:

	z	x_1	x'_2	x'_3	x''_3	s_1	e_2	a_2	RHS
	1	0	8	0	0	3	2	$M-2$	-8
x_1	0	1	-2	0	0	-1	-1	1	6
x''_3	0	0	-3	-1	1	-2	-1	1	2

No negative coefficient in row 0. The tableau is final. (a_2 is not in the basis so remove the column a_2 from the tableau)

Then,

$$x^* = (x_1, x'_2, x'_3, x''_3, s_1, e_2) = (6, 0, 0, 2, 0, 0)$$

and $z^* = -8$. Note that $x_2 = -x'_2 = 0$, $x_3 = x'_3 - x''_3 = 0 - 2 = -2$.

Question 8.

(a) **Parameters:**

f_i = fixed cost of plant i , $i = 1, \dots, 4$

c_{ij} = cost of producing car j at plant i $i = 1, \dots, 4$, $j = 1, 2, 3$

Decision Variables:

$$y_i = \begin{cases} 1 & \text{if plant } i \text{ is used } i = 1, \dots, 4 \\ 0 & \text{o.w} \end{cases}$$

$$x_{ij} = \begin{cases} 1 & \text{if car } j \text{ is produced at plant } i \ i = 1, \dots, 4, \ j = 1, 2, 3 \\ 0 & \text{o.w} \end{cases}$$

Model:

$$\begin{aligned} \min \quad & \sum_{i=1}^4 f_i y_i + \sum_{i=1}^4 \sum_{j=1}^3 c_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{j=1}^3 x_{ij} \leq 1 & i = 1, 2, 3, 4 \end{aligned} \quad (1)$$

$$\sum_{i=1}^4 x_{ij} = 1 \quad j = 1, 2, 3 \quad (2)$$

$$y_3 + y_4 \leq 1 + y_1 \quad (3)$$

$$x_{ij} \leq y_i \quad i = 1, 2, 3, 4 \ j = 1, 2, 3$$

$$x_{ij} \in \{0, 1\} \quad i = 1, 2, 3, 4 \ j = 1, 2, 3$$

$$y_j \in \{0, 1\} \quad j = 1, 2, 3$$

Constraint (1): Each plant can produce one type of car.

Constraint (2): The total production of each type of car must be at a single plant; that is, for example, if any Tauruses are made at plant 1, then all Tauruses must be made there.

Constraint (3): If plants 3 and 4 are used, then plant 1 must also be used.

(b)

$$\text{either} \quad y_1 + y_2 \leq 1 \quad (*)$$

$$\text{or} \quad y_3 + y_4 \geq 1 \quad (**)$$

In order to linearize the constraint, add the following decision variable:

$$w = \begin{cases} 1 & \text{if constraint } (*) \text{ holds} \\ 0 & \text{if constraint } (**) \text{ holds} \end{cases}$$

Then add the following constraints to the model:

$$\begin{aligned}y_1 + y_2 &\leq 1w - (1 - w)M \\y_3 + y_4 &\geq 1(1 - w) + Mw \\w &\in \{0, 1\}\end{aligned}$$

where M is a sufficiently large number.

(For $w = 1$, the constraint (*) holds and (**) becomes redundant. For $w = 0$, the constraint (**) holds and (*) becomes redundant.)

Question 9.

Decision Variables:

$$x_{ij} = \begin{cases} 1 & \text{if astronaut } i \text{ is assigned to post } j \ i = 1, 2, 3, 4, \ j = 1, 2, 3, 4 \\ 0 & \text{o.w} \end{cases}$$

Model:

$$\begin{aligned} \max \quad & \sum_{i=1}^4 \sum_{j=1}^4 p_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{i=1}^4 x_{ij} = 1 \quad j = 1, 2, 3, 4 \end{aligned} \tag{1}$$

$$\sum_{j=1}^4 x_{ij} = 1 \quad i = 1, 2, 3, 4 \tag{2}$$

$$x_{ij} \in \{0, 1\} \quad i = 1, 2, 3, 4 \ j = 1, 2, 3, 4$$

- Constraint (1) indicates that each post should have a single astronaut.
- Constraint (2) indicates that each astronaut should be assigned to a single post.

Question 10.

Parameters: As given in the question

Decision Variables: As given in the question

Model:

$$\begin{aligned}
 \min \quad & \sum_{i=1}^M x_i \\
 \text{s.t.} \quad & \sum_{i=1}^M y_{ik} = d_k & k = 1, \dots, n \\
 & \sum_{k=1}^n y_{ik} \cdot l_k \leq l \cdot x_i & i = 1, \dots, M \\
 & x_i \in \{0, 1\} & i = 1, \dots, M \\
 & y_{ik} \geq 0 & i = 1, \dots, M \quad k = 1, \dots, n \\
 & y_{ik} \text{ integer} & i = 1, \dots, M \quad k = 1, \dots, n
 \end{aligned}$$