IE 400: Principles of Engineering Management
Study Set 3 Solutions

Spring 2021

Question 1.

First Tableau:

x_1	x_2	e_1	e_2	x_3	x_4	RHS
-2	-3	0	0	-M	-M	0
2	1	-1	0	1	0	4
1	-1	0	-1	0	1	1

x_3 and x_4 are basic variables but their row zero coefficients are not zero so we need to add the first and the second row to row zero after multiplying by M:

x_1	x_2	e_1	e_2	x_3	x_4	RHS
3M-2	-3	-M	-M	0	0	5M
2	1	-1	0	1	0	4
1	-1	0	-1	0	1	1

Then, x_1 enters x_4 leaves.

Second Tableau:

x_1	x_2	s_1	e_1	x_3	x_4	RHS
0	0	0	0	-1	-1	0
1	1	1	0	0	0	3
2	1	0	-1	1	0	4
1	1	0	0	0	1	3

x_3 and x_4 are basic variables but their row zero coefficients are not zero so we need to add the second and the third row to row zero:

x_1	x_2	s_1	e_1	x_3	x_4	RHS
3	2	0	-1	0	0	7
1	1	1	0	0	0	3
2	1	0	-1	1	0	4
1	1	0	0	0	1	3

Then, x_1 enters x_3 leaves.

Third Tableau:

x_1	x_2	x_3	s_1	s_2	e_1	RHS
-2	-1	-1	0	0	0	0
1/3	2/3	0	1	0	1/3	1
2	1	0	0	1	0	3
2/3	1/3	1	0	0	-1/3	1

x_3 is a basic variable but its row zero coefficient is not zero so we need to add the third row to row zero:

x_1	x_2	x_3	s_1	s_2	e_1	RHS
-4/3	-2/3	0	0	0	-1/3	1
1/3	2/3	0	1	0	1/3	1
2	1	0	0	1	0	3
2/3	1/3	1	0	0	-1/3	1

Then, x_1 enters and s_2 leaves (or x_3 leaves).

Question 2. Give general conditions on each of the unknowns A-F such that each of the following statements is true.

(a) The tableau is final and there exists a unique optimal solution.

$$A < 0,\ \ B = 0,\ \ C \text{ any}, D \geq 0, E = 1,\ \ F \text{ any}$$

(b) The simplex method determines an unbounded solution from this tableau.

$$A > 0,\ \ B = 0,\ \ C \leq 0,\ \ D \geq 0, E = 1,\ \ F \text{ any}$$

(c) The current bfs is degenerate (not necessarily optimal).

$$A \text{ any},\ \ B = 0,\ \ C \text{ any},\ \ D = 0,\ \ E = 1,\ \ F \text{ any}$$

(d) d) The current solution is optimal, there are alternative optimal solutions but no alternative optimal bfs.

$$A = 0,\ \ B = 0,\ \ C \leq 0,\ \ D \geq 0,\ \ E = 1,\ \ F \text{ any}$$

Question 3. Convert into standard form and add artificial variable.

$$\begin{aligned} \min\ & -3x_1 + 4x_2 \\ \text{s.t. } & x_1 + x_2 + s_1 = 4 \\ & 2x_1 + 3x_2 - e_1 + a_1 = 18 \\ & x_1, x_2, s_1, e_1, a_1 \geq 0 \end{aligned}$$

PHASE 1

	z	x_1	x_2	s_1	e_1	a_1	RHS
1	1	0	0	0	0	-1	0
s_1	0	1	1	1	0	0	4
a_1	0	2	3	0	-1	1	18

Add second row to row 0.

	z	x_1	x_2	s_1	e_1	a_1	RHS
1	1	2	3	0	-1	0	18
s_1	0	1	1	1	0	0	4
a_1	0	2	3	0	-1	1	18

x_2 enters. Apply MRT: $\min\{4/1, 18/2\} = 4$. Then, s_1 leaves the basis.

	z	x_1	x_2	s_1	e_1	a_1	RHS
1	1	-1	0	-3	-1	0	6
x_2	0	1	1	1	0	0	4
a_1	0	-1	0	-3	-1	1	6

The tableau is optimal, but a_1 is still in the basis. Hence, the Phase 1 problem is infeasible.

Question 4. Convert the following LP to standard form and solve with the Big-M method.

$$
\begin{aligned}
\max\ & -x_1 + 3x_2 + x_3 \\
\text{s.t.}\ & -x_1 + x_2 - x_3 \geq -4 \\
& 2x_1 + x_2 + x_3 \geq 10 \\
& x_1 \geq 0 \\
& x_2 \leq 0 \\
& x_3 \text{ free}
\end{aligned}
$$

Let $x_3 = x_3' - x_3''$ where $x_3' \geq 0$ and $x_3'' \geq 0$. Let $-x_2 = x_2'$. Introduce slack variable s_1 and excess variable e_2.

$$\begin{aligned} \max\ & -x_1 - 3x_2' + (x_3' - x_3'') - Ma_2 \\ \text{s.t. } & x_1 + x_2' + (x_3' - x_3'') + s_1 = 4 \\ & 2x_1 - x_2' + (x_3' - x_3'') - e_2 + a_2 = 10 \\ & x_1, x_2', x_3', x_3'', s_1, e_2 \geq 0 \end{aligned}$$

	z	x_1	x_2'	x_3'	x_3''	s_1	e_2	a_2	RHS
	1	1	3	-1	1	0	0	M	0
s_1	0	1	1	1	-1	1	0	0	4
a_1	0	2	-1	1	-1	0	-1	1	10

Multiply the second row with $-M$ and add it to row 0:

	z	x_1	x_2'	x_3'	x_3''	s_1	e_2	a_2	RHS
	1	-2M+1	M+3	-M-1	M+1	0	M	0	-10M
s_1	0	1	1	1	-1	1	0	0	4
a_1	0	2	-1	1	-1	0	-1	1	10

x_1 enters the basis. By minimum ratio test: $\min\{4/1, 10/2\} = 4$, s_1 leaves:

	z	x_1	x_2'	x_3'	x_3''	s_1	e_2	a_2	RHS
	1	0	3M+2	M-2	-M+2	2M-1	M	0	-2M-4
x_1	0	1	1	1	-1	1	0	0	4
a_1	0	0	-3	-1	1	-2	-1	1	2

x_3'' enters the basis. Apply MRT: $\min\{2/1\}$. Then, a_1 leaves:

	z	x_1	x_2'	x_3'	x_3''	s_1	e_2	a_2	RHS
	1	0	8	0	0	3	2	M-2	-8
x_1	0	1	-2	0	0	-1	-1	1	6
x_3''	0	0	-3	-1	1	-2	-1	1	2

No negative coefficient in row 0. The tableau is final. (a_2 is not in the basis so remove the column a_2 from the tableau)

Then,

$$x^* = (x_1, x_2', x_3', x_3'', s_1, e_2) = (6, 0, 0, 2, 0, 0)$$

and $z^* = -8$. Note that $x_2 = -x_2' = 0$, $x_3 = x_3' - x_3'' = 0 - 2 = -2$.

Question 5. Consider the following simplex tableau of a given **maximization** LP problem.

	z	x_1	x_2	x_3	x_4	x_5	RHS
1	1	2	**A**	0	0	0	12
x_3	0	**B**	-1	1	0	0	5
x_4	0	-6	**C**	0	1	0	1
x_5	0	-11	**D**	0	0	1	**E**

Give general conditions on each of the unknowns A-E such that each of the following statements is true. Even if the statement holds independently of the values of a specific variable, you should still mention that that variable can take any value. So, for each part you should present 5 conditions, one for each unknown.

(a) (*5 pts*) The tableau is final and there exists a unique optimal solution.

$A > 0$, B any, C any, D any, $E \geq 0$

(b) (*5 pts*) The simplex method determines an unbounded solution from this tableau. $A < 0$, B any, $C \leq 0$, $D \leq 0$, $E \geq 0$

(c) (*5 pts*) The current bfs is degenerate (not necessarily optimal).

A any, B any, C any, D any, $E = 0$

(d) (*5 pts*) The current solution is optimal, there are alternative optimal solutions but no alternative optimal bfs.

$A = 0$, B any, $C \leq 0$, $D \leq 0$, $E \geq 0$

Question 6. Consider the following LP:

$$\begin{aligned} \max\ & x_1 + 2x_2 + 4x_3 \\ \text{s.t.}\ & -x_1 + x_2 - x_3 \geq 2 \\ & -2x_1 + x_2 + x_3 \leq 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solve the LP using big M method. Could you find an optimal solution? If yes, report the optimal solution. If no, state why.

Convert the LP to standard form:

$$\begin{aligned} \max\ & x_1 + 2x_2 + 4x_3 - Ma_1 \\ \text{s.t.}\ & -x_1 + x_2 - x_3 - e_1 + a_1 = 2 \\ & -2x_1 + x_2 + x_3 + s_2 = 1 \\ & x_1, x_2, x_3, e_1, a_1, s_2 \geq 0 \end{aligned}$$

	z	x_1	x_2	x_3	e_1	a_1	s_2	RHS
z	1	-1	-2	-4	0	M	0	0
a_1	0	-1	1	-1	-1	1	0	2
s_2	0	-2	1	1	0	0	1	1

	z	x_1	x_2	x_3	e_1	a_1	s_2	RHS
z	1	M-1	-M-2	M-4	M	0	0	-2M
a_1	0	-1	1	-1	-1	1	0	2
s_2	0	-2	1	1	0	0	1	1

x_2 enters, s_2 leaves.

	z	x_1	x_2	x_3	e_1	a_1	s_2	RHS
z	1	-M-5	0	2M-2	M	0	M+2	-M+2
a_1	0	1	0	-2	-1	1	-1	1
x_2	0	-2	1	1	0	0	1	1

x_1 enters, a_1 leaves.

	z	x_1	x_2	x_3	e_1	a_1	s_2	RHS
z	1	0	0	-12	-5	M+5	-3	7
x_1	0	1	0	-2	-1	1	-1	1
x_2	0	0	1	-3	-2	2	-1	3

x_3 enters, but its coefficients on the constraints are negative. Then, there is no leaving variable. Hence, the problem is **unbounded**.