IE 400: Principles of Engineering Management
Study Set 3 Solutions

Fall 2020

Question 1.

(a) **Parameters:**

w_j= weight capacity of compartment j (in tons)

s_j= space capacity of compartment j (in cubic meters)

wt_i= weight of cargo i (in tons)

v_i= weight of cargo i (in cubic meters/ton)

p_i= profit gained from cargo i (TL/ton)

Decision Variables:

x_{ij}= amount of cargo i distributed to compartment j in tons

$$
\begin{aligned}
\max \ & \sum_{i=1}^{4}\sum_{j=1}^{3} p_i x_{ij} & & \\
\text{s.t.} \ & \sum_{j=1}^{3} x_{ij} \le wt_i & i = 1,2,3,4 & \quad (1)\\
& \sum_{i=1}^{4} v_i x_{ij} \le s_j & j = 1,2,3 & \quad (2)\\
& \sum_{i=1}^{4} x_{ij} \le w_j & j = 1,2,3 & \quad (3)\\
& \frac{\sum_{i=1}^{4} x_{i1}}{w_1} = \frac{\sum_{i=1}^{4} x_{i2}}{w_2} = \frac{\sum_{i=1}^{4} x_{i3}}{w_3} & & \quad (4)\\
& x_{ij} \ge 0 & i = 1,2,3,4 \quad j = 1,2,3 &
\end{aligned}
$$

(b) Replace constraint (4) with

$$\left| \frac{\sum_{i=1}^{4} x_{i1}}{w_1} - \frac{\sum_{i=1}^{4} x_{i2}}{w_2} \right| \le 0.1$$

$$\left| \frac{\sum_{i=1}^{4} x_{i1}}{w_1} - \frac{\sum_{i=1}^{4} x_{i3}}{w_3} \right| \le 0.1$$

$$\left| \frac{\sum_{i=1}^{4} x_{i2}}{w_2} - \frac{\sum_{i=1}^{4} x_{i3}}{w_3} \right| \le 0.1$$

You need to linearize these constraints before adding to the model:

$$-0.1 \le \frac{\sum_{i=1}^{4} x_{i1}}{w_1} - \frac{\sum_{i=1}^{4} x_{i2}}{w_2} \le 0.1$$

$$-0.1 \le \frac{\sum_{i=1}^{4} x_{i1}}{w_1} - \frac{\sum_{i=1}^{4} x_{i3}}{w_3} \le 0.1$$

$$-0.1 \le \frac{\sum_{i=1}^{4} x_{i2}}{w_2} - \frac{\sum_{i=1}^{4} x_{i3}}{w_3} \le 0.1$$

(c) (b) is more likely to have a larger optimal value since the feasible region in (b) includes the feasible region in (a).

(d) **Decision Variables:**

x_{ij}= amount of cargo i distributed to compartment j in tons

$$y_{ij} = \begin{cases} 1 & \text{if cargo i is shipped in compartment j} \\ 0 & \text{o.w} \end{cases}$$

Model: All constraints in (b) +

$$\begin{aligned}
& x_{ij} \le w_j y_{ij} && i = 1,2,3,4 \quad j = 1,2,3 \\
& \sum_{j=1}^{3} y_{ij} \le 1 && i = 1,2,3,4 \\
& y_{ij} \in \{0,1\} && i = 1,2,3,4 \quad j = 1,2,3 \\
& x_{ij} \text{ integer} && i = 1,2,3,4 \quad j = 1,2,3
\end{aligned}$$

Question 2.

(a) Formulate the model to determine which trucks to use so as to minimize the cost of delivering to all the customers.

Decision Variables:

$$x_k = \begin{cases} 1 & \text{if truck k is used } k = 1, \ldots, 7 \\ 0 & \text{o.w} \end{cases}$$

$$y_{jk} = \begin{cases} 1 & \text{if customer j is visited by truck k } j = 1, \ldots, 20 \quad k = 1, \ldots, 7 \\ 0 & \text{o.w} \end{cases}$$

z_{jk} =Amount of good is carried by truck k for customer j
$j = 1, \ldots, 20 \quad k = 1, \ldots, 7$

Model:

$$\begin{aligned}
\min \ & \sum_{k=1}^{7} q_k x_k \\
\text{s.t.} \ & \sum_{k=1}^{7} z_{jk} = d_j && j = 1, \ldots, 20 \\
& \sum_{j=1}^{20} z_{jk} \leq C_k x_k && k = 1, \ldots, 7 \\
& z_{jk} \leq d_j y_{jk} && j = 1, \ldots, 20 \quad k = 1, \ldots, 7 \\
& y_{jk} \leq x_k && j = 1, \ldots, 20 \quad k = 1, \ldots, 7 \\
& x_k \in \{0, 1\} && k = 1, \ldots, 7 \\
& y_{jk} \in \{0, 1\} && j = 1, \ldots, 20 \quad k = 1, \ldots, 7 \\
& z_{jk} \geq 0 && j = 1, \ldots, 20 \quad k = 1, \ldots, 7
\end{aligned}$$

(b) Add the following requirements to your model:

- A single truck cannot deliver to more than six customers.

$$\sum_{j=1}^{20} y_{jk} \leq 6 \quad k = 1, \ldots, 7$$

- Customer 1 and customer 7 cannot be visited by the same truck.

$$y_{1k} + y_{7k} \leq 1 \quad k = 1, \ldots, 7$$

- Both customer 2 and customer 12 should be visited by the same truck(s).

$$y_{2,k} = y_{12,k} \quad k = 1, \ldots, 7$$

- If a truck visits customer 4 then it has to visit customer 19 or customer 20. (Assume demands of these customers do not exceed truck capacities.)

$$y_{4,k} \leq y_{19,k} + y_{20,k} \quad k = 1, \ldots, 7$$

- Either at least two of the customers 5, 6 and 7 or at most three of customers 11, 12, 13 and 14 should be visited by truck 4.

$$\text{either } y_{5,4} + y_{6,4} + y_{7,4} \geq 2 \tag{1}$$
$$\text{or } y_{11,4} + y_{12,4} + y_{13,4} + y_{14,4} \leq 3 \tag{2}$$

In order to linearize the constraint, add the following decision variable:

$$w = \begin{cases} 1 & \text{if constraint 1 holds} \\ 0 & \text{if constraint 2 holds} \end{cases}$$

Then add the following constraints to the model:

$$\begin{aligned} & y_{5,4} + y_{6,4} + y_{7,4} \geq 2w - (1 - w)M \\ & y_{11,4} + y_{12,4} + y_{13,4} + y_{14,4} \leq 3(1 - w) + Mw \\ & w \in \{0, 1\} \end{aligned}$$

Question 3.

Decision Variables:

$$y_i = \begin{cases} 1 & \text{if constraint i is satisfied} \\ 0 & \text{o.w} \end{cases}$$

$$z_1 = \begin{cases} 1 & \text{if at least one of 7,8,9 is satisfied} \\ 0 & \text{o.w} \end{cases}$$

$$z_2 = \begin{cases} 1 & \text{if at least two of 10,11,12,13 are satisfied} \\ 0 & \text{o.w} \end{cases}$$

Model:

$$\begin{aligned}
& \min 0 \\
& \text{s.t.} \sum_{j=1}^{n} a_{ij}x_j \le b_i + M(1 - y_i) && i = 1, \ldots, 20 && (1) \\
& \sum_{j=1}^{n} a_{ij}x_j \ge b_i - My_i && i = 1, \ldots, 20 && (2) \\
& \sum_{i=1}^{20} y_i \ge 10 && && (3) \\
& y_1 \le y_4 && && (4) \\
& y_2 + y_3 \le 1 && && (5) \\
& y_5 = y_6 && && (6) \\
& y_7 + y_8 + y_9 \ge z_1 && && (7) \\
& y_{10} + y_{11} + y_{12} + y_{13} \ge 2z_2 && && (8) \\
& z_1 + z_2 \ge 1 && && (9) \\
& z_1, z_2 \in \{0, 1\} && && (10) \\
& y_i \in \{0, 1\} && i = 1, \ldots, 20 && (11)
\end{aligned}$$

where M is a sufficiently large number. Observe that we minimize zero, which means that the model is a feasibility problem. If a feasible solution is found for the above model, the objective will take value 0.

Let's take a closer look at constraints (1) and (2):

- If $y_i = 1$:

  Constraint (1) becomes $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ which means that the constraint i is satisfies.
  Constraint (2) becomes $\sum_{j=1}^{n} a_{ij}x_j \geq b_i - M$. This makes the constraint redundant as $-M$ is a really small number.

- If $y_i = 0$:

  Constraint (1) becomes $\sum_{j=1}^{n} a_{ij}x_j \leq b_i + M$ which makes the constraint redundant as M is a really large number.
  Constraint (2) becomes $\sum_{j=1}^{n} a_{ij}x_j \geq b_i$. This means constraint is not satisfied.

(3): Point x should satisfy at least 10 of the 20 constraints.
(4): If point x satisfies constraint 1, then it should also satisfy constraint 4.
(5): If point x satisfies constraint 2, then it should not satisfy constraint 3.
(6): If point x satisfies either both of the constraints 5 and 6 or neither of them.
(7)-(8)-(9): Point x should satisfy either at least one of the constraints 7,8,9 or at least two of constraints 10,11,12,13.

Question 4.

Decision Variables:

x_i : Number of workers employed on line i $\;i = 1, 2$

$$y_i = \begin{cases} 1 & \text{if line i is used} \quad i = 1, 2 \\ 0 & \text{o.w} \end{cases}$$

Model:

$$\begin{aligned} \min\ & 1000y_1 + 2000y_2 + 500x_1 + 900x_2 \\ \text{s.t.}\ & 20x_1 + 50x_2 \geq 120 \\ & 30x_1 + 35x_2 \geq 150 \\ & 40x_1 + 45x_2 \geq 200 \\ & x_1 \leq 7y_1 \\ & x_2 \leq 7y_2 \\ & x_1, x_2 \geq 0 \\ & y_1, y_2 \in \{0, 1\} \end{aligned}$$

Question 5.

<u>Two-Phase Method:</u>

The phase-I problem is the following:

$$
\begin{array}{llllllll}
\min & a_1 & + a_2 & & & & & \\
\text{s.t.} & & & & & & & \\
& 2x_1 & + x_2 & + 3x_3 & & + a_1 & & = 60 \\
& 3x_1 & + 3x_2 & + 5x_3 & - e_1 & & + a_2 & = 120 \\
& x_1 , & x_2 , & x_3 , & e_1 , & a_1 , & a_2 & \geq 0
\end{array}
$$

Now we need to solve phase-I LP:

Tableau 0:

z	x_1	x_2	x_3	e_1	a_1	a_2	RHS
1	0	0	0	0	-1	-1	0
0	2	1	3	0	1	0	60
0	3	3	5	-1	0	1	120

The simplex tableau above is not in proper format since <u>the row zero coefficients of basic variables a_1 and a_2</u> are not zero. We add the first and second row to row zero to make these coefficients zero:

Tableau 0:

z	x_1	x_2	x_3	e_1	a_1	a_2	RHS	
1	5	4	8	-1	0	0	180	
0	2	1	3	0	1	0	60	➡ 60/3
0	3	3	5	-1	0	1	120	120/5

Now we can start simplex iterations: since it is a minimization problem we are looking for positive coefficients to decide on entering variables.

x_3 enters and a_1 leaves:

Tableau 1:

z	x_1	x_2	x_3	e_1	a_1	a_2	RHS	
1	-1/3	4/3	0	-1	-8/3	0	20	
0	2/3	1/3	1	0	1/3	0	20	20/(1/3)
0	-1/3	4/3	0	-1	-5/3	1	20	➡ 20/(4/3)

x_2 enters and a_2 leaves:

Tableau 2:

z	x_1	x_2	x_3	e_1	a_1	a_2	RHS
1	0	0	0	0	-1	-1	0
0	3/4	0	1	1/4	3/4	-1/4	15
0	-1/4	1	0	-3/4	-5/4	3/4	15

Both a_1 and a_2 left the basis so we can delete their columns from the tableau and move onto phase-II. We write the original objective function coefficients to the tableau:

Tableau 0:

z	x_1	x_2	x_3	e_1	RHS
1	-3	-2	-4	0	0
0	3/4	0	1	1/4	15
0	-1/4	1	0	-3/4	15

Again, notice that this tableau is not in proper format, we need to make row zero coefficients of basic variables x_2 and x_3 zero:

Tableau 0:

z	x_1	x_2	x_3	e_1	RHS
1	-1/2	0	0	-1/2	90
0	3/4	0	1	1/4	15
0	-1/4	1	0	-3/4	15

Stop. The tableau is optimal since there is no positive entry in row zero.

The optimal solution is $x_1^*=0$, $x_2^*=x_3^*=15$ with objective value 90.

Now let's solve the same problem with Big-M method.

<u>Big-M method:</u>

The LP is the following:

$$
\begin{array}{llllllllll}
\min & 3x_1 & +\ 2x_2 & +\ 4x_3 & & +\ Ma_1 & +\ Ma_2 & & \\
\text{s.t.} & & & & & & & & \\
 & 2x_1 & +\ x_2 & +\ 3x_3 & & +\ a_1 & & = & 60 \\
 & 3x_1 & +\ 3x_2 & +\ 5x_3 & -\ e_1 & & +\ a_2 & = & 120 \\
 & x_1 & ,\ x_2 & ,\ x_3 & ,\ e_1 & ,\ a_1 & ,\ a_2 & \geq & 0
\end{array}
$$

Tableau 0:

z	x_1	x_2	x_3	e_1	a_1	a_2	RHS
1	-3	-2	-4	0	-M	-M	0
0	2	1	3	0	1	0	60
0	3	3	5	-1	0	1	120

Let's put the tableau in the proper format first:

Tableau 0:

z	x_1	x_2	x_3	e_1	a_1	a_2	RHS	
1	5M-3	4M-2	8M-4	-M	0	0	180M	
0	2	1	3	0	1	0	60	60/3
0	3	3	5	-1	0	1	120	120/5

x_3 enters and a_1 leaves:

Tableau 1:

z	x_1	x_2	x_3	e_1	a_1	a_2	RHS	
1	-(M+1)/3	(4M-2)/3	0	-M	(4-8M)/3	0	80 +20M	
0	2/3	1/3	1	0	1/3	0	20	20/(1/3)
0	-1/3	4/3	0	-1	-5/3	1	20	20/(4/3)

x_2 enters and a_2 leaves:

Tableau 2:

z	x_1	x_2	x_3	e_1	a_1	a_2	RHS
1	-1/2	0	0	-1/2	1/2-M	1/2-M	90
0	3/4	0	1	1/4	3/4	-1/4	15
0	-1/4	1	0	-3/4	-5/4	3/4	15

Stop. The tableau is optimal since there is no positive entry in row zero.

The optimal solution is $x_1^*=0$, $x_2^*=x_3^*=15$ with objective value 90.