IE 400: Principles of Engineering Management
Study Set 2 Solutions

Spring 2021

Question 1.

- x_1, x_2 basic,
  x_3, x_4 nonbasic $(x_3 = 0,\ x_4 = 0)$
  Then,

  $$x_1 - x_2 = 15$$
  $$x_1 + x_2 = 9$$

  Solving this system, we obtain $x = (12, -3, 0, 0)$. This is basic but not feasible since $x_2 = -3 < 0$.

- x_1, x_3 basic,
  x_2, x_4 nonbasic $(x_2 = 0,\ x_4 = 0)$
  Then,

  $$x_1 + 3x_3 = 15$$
  $$x_1 = 9$$

  Solving this system, we obtain $x = (9, 0, 2, 0)$. This is a basic feasible solution.

- x_1, x_4 basic,
  x_2, x_3 nonbasic ($x_2 = 0,\ x_3 = 0$)
  Then,

  $$\begin{aligned} x_1 + 5x_4 &= 15 \\ x_1 + x_4 &= 9 \end{aligned}$$

  Solving this system, we obtain $x = (15/2, 0, 0, 3/2)$. This is a basic feasible solution.

- x_2, x_3 basic,
  x_1, x_4 nonbasic ($x_1 = 0,\ x_4 = 0$)
  Then,

  $$\begin{aligned} -x_2 + 3x_3 &= 15 \\ x_2 &= 9 \end{aligned}$$

  Solving this system, we obtain $x = (0, 9, 8, 0)$. This is a basic feasible solution.

- x_2, x_4 basic,
  x_1, x_3 nonbasic ($x_1 = 0,\ x_3 = 0$)
  Then,

  $$\begin{aligned} -x_2 + 5x_4 &= 15 \\ x_2 + x_4 &= 9 \end{aligned}$$

  Solving this system, we obtain $x = (0, 5, 0, 4)$. This is a basic feasible solution.

- x_3, x_4 basic,
  x_1, x_2 nonbasic ($x_1 = 0,\ x_2 = 0$)
  Then,

  $$\begin{aligned} 3x_3 + 5x_4 &= 15 \\ x_4 &= 9 \end{aligned}$$

  Solving this system, we obtain $x = (0, 0, -10, 9)$. This is basic but not feasible since $x_3 = -10 < 0$.

Question 2. First convert the LP into standard form:

$$\begin{aligned} \max\ & 5x_1 + 3x_2 + x_3 \\ \text{s.t. } & x_1 + x_2 + 3x_3 + s_1 = 6 \\ & 5x_1 + 3x_2 + 6x_3 + s_2 = 15 \\ & x_i \geq 0 \ i = 1, 2, 3 \\ & s_1, s_2 \geq 0 \end{aligned}$$

	z	x_1	x_2	x_3	s_1	s_2	RHS
1	1	-5	-3	-1	0	0	0
s_1	0	1	1	3	1	0	6
s_2	0	5	3	6	0	1	15

x_1 enters. Apply MRT: $\min\{6/1, 15/5\}$=3 Hence, s_2 leaves.

	z	x_1	x_2	x_3	s_1	s_2	RHS
1	1	0	0	5	0	1	15
s_1	0	0	2/5	9/5	1	-1/5	3
x_1	0	1	3/5	6/5	0	1/5	3

Since there is no negative row 0 coefficient, the solution is optimal: $x^* = (3, 0, 0)$ and $z^* = 15$.

Since there exists 0 in row 0, there may be alternative optimal solutions: Let x_2 enter the basis. Apply MRT: $\min\{\frac{3}{2/5}, \frac{3}{3/5}\} = 5$. Then x_1 leaves.

	z	x_1	x_2	x_3	s_1	s_2	RHS
1	1	0	0	5	0	1	15
s_1	0	-2/3	0	1	1	-1/3	1
x_2	0	5/3	1	2	0	1/3	5

$x^* = (0, 5, 0)$ and z^*=15.
(If x_1 enters again, you turn back to the previous tableau!)

When x_3 is unrestricted remember the following x_3=x_3'-x_3'' where x_3'' and x_3' are non-negative. Use the similar steps applied first part of the question.

when x_3 **urs** the standard form will be as following.

$$\begin{aligned}
\max\ & 5x_1 + 3x_2 + x_3' - x_3'' \\
\text{s.t. } & x_1 + x_2 + 3x_3' - 3x_3'' + s_1 = 6 \\
& 5x_1 + 3x_2 + 6x_3' - 6x_3'' + s_2 = 15 \\
& x_i \geq 0 \ i = 1, 2 \\
& s_1, s_2 \geq 0 \\
& x_3', x_3'' \geq 0
\end{aligned}$$

We then write the simplex tableau.

	z	x_1	x_2	x_3'	x_3''	s_1	s_2	RHS
1	1	-5	-3	-1	1	0	0	0
s_1	0	1	1	3	-3	1	0	6
s_2	0	5	3	6	-6	0	1	15

x_1 enters. Apply MRT: $\min\{6/1, 15/5\}$=3 Hence, s_2 leaves.

	z	x_1	x_2	x_3'	x_3''	s_1	s_2	RHS
1	1	0	0	5	-5	0	1	15
s_1	0	0	0.4	1.8	-1.8	1	-0.2	3
s_2	0	1	0.6	1.2	-1.2	0	0.2	3

x_3'' enters. However all ratio tests are negative. We cannot find any leaving variable. Then we cannot continue. Simplex Algorithm stops.

Question 3. Introduce slack variables s_1, s_2 and s_3. Then the LP in the standard form becomes:

$$
\begin{aligned}
\max\ & 2x_1 + x_2 + 6x_3 - 4x_4 \\
\text{s.t. } & x_1 + 2x_2 + 4x_3 - x_4 + s_1 = 6 \\
& 2x_1 + 3x_2 - x_3 + x_4 + s_2 = 12 \\
& x_1 + x_3 + x_4 + s_3 = 6 \\
& x_i \geq 0 \ i = 1, 2, 3, 4, 5, 6, 7
\end{aligned}
$$

Now construct the simplex tableau for initial basis s_1, s_2 and s_3. Now we will add the required variables to basis one by one. (Be careful! The following are not Simplex iterations to find the optimal solution!)

	z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
1	1	-2	-1	-6	4	0	0	0	0
s_1	0	1	2	4	-1	1	0	0	6
s_2	0	2	3	-1	1	0	1	0	12
s_3	0	1	0	1	1	0	0	1	6

Let x_1 enter and s_1 leave.

	z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
1	1	0	3	2	2	2	0	0	12
x_1	0	1	2	4	-1	1	0	0	6
s_2	0	0	-1	-9	3	-2	1	0	0
s_3	0	0	-2	-3	2	-1	0	1	0

Let x_2 enter and s_2 leave.

	z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
1	1	0	0	-25	11	-4	3	0	12
x_1	0	1	0	-14	5	-3	2	0	6
x_2	0	0	1	9	-3	2	-1	0	0
s_3	0	0	0	15	-4	3	-2	1	0

Let x_4 enter and s_3 leave.

	z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
1	1	0	0	65/4	0	17/4	-5/2	11/4	12
x_1	0	1	0	19/4	0	3/4	-1/2	5/4	6
x_2	0	0	1	-9/4	0	-1/4	1/2	-3/4	0
x_4	0	0	0	-15/4	1	-3/4	1/2	-1/4	0

The Simplex tableau is constructed for basis x_1, x_2 and x_4. This tableau is not optimal because there is a negative coefficient in row zero. Hence, s_2 enters.

To find the variable that leaves the basis, apply MRT: $\min\{0, 0\} = 0$ Hence, x_2 (or x_4) leaves.

Question 4. The LP in the standard form:

$$\begin{aligned} &\max\ 10x_1 + 8x_2 - 3x_3 + 3x_4 \\ &\text{s.t. } 2x_1 + 4x_2 - 0.5x_3 + 0.5x_4 + s_1 = 3 \\ &\quad -2x_1 + 6x_2 - 4.5x_3 + 4.5x_4 + s_2 = 7 \\ &\quad x_1, x_2, x_3, x_4, s_1, s_2 \geq 0 \end{aligned}$$

	z	x_1	x_2	x_3	x_4	s_1	s_2	RHS
z	1	-10	-8	3	-3	0	0	0
s_1	0	2	4	-0.5	0.5	1	0	3
s_2	0	-2	6	-4.5	4.5	0	1	7

x_1 enters, s_1 leaves.

	z	x_1	x_2	x_3	x_4	s_1	s_2	RHS
z	1	0	12	0.5	-0.5	5	0	15
x_1	0	1	2	-0.25	0.25	0.5	0	3/2
s_2	0	0	10	-5	5	1	1	10

x_4 enters, s_2 leaves.

	z	x_1	x_2	x_3	x_4	s_1	s_2	RHS
z	1	0	13	0	0	5.1	0.1	16
x_1	0	1	1.5	0	0	0.45	-0.05	1
x_4	0	0	2	-1	1	0.2	0.2	2

(a) Yes, it is a basic solution. Any tableu solution in proper form will be basic, since it corresponds to a basis B.

(b) Yes, it is a basic feasible solution. All RHS values are nonnegative.

(c) Yes, it is an optimal BFS. All row zero elements are ≥ 0.

Question 5. Decision Variables:
x_1: Amount of TLs invested (in millions) x_2: Amount of TLs borrowed (in millions)

Model:

$$\begin{aligned} \max\ & -0.1x_1 + 0.5x_2 \\ \text{s.t. } & x_1 \leq 1 \\ & x_2 \leq 0.4x_1 \\ & x_1 - x_2 \leq 0.8 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Now, apply simplex method. The LP in standard form is:

$$\begin{aligned} \max\ & -0.1x_1 + 0.5x_2 \\ \text{s.t. } & x_1 + s_1 = 1 \\ & x_2 - 0.4x_1 + s_2 = 0 \\ & x_1 - x_2 + s_3 = 0.8 \\ & x_1, x_2, s_1, s_2, s_3 \geq 0 \end{aligned}$$

	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	0.1	-0.5	0	0	0	0
s_1	0	1	0	1	0	0	1
s_2	0	-0.4	1	0	1	0	0
s_3	0	1	-1	0	0	1	0.8

x_2 enters, s_2 leaves.

	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	-0.1	0	0	0.5	0	0
s_1	0	1	0	1	0	0	1
x_2	0	-0.4	1	0	1	0	0
s_3	0	0.6	0	0	1	1	0.8

x_1 enters, s_1 leaves.

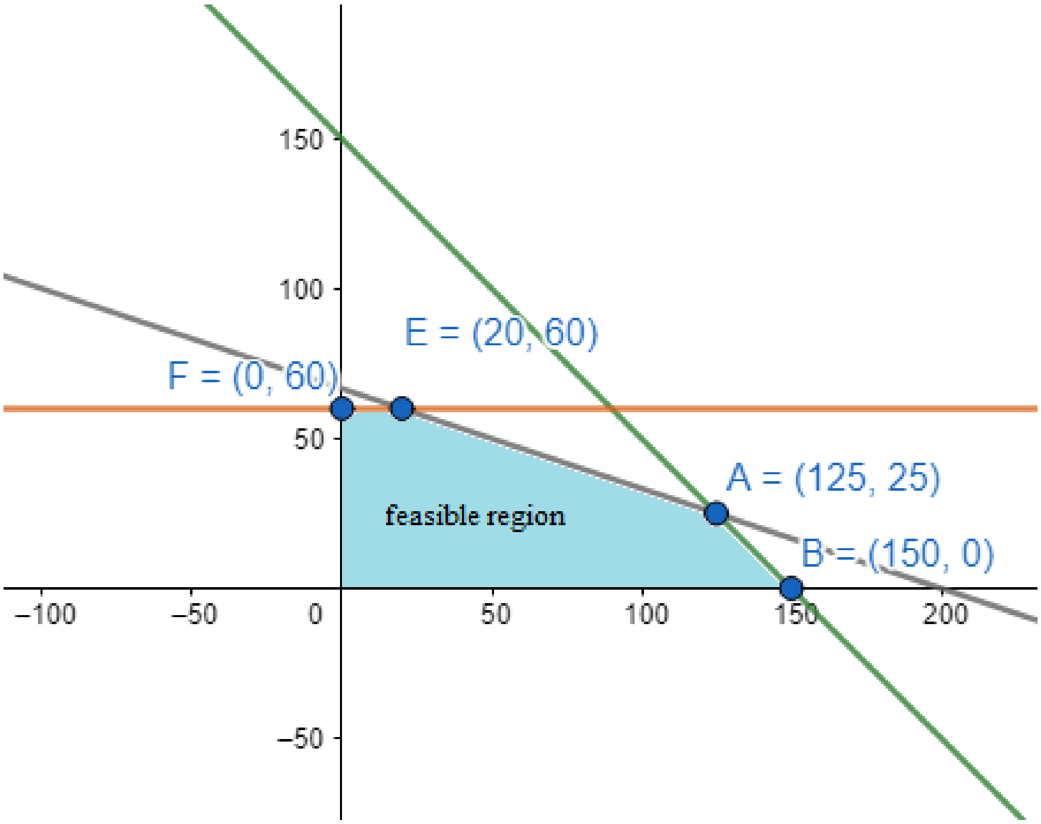
	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	0	0	0.1	0.5	0	0.1
x_1	0	1	0	1	0	0	1
x_2	0	0	1	0.4	1	0	0.4
s_3	0	0	0	-0.6	1	1	0.2

The optimal solution is $x^* = (1, 0.4)$, and $z^* = 0.1$.

Question 6. Decision Variables:
x_1: Amount of product 1 manufactured
x_2: Amount of product 2 manufactured

Then let us write the constraints and the objective function
Model:

$$\begin{aligned} \max\ & x_1 + 2x_2 \\ \text{s.t.}\ & x_1 + 3x_2 \leq 200 \\ & 2x_1 + 2x_2 \leq 300 \\ & x_2 \leq 60 \\ & x_1, x_2 \geq 0 \end{aligned}$$



Therefore there are 5 extreme points in the feasible region.A(125,25), B(150,0), E(20,60), F(0,60) and O(0,0).

So the optimum profit is 175 dollar. Solution is $x_1 = 125$ and $x_2 = 25$.

Question 7.

First Tableau:

x_1	x_2	e_1	e_2	x_3	x_4	RHS
-2	-3	0	0	-M	-M	0
2	1	-1	0	1	0	4
1	-1	0	-1	0	1	1

x_3 and x_4 are basic variables but their row zero coefficients are not zero so we need to add the first and the second row to row zero after multiplying by M:

x_1	x_2	e_1	e_2	x_3	x_4	RHS
3M-2	-3	-M	-M	0	0	5M
2	1	-1	0	1	0	4
1	-1	0	-1	0	1	1

Then, x_1 enters x_4 leaves.

Second Tableau:

x_1	x_2	s_1	e_1	x_3	x_4	RHS
0	0	0	0	-1	-1	0
1	1	1	0	0	0	3
2	1	0	-1	1	0	4
1	1	0	0	0	1	3

x_3 and x_4 are basic variables but their row zero coefficients are not zero so we need to add the second and the third row to row zero:

x_1	x_2	s_1	e_1	x_3	x_4	RHS
3	2	0	-1	0	0	7
1	1	1	0	0	0	3
2	1	0	-1	1	0	4
1	1	0	0	0	1	3

Then, x_1 enters x_3 leaves.

Third Tableau:

x_1	x_2	x_3	s_1	s_2	e_1	RHS
-2	-1	-1	0	0	0	0
1/3	2/3	0	1	0	1/3	1
2	1	0	0	1	0	3
2/3	1/3	1	0	0	-1/3	1

x_3 is a basic variable but its row zero coefficient is not zero so we need to add the third row to row zero:

x_1	x_2	x_3	s_1	s_2	e_1	RHS
-4/3	-2/3	0	0	0	-1/3	1
1/3	2/3	0	1	0	1/3	1
2	1	0	0	1	0	3
2/3	1/3	1	0	0	-1/3	1

Then, x_1 enters and s_2 leaves (or x_3 leaves).

Please do not consider the First Tableau. Second tableau is treated as a minimization problem. So here for maximization instead of x_1 enters x_3 leaves, we have an optimal solution. Since all row zero coefficients are positive.