IE 400: Principles of Engineering Management
Study Set 2 Solutions

Fall 2020

Question 1.

- x_1, x_2 basic,
  x_3, x_4 nonbasic ($x_3 = 0,\ x_4 = 0$)
  Then,

  $$\begin{aligned} x_1 - x_2 &= 15 \\ x_1 + x_2 &= 9 \end{aligned}$$

  Solving this system, we obtain $x = (12, -3, 0, 0)$. This is basic but not feasible since $x_2 = -3 < 0$.

- x_1, x_3 basic,
  x_2, x_4 nonbasic ($x_2 = 0,\ x_4 = 0$)
  Then,

  $$\begin{aligned} x_1 + 3x_3 &= 15 \\ x_1 &= 9 \end{aligned}$$

  Solving this system, we obtain $x = (9, 0, 2, 0)$. This is a basic feasible solution.

- x_1, x_4 basic,
  x_2, x_3 nonbasic ($x_2 = 0,\ x_3 = 0$)
  Then,
  $$x_1 + 5x_4 = 15$$
  $$x_1 + x_4 = 9$$
  Solving this system, we obtain $x = (15/2, 0, 0, 3/2)$. This is a basic feasible solution.

- x_2, x_3 basic,
  x_1, x_4 nonbasic ($x_1 = 0,\ x_4 = 0$)
  Then,
  $$-x_2 + 3x_3 = 15$$
  $$x_2 = 9$$
  Solving this system, we obtain $x = (0, 9, 8, 0)$. This is a basic feasible solution.

- x_2, x_4 basic,
  x_1, x_3 nonbasic ($x_1 = 0,\ x_3 = 0$)
  Then,
  $$-x_2 + 5x_4 = 15$$
  $$x_2 + x_4 = 9$$
  Solving this system, we obtain $x = (0, 5, 0, 4)$. This is a basic feasible solution.

- x_3, x_4 basic,
  x_1, x_2 nonbasic ($x_1 = 0,\ x_2 = 0$)
  Then,
  $$3x_3 + 5x_4 = 15$$
  $$x_4 = 9$$
  Solving this system, we obtain $x = (0, 0, -10, 9)$. This is basic but not feasible since $x_3 = -10 < 0$.

Question 2.

First Tableau:

x_1	x_2	e_1	e_2	x_3	x_4	RHS
-2	-3	0	0	-M	-M	0
2	1	-1	0	1	0	4
1	-1	0	-1	0	1	1

x_3 and x_4 are basic variables but their row zero coefficients are not zero so we need to add the first and the second row to row zero after multiplying by M:

x_1	x_2	e_1	e_2	x_3	x_4	RHS
3M-2	-3	-M	-M	0	0	5M
2	1	-1	0	1	0	4
1	-1	0	-1	0	1	1

Then, x_1 enters x_4 leaves.

Second Tableau:

x_1	x_2	s_1	e_1	x_3	x_4	RHS
0	0	0	0	-1	-1	0
1	1	1	0	0	0	3
2	1	0	-1	1	0	4
1	1	0	0	0	1	3

x_3 and x_4 are basic variables but their row zero coefficients are not zero so we need to add the second and the third row to row zero:

x_1	x_2	s_1	e_1	x_3	x_4	RHS
3	2	0	-1	0	0	7
1	1	1	0	0	0	3
2	1	0	-1	1	0	4
1	1	0	0	0	1	3

Then, x_1 enters x_3 leaves.

Third Tableau:

x_1	x_2	x_3	s_1	s_2	e_1	RHS
-2	-1	-1	0	0	0	0
1/3	2/3	0	1	0	1/3	1
2	1	0	0	1	0	3
2/3	1/3	1	0	0	-1/3	1

x_3 is a basic variable but its row zero coefficient is not zero so we need to add the third row to row zero:

x_1	x_2	x_3	s_1	s_2	e_1	RHS
-4/3	-2/3	0	0	0	-1/3	1
1/3	2/3	0	1	0	1/3	1
2	1	0	0	1	0	3
2/3	1/3	1	0	0	-1/3	1

Then, x_1 enters and s_2 leaves (or x_3 leaves).

Question 3. First convert the LP into standard form:

$$\begin{aligned} \max\ & 5x_1 + 3x_2 + x_3 \\ \text{s.t. } & x_1 + x_2 + 3x_3 + s_1 = 6 \\ & 5x_1 + 3x_2 + 6x_3 + s_2 = 15 \\ & x_i \geq 0 \ i = 1, 2, 3 \\ & s_1, s_2 \geq 0 \end{aligned}$$

	z	x_1	x_2	x_3	s_1	s_2	RHS
1	1	-5	-3	-1	0	0	0
s_1	0	1	1	3	1	0	6
s_2	0	5	3	6	0	1	15

x_1 enters. Apply MRT: min$\{6/1, 15/5\}$=3 Hence, s_2 leaves.

	z	x_1	x_2	x_3	s_1	s_2	RHS
1	1	0	0	5	0	1	15
s_1	0	0	2/5	9/5	1	-1/5	3
x_1	0	1	3/5	6/5	0	1/5	3

Since there is no negative row 0 coefficient, the solution is optimal: $x^* = (3, 0, 0)$ and $z^* = 15$.

Since there exists 0 in row 0, there may be alternative optimal solutions: Let x_2 enter the basis. Apply MRT: $\min\{\frac{3}{2/5}, \frac{3}{3/5}\} = 5$. Then x_1 leaves.

	z	x_1	x_2	x_3	s_1	s_2	RHS
1	1	0	0	5	0	1	15
s_1	0	-2/3	0	1	1	-1/3	1
x_2	0	5/3	1	2	0	1/3	5

$x^* = (0, 5, 0)$ and z^*=15.
(If x_1 enters again, you turn back to the previous tableau!)

Question 4. Introduce slack variables s_1, s_2 and s_3. Then the LP in the standard form becomes:

$$\begin{aligned}
\max\ & 2x_1 + x_2 + 6x_3 - 4x_4 \\
\text{s.t. } & x_1 + 2x_2 + 4x_3 - x_4 + s_1 = 6 \\
& 2x_1 + 3x_2 - x_3 + x_4 + s_2 = 12 \\
& x_1 + x_3 + x_4 + s_3 = 6 \\
& x_i \geq 0 \ i = 1, 2, 3, 4, 5, 6, 7
\end{aligned}$$

Now construct the simplex tableau for initial basis s_1, s_2 and s_3. Now we will add the required variables to basis one by one. (Be careful! The following are not Simplex iterations to find the optimal solution!)

	z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
1	1	-2	-1	-6	4	0	0	0	0
s_1	0	1	2	4	-1	1	0	0	6
s_2	0	2	3	-1	1	0	1	0	12
s_3	0	1	0	1	1	0	0	1	6

Let x_1 enter and s_1 leave.

	z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
1	1	0	3	2	2	2	0	0	12
x_1	0	1	2	4	-1	1	0	0	6
s_2	0	0	-1	-9	3	-2	1	0	0
s_3	0	0	-2	-3	2	-1	0	1	0

Let x_2 enter and s_2 leave.

	z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
1	1	0	0	-25	11	-4	3	0	12
x_1	0	1	0	-14	5	-3	2	0	6
x_2	0	0	1	9	-3	2	-1	0	0
s_3	0	0	0	15	-4	3	-2	1	0

Let x_4 enter and s_3 leave.

	z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
1	1	0	0	65/4	0	17/4	-5/2	11/4	12
x_1	0	1	0	19/4	0	3/4	-1/2	5/4	6
x_2	0	0	1	-9/4	0	-1/4	1/2	-3/4	0
x_4	0	0	0	-15/4	1	-3/4	1/2	-1/4	0

The Simplex tableau is constructed for basis x_1, x_2 and x_4. This tableau is not optimal because there is a negative coefficient in row zero. Hence, s_2 enters.

To find the variable that leaves the basis, apply MRT: $\min\{0, 0\} = 0$ Hence, x_2 (or x_4) leaves.

Question 5.

(a) The tableau is final and there exists a unique optimal solution.

$$A < 0,\ \ B = 0,\ \ C \text{ any}, D \geq 0, E = 1,\ \ F \text{ any}$$

(b) The simplex method determines an unbounded solution from this tableau.

$$A > 0,\ B = 0,\ C \leq 0,\ D \geq 0, E = 1,\ F \text{ any}$$

(c) The current bfs is degenerate (not necessarily optimal).

$$A \text{ any},\ B = 0,\ C \text{ any},\ D = 0,\ E = 1,\ F \text{ any}$$

(d) d) The current solution is optimal, there are alternative optimal solutions but no alternative optimal bfs.

$$A = 0,\ B = 0,\ C \leq 0,\ D \geq 0,\ E = 1,\ F \text{ any}$$

Question 6. Convert into standard form and add artificial variable.

$$\begin{aligned} \min\ & -3x_1 + 4x_2 \\ \text{s.t. } & x_1 + x_2 + s_1 = 4 \\ & 2x_1 + 3x_2 - e_1 + a_1 = 18 \\ & x_1, x_2, s_1, e_1, a_1 \geq 0 \end{aligned}$$

PHASE 1

	z	x_1	x_2	s_1	e_1	a_1	RHS
1	1	0	0	0	0	-1	0
s_1	0	1	1	1	0	0	4
a_1	0	2	3	0	-1	1	18

Add second row to row 0.

	z	x_1	x_2	s_1	e_1	a_1	RHS
1	1	2	3	0	-1	0	18
s_1	0	1	1	1	0	0	4
a_1	0	2	3	0	-1	1	18

x_2 enters. Apply MRT: $\min\{4/1, 18/2\} = 4$. Then, s_1 leaves the basis.

	z	x_1	x_2	s_1	e_1	a_1	RHS
1	1	-1	0	-3	-1	0	6
x_2	0	1	1	1	0	0	4
a_1	0	-1	0	-3	-1	1	6

The tableau is optimal, but a_1 is still in the basis. Hence, the Phase 1 problem is infeasible.

Question 7. Convert the following LP to standard form and solve with the Big-M method.

$$\begin{aligned}
\max\ & -x_1 + 3x_2 + x_3 \\
\text{s.t.}\ & -x_1 + x_2 - x_3 \geq -4 \\
& 2x_1 + x_2 + x_3 \geq 10 \\
& x_1 \geq 0 \\
& x_2 \leq 0 \\
& x_3 \text{ free}
\end{aligned}$$

Let $x_3 = x_3' - x_3''$ where $x_3' \geq 0$ and $x_3'' \geq 0$. Let $-x_2 = x_2'$. Introduce slack variable s_1 and excess variable e_2.

$$\begin{aligned}
\max\ & -x_1 - 3x_2' + (x_3' - x_3'') - Ma_2 \\
\text{s.t.}\ & x_1 + x_2' + (x_3' - x_3'') + s_1 = 4 \\
& 2x_1 - x_2' + (x_3' - x_3'') - e_2 + a_2 = 10 \\
& x_1, x_2', x_3', x_3'', s_1, e_2 \geq 0
\end{aligned}$$

	z	x_1	x_2'	x_3'	x_3''	s_1	e_2	a_2	RHS
	1	1	3	-1	1	0	0	M	0
s_1	0	1	1	1	-1	1	0	0	4
a_1	0	2	-1	1	-1	0	-1	1	10

Multiply the second row with $-M$ and add it to row 0:

	z	x_1	x_2'	x_3'	x_3''	s_1	e_2	a_2	RHS
	1	-2M+1	M+3	-M-1	M+1	0	M	0	-10M
s_1	0	1	1	1	-1	1	0	0	4
a_1	0	2	-1	1	-1	0	-1	1	10

x_1 enters the basis. By minimum ratio test: $\min\{4/1, 10/2\} = 4$, s_1 leaves:

	z	x_1	x_2'	x_3'	x_3''	s_1	e_2	a_2	RHS
	1	0	3M+2	M-2	-M+2	2M-1	M	0	-2M-4
x_1	0	1	1	1	-1	1	0	0	4
a_1	0	0	-3	-1	1	-2	-1	1	2

x_3'' enters the basis. Apply MRT: $\min\{2/1\}$. Then, a_1 leaves:

	z	x_1	x_2'	x_3'	x_3''	s_1	e_2	a_2	RHS
	1	0	8	0	0	3	2	M-2	-8
x_1	0	1	-2	0	0	-1	-1	1	6
x_3''	0	0	-3	-1	1	-2	-1	1	2

No negative coefficient in row 0. The tableau is final. (a_2 is not in the basis so remove the column a_2 from the tableau)

Then,

$$x^* = (x_1, x_2', x_3', x_3'', s_1, e_2) = (6, 0, 0, 2, 0, 0)$$

and $z^* = -8$. Note that $x_2 = -x_2' = 0$, $x_3 = x_3' - x_3'' = 0 - 2 = -2$.

Question 8.

(a) **Parameters:**

f_i= fixed cost of plant i, $i = 1, \dots, 4$

c_{ij}= cost of producing car j at plant i $i = 1, \dots, 4$, $j = 1, 2, 3$

Decision Variables:

$$y_i = \begin{cases} 1 & \text{if plant i is used} \quad i = 1, \ldots, 4 \\ 0 & \text{o.w} \end{cases}$$

$$x_{ij} = \begin{cases} 1 & \text{if car j is produced at plant i} \quad i = 1, \ldots, 4, \quad j = 1, 2, 3 \\ 0 & \text{o.w} \end{cases}$$

Model:

$$\begin{aligned}
\min \; & \sum_{i=1}^{4} f_i y_i + \sum_{i=1}^{4} \sum_{j=1}^{3} c_{ij} x_{ij} & & \\
\text{s.t.} \; & \sum_{j=1}^{3} x_{ij} \le 1 & i = 1, 2, 3, 4 & \qquad (1) \\
& \sum_{i=1}^{4} x_{ij} = 1 & j = 1, 2, 3 & \qquad (2) \\
& y_3 + y_4 \le 1 + y_1 & & \qquad (3) \\
& x_{ij} \le y_i & i = 1, 2, 3, 4 \quad j = 1, 2, 3 & \\
& x_{ij} \in \{0, 1\} & i = 1, 2, 3, 4 \quad j = 1, 2, 3 & \\
& y_j \in \{0, 1\} & j = 1, 2, 3 &
\end{aligned}$$

Constraint (1): Each plant can produce one type of car.
Constraint (2): The total production of each type of car must be at a single plant; that is, for example, if any Tauruses are made at plant 1, then all Tauruses must be made there.
Constraint (3): If plants 3 and 4 are used, then plant 1 must also be used.

(b)

$$\begin{aligned}
&\text{either} & y_1 + y_2 \le 1 & \qquad (*) \\
&\text{or} & y_3 + y_4 \ge 1 & \qquad (**)
\end{aligned}$$

In order to linearize the constraint, add the following decision variable:

$$w = \begin{cases} 1 & \text{if constraint (*) holds} \\ 0 & \text{if constraint (**) holds} \end{cases}$$

Then add the following constraints to the model:

$$
\begin{aligned}
&y_1 + y_2 \leq 1w - (1 - w)M \\
&y_3 + y_4 \geq 1(1 - w) + Mw \\
&w \in \{0, 1\}
\end{aligned}
$$

where M is a sufficiently large number.

(For $w = 1$, the constraint (*) holds and (**) becomes redundant. For $w = 0$, the constraint (**) holds and (*) becomes redundant.)

Question 9.

Decision Variables:

$$x_{ij} = \begin{cases} 1 & \text{if astronaut i is assigned to post j } i = 1,2,3,4,\ \ j = 1,2,3,4 \\ 0 & \text{o.w} \end{cases}$$

Model:

$$\begin{aligned} \max \ & \sum_{i=1}^{4}\sum_{j=1}^{4} p_{ij}x_{ij} & & \\ \text{s.t.} \ & \sum_{i=1}^{4} x_{ij} = 1 & j = 1,2,3,4 & \qquad (1) \\ & \sum_{j=1}^{4} x_{ij} = 1 & i = 1,2,3,4 & \qquad (2) \\ & x_{ij} \in \{0,1\} & i = 1,2,3,4 \ \ j = 1,2,3,4 & \end{aligned}$$

- Constraint (1) indicates that each post should have a single astronaut.
- Constraint (2) indicates that each astronaut should be assigned to a single post.

Question 10.

Parameters: As given in the question

Decision Variables: As given in the question

Model:

$$
\begin{aligned}
\min \ & \sum_{i=1}^{M} x_i \\
\text{s.t.} \ & \sum_{i=1}^{M} y_{ik} = d_k && k = 1, \cdots, n \\
& \sum_{k=1}^{n} y_{ik} \cdot l_k \le l \cdot x_i && i = 1, \cdots, M \\
& x_i \in \{0, 1\} && i = 1, \cdots, M \\
& y_{ik} \ge 0 && i = 1, \cdots, M && k = 1, \cdots, n \\
& y_{ik} \text{ integer} && i = 1, \cdots, M && k = 1, \cdots, n
\end{aligned}
$$