IE 400: Principles of Engineering Management
Study Set 5 Solutions

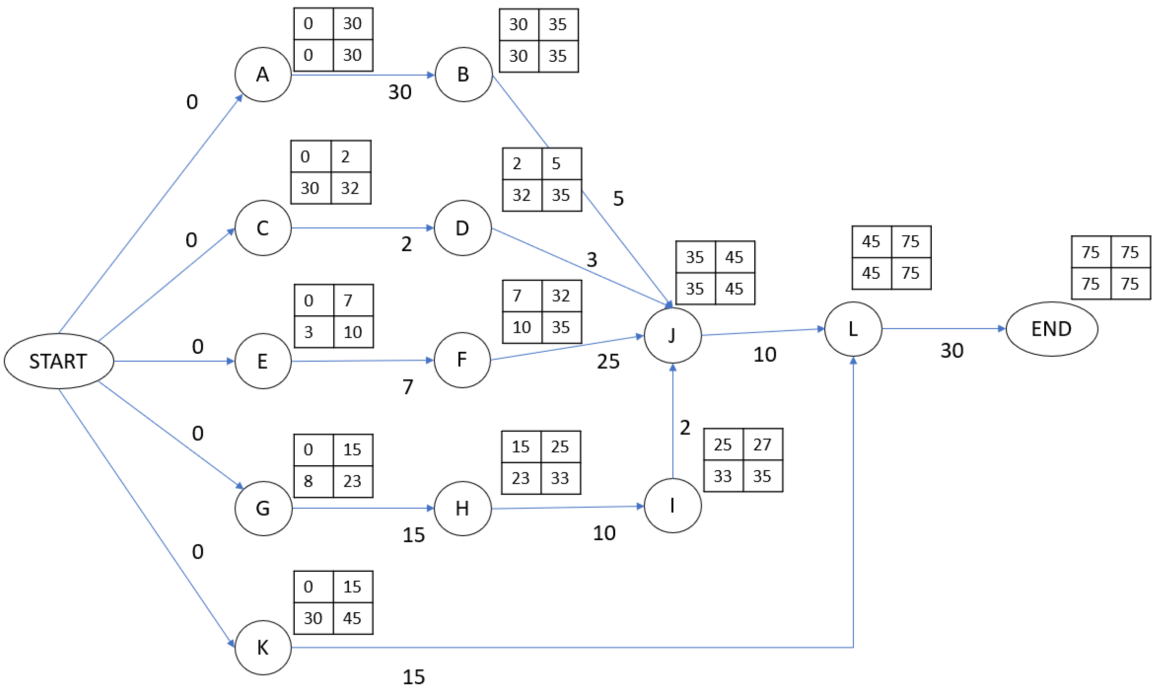
Fall 2020

Question 1.

ES: Earliest Start, EF: Earliest Finish, LS: Latest Start, LF: Latest Finish

ES	EF
LS	LF

(a)-(b)



Activity	Slack
A	0
B	0
C	30
D	30
E	3
F	3
G	8
H	8
I	8
J	0
K	30
L	0

(c) Since the slack for cutting the onions and mushrooms is 3, the dinner will be delayed 6-3=3 minutes.

If the cutting time is 2 minutes, slack is 10 (latest finish time) – 2 (task duration) = 8 minutes. Since the slack time is more than the delay time, it does not delay the dinner.

(d) Let N be the set of activities and A be the set of arc in the project network.

Parameters:

a_i : Duration of activity i, for all i $\in N$

d : Project duration (75)

Decision Variables:

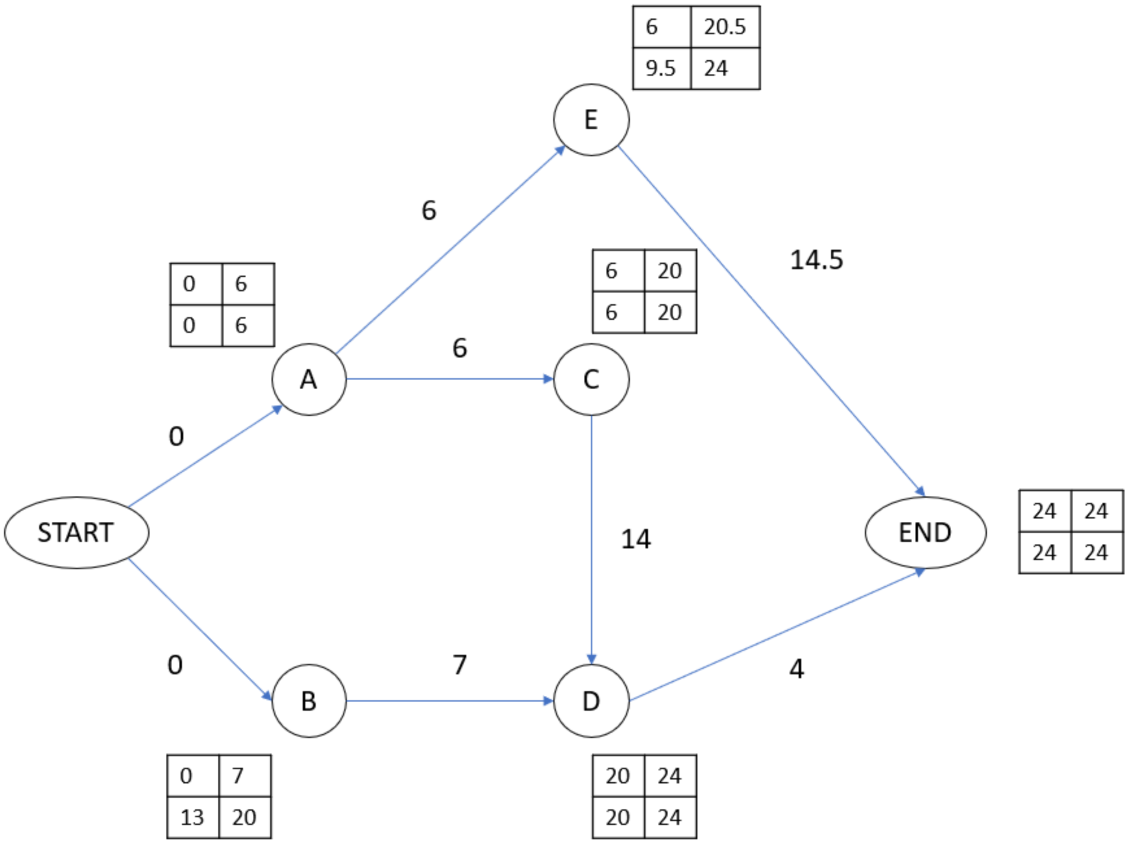
x_i : Start time of activity i, for all $i \in N$

Model:

$$
\begin{aligned}
\max \; & \sum_{i \in N} x_i \\
\text{s.t. } & x_j \geq x_i + a_i && (i,j) \in A \\
& x_{end} \leq d \\
& x_i \geq 0 && i \in N \\
& x_{start} = 0
\end{aligned}
$$

Question 2. (a)

Activity	**Mean**	**Variance**
A	6	4/36
B	7	196/36
C	14	256/36
D	4	4/36
E	14.5	169/36



The critical path is A-C-D.

(b) Note that

$$\mathbb{P}(18 \leq t \leq 26) = \mathbb{P}(t \leq 26) - \mathbb{P}(t \leq 18)$$

We have, $\sigma_p = 2.708$

Note that $Z_{18} = \frac{18-24}{2.708} = -2.2156$

Then, $\mathbb{P}(t \leq 18) = \mathbb{P}(Z \leq -2.2156) = 0.013$

$Z_{26} = \frac{26-24}{2.708} = 0.739$

$\mathbb{P}(t \leq 26) = \mathbb{P}(Z \leq 0.739) = 0.770$

Hence,

$$\mathbb{P}(18 \leq t \leq 26) = \mathbb{P}(t \leq 26) - \mathbb{P}(t \leq 18) = 0.757$$

Question 3. The model is in the following form:

$$
\begin{aligned}
\max \sum_{j=1}^{n} c_j x_j & \\
\text{s.t.} \sum_{j=1}^{n} \alpha_j x_j \le b_1 & \\
\sum_{j=1}^{n} d_j x_j \le b_2 & \\
x_j \ge 0 & \qquad\qquad j = 1, \ldots, n
\end{aligned}
$$

x_j : Decision at stage j, $j = 1, \ldots, n$ (amount of type i items taken)
States: R_1, R_2 (Amount of resources available)
$f_j(R_1, R_2)$: Maximum value of the objective function considering stages (items) $j, j+1, \ldots, n$ with available resources R_1 and R_2.

So the **recursion function** is as follows:

$$f_j(R_1, R_2) = \max_{\substack{\alpha_j x_j \le R_1 \\ d_j x_j \le R_2 \\ x_j \ge 0}} \{c_j x_j + f_{j+1}(R_1 - a_j x_j, R_2 - d_j x_j)\} j = 1, \ldots, n$$

Base Case: $f_{n+1}(k_1, k_2) = 0$ for all k_1, k_2
Optimal Solution Value: $f_1(b_1, b_2)$

Then, for the LP given in the question;
Base Case: $f_3(430 - 2x_1 - x_2, 230 - x_2) = 0$
Optimal Solution Value: $f_1(430, 230)$

$$\begin{aligned}
f_2(430-2x_1,230) &= \max_{\substack{x_2\leq 430-2x_1\\ x_2\leq 230\\ x_2\geq 0}} \{5x_2 + \underbrace{f_3(430-2x_1-x_2, 230-x_2)}_{0}\} \\
&= \max_{\substack{x_2\leq 430-2x_1\\ x_2\leq 230\\ x_2\geq 0}} \{5x_2\} \\
&= \min\{5(430-2x_1), 5\cdot 230\} \\
&= \begin{cases} -10x_1+2150 & \text{if } 100\leq x_1\leq 215 \\ 1150 & \text{if } 0\leq x_1\leq 100 \end{cases}
\end{aligned}$$

The minimization in the third equality follows as x_2 cannot be greater than the minimum of $430-2x_1$ and 230.

$$\begin{aligned}
f_1(430,230) &= \max_{\substack{2x_1\leq 430\\ x_1\geq 0}} \{2x_1 + f_2(430-2x_1,230)\} \\
&= \max_{x_1} \begin{cases} 2x_1+1150 & \text{if } 0\leq x_1\leq 100 \\ -8x_1+2150 & \text{if } 100\leq x_1\leq 215 \end{cases} \\
&= 1350
\end{aligned}$$

(You can also graphically verify that $x_1^*=100$ and $x_2^*=230$. Therefore, $f_1(430,230)=2x_1^*+5x_2^*$.)

Question 4.
State: s (Capacity available in the beginning of a year)
Stage: t (Years $1, \ldots, T$)
Decision: x (Change in the capacity)

$f_t(s)$: the minimum cost of meeting power requirements from year t to T having capacity s in the beginning of year t.

Base Case: $f_{T+1}(s) = 0$ for all $s \geq 0$.
Optimal Solution Value: $f_1(100)$
Recursion Function: $f_t(s) = \min_{s+x \geq r_t}\{c_t(x) + m_t(s+x) + f_{t+1}(0.9(s+x))\}$

Question 5.
b_i: the minimum number of workforce needed for year i
x_i: the number of workers in year i
Salaries: $\$30000x_i$
Hiring cost: $\$10000(x_i - \lfloor 0.9x_{i-1} \rfloor)^+$
Firing cost: $\$20000(\lfloor 0.9x_{i-1} \rfloor - x_i)^+$
Stages: Years $i = 1, \ldots, 5$
States: Number of workers in the previous year
$f_i(x_{i-1})$: minimal cost of maintaining a workforce from year i to 5 with workforce (x_{i-1}) in the previous year

Base Case: $f_6(x_5) = 0$
Optimal Solution Value: $f_1(20)$ (Since we started with 20 workers)
Recursion Function:

$$f_i(x_{i-1}) = \min_{b_i \leq x_i \leq \max\{b_1, \ldots, b_n\}} \{f_{i+1}(x_i) + 30000x_i + \$10000(x_i - \lfloor 0.9x_{i-1} \rfloor)^+ + \$20000(\lfloor 0.9x_{i-1} \rfloor - x_i)^+\}$$

for $i = 1, \ldots, 5$