IE 400: Principles of Engineering Management
Study Set 4 Solutions

Spring 2021

Question 1.

(a) **Parameters:**

f_i= fixed cost of plant i, $i = 1, \dots, 4$

c_{ij}= cost of producing car j at plant i $i = 1, \dots, 4$, $j = 1, 2, 3$

Decision Variables:

$$y_i = \begin{cases} 1 & \text{if plant i is used} \quad i = 1, \dots, 4 \\ 0 & \text{o.w} \end{cases}$$

$$x_{ij} = \begin{cases} 1 & \text{if car j is produced at plant i} \quad i = 1, \dots, 4, \quad j = 1, 2, 3 \\ 0 & \text{o.w} \end{cases}$$

Model:

$$\min \sum_{i=1}^{4} f_i y_i + \sum_{i=1}^{4} \sum_{j=1}^{3} c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{3} x_{ij} \leq 1 \qquad i = 1, 2, 3, 4 \qquad (1)$$

$$\sum_{i=1}^{4} x_{ij} = 1 \qquad j = 1, 2, 3 \qquad (2)$$

$$y_3 + y_4 \leq 1 + y_1 \qquad (3)$$

$$x_{ij} \leq y_i \qquad i = 1, 2, 3, 4 \quad j = 1, 2, 3$$

$$x_{ij} \in \{0, 1\} \qquad i = 1, 2, 3, 4 \quad j = 1, 2, 3$$

$$y_j \in \{0, 1\} \qquad j = 1, 2, 3$$

Constraint (1): Each plant can produce one type of car.
Constraint (2): The total production of each type of car must be at a single plant; that is, for example, if any Tauruses are made at plant 1, then all Tauruses must be made there.
Constraint (3): If plants 3 and 4 are used, then plant 1 must also be used.

(b)

$$\begin{array}{lll} \text{either} & y_1 + y_2 \leq 1 & (*) \\ \text{or} & y_3 + y_4 \geq 1 & (**) \end{array}$$

In order to linearize the constraint, add the following decision variable:

$$w = \begin{cases} 0 & \text{if constraint } (*) \text{ holds} \\ 1 & \text{if constraint } (**) \text{ holds} \end{cases}$$

Then add the following constraints to the model:

$$\begin{aligned} & y_1 + y_2 - 1 \leq Mw \\ & -y_3 - y_4 + 1 \leq (1-w)M \\ & w \in \{0,1\} \end{aligned}$$

where M is a sufficiently large number i.e. we can take M=1. Then;

$$\begin{aligned} & y_1 + y_2 \leq 1 + w \\ & y_3 + y_4 \geq w \\ & w \in \{0,1\} \end{aligned}$$

(For $w = 0$, the constraint (*) holds and (**) becomes redundant. For $w = 1$, the constraint (**) holds and (*) becomes redundant.)

Question 2.

Decision Variables:

$$x_{ij} = \begin{cases} 1 & \text{if astronaut i is assigned to post j } i = 1,2,3,4,\ \ j = 1,2,3,4 \\ 0 & \text{o.w} \end{cases}$$

Model:

$$\begin{aligned}
\max \; & \sum_{i=1}^{4}\sum_{j=1}^{4} p_{ij}x_{ij} \\
\text{s.t.} \; & \sum_{i=1}^{4} x_{ij} = 1 && j = 1,2,3,4 && (1) \\
& \sum_{j=1}^{4} x_{ij} = 1 && i = 1,2,3,4 && (2) \\
& x_{ij} \in \{0,1\} && i = 1,2,3,4 \;\; j = 1,2,3,4
\end{aligned}$$

- Constraint (1) indicates that each post should have a single astronaut.
- Constraint (2) indicates that each astronaut should be assigned to a single post.

Question 3.

Parameters: As given in the question

Decision Variables: As given in the question

Model:

$$
\begin{aligned}
\min \ & \sum_{i=1}^{M} x_i && && \\
\text{s.t.} \ & \sum_{i=1}^{M} y_{ik} = d_k && k = 1, \cdots, n && \\
& \sum_{k=1}^{n} y_{ik} \cdot l_k \leq l \cdot x_i && i = 1, \cdots, M && \\
& x_i \in \{0, 1\} && i = 1, \cdots, M && \\
& y_{ik} \geq 0 && i = 1, \cdots, M && k = 1, \cdots, n \\
& y_{ik} \text{ integer} && i = 1, \cdots, M && k = 1, \cdots, n
\end{aligned}
$$

Question 4.

(a) **Parameters:**

w_j= weight capacity of compartment j (in tons)

s_j= space capacity of compartment j (in cubic meters)

wt_i= weight of cargo i (in tons)

v_i= weight of cargo i (in cubic meters/ton)

p_i= profit gained from cargo i (TL/ton)

Decision Variables:

x_{ij}= amount of cargo i distributed to compartment j in tons

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{4}\sum_{j=1}^{3} p_i x_{ij} \\
\text{s.t.} \quad & \sum_{j=1}^{3} x_{ij} \le wt_i && i = 1,2,3,4 && (1)\\
& \sum_{i=1}^{4} v_i x_{ij} \le s_j && j = 1,2,3 && (2)\\
& \sum_{i=1}^{4} x_{ij} \le w_j && j = 1,2,3 && (3)\\
& \frac{\sum_{i=1}^{4} x_{i1}}{w_1} = \frac{\sum_{i=1}^{4} x_{i2}}{w_2} = \frac{\sum_{i=1}^{4} x_{i3}}{w_3} && && (4)\\
& x_{ij} \ge 0 && i = 1,2,3,4 \;\; j = 1,2,3
\end{aligned}
$$

(b) Replace constraint (4) with

$$
\left|\frac{\sum_{i=1}^{4} x_{i1}}{w_1} - \frac{\sum_{i=1}^{4} x_{i2}}{w_2}\right| \le 0.1
$$

$$
\left|\frac{\sum_{i=1}^{4} x_{i1}}{w_1} - \frac{\sum_{i=1}^{4} x_{i3}}{w_3}\right| \le 0.1
$$

$$
\left|\frac{\sum_{i=1}^{4} x_{i2}}{w_2} - \frac{\sum_{i=1}^{4} x_{i3}}{w_3}\right| \le 0.1
$$

You need to linearize these constraints before adding to the model:

$$
\begin{aligned}
-0.1 \le \frac{\sum_{i=1}^{4} x_{i1}}{w_1} - \frac{\sum_{i=1}^{4} x_{i2}}{w_2} \le 0.1 \\
-0.1 \le \frac{\sum_{i=1}^{4} x_{i1}}{w_1} - \frac{\sum_{i=1}^{4} x_{i3}}{w_3} \le 0.1 \\
-0.1 \le \frac{\sum_{i=1}^{4} x_{i2}}{w_2} - \frac{\sum_{i=1}^{4} x_{i3}}{w_3} \le 0.1
\end{aligned}
$$

(c) (b) is more likely to have a larger optimal value since the feasible region in (b) includes the feasible region in (a).

(d) **Decision Variables:**

x_{ij}= amount of cargo i distributed to compartment j in tons

$$y_{ij} = \begin{cases} 1 & \text{if cargo i is shipped in compartment j} \\ 0 & \text{o.w} \end{cases}$$

Model: All constraints in (b) +

$$\begin{aligned}
& x_{ij} \leq w_j y_{ij} && i = 1,2,3,4 \quad j = 1,2,3 \\
& \sum_{j=1}^{3} y_{ij} \leq 1 && i = 1,2,3,4 \\
& y_{ij} \in \{0,1\} && i = 1,2,3,4 \quad j = 1,2,3 \\
& x_{ij} \text{ integer} && i = 1,2,3,4 \quad j = 1,2,3
\end{aligned}$$

Question 5.

(a) Formulate the model to determine which trucks to use so as to minimize the cost of delivering to all the customers.

Decision Variables:

$$x_k = \begin{cases} 1 & \text{if truck k is used } k = 1, \ldots, 7 \\ 0 & \text{o.w} \end{cases}$$

$$y_{jk} = \begin{cases} 1 & \text{if customer j is visited by truck k } j = 1, \ldots, 20 \quad k = 1, \ldots, 7 \\ 0 & \text{o.w} \end{cases}$$

$$z_{jk} = \text{Amount of good is carried by truck k for customer j}$$
$$j = 1, \ldots, 20 \quad k = 1, \ldots, 7$$

Model:

$$\begin{aligned}
\min \ & \sum_{k=1}^{7} q_k x_k \\
\text{s.t.} \ & \sum_{k=1}^{7} z_{jk} = d_j && j = 1, \ldots, 20 \\
& \sum_{j=1}^{20} z_{jk} \le C_k x_k && k = 1, \ldots, 7 \\
& z_{jk} \le d_j y_{jk} && j = 1, \ldots, 20 \quad k = 1, \ldots, 7 \\
& y_{jk} \le x_k && j = 1, \ldots, 20 \quad k = 1, \ldots, 7 \\
& x_k \in \{0, 1\} && k = 1, \ldots, 7 \\
& y_{jk} \in \{0, 1\} && j = 1, \ldots, 20 \quad k = 1, \ldots, 7 \\
& z_{jk} \ge 0 && j = 1, \ldots, 20 \quad k = 1, \ldots, 7
\end{aligned}$$

(b) Add the following requirements to your model:

- A single truck cannot deliver to more than six customers.

$$\sum_{j=1}^{20} y_{jk} \leq 6 \quad k = 1, \ldots, 7$$

- Customer 1 and customer 7 cannot be visited by the same truck.

$$y_{1k} + y_{7k} \leq 1 \quad k = 1, \ldots, 7$$

- Both customer 2 and customer 12 should be visited by the same truck(s).

$$y_{2,k} = y_{12,k} \quad k = 1, \ldots, 7$$

- If a truck visits customer 4 then it has to visit customer 19 or customer 20. (Assume demands of these customers do not exceed truck capacities.)

$$y_{4,k} \leq y_{19,k} + y_{20,k} \quad k = 1, \ldots, 7$$

- Either at least two of the customers 5, 6 and 7 or at most three of customers 11, 12, 13 and 14 should be visited by truck 4.

$$\text{either } y_{5,4} + y_{6,4} + y_{7,4} \geq 2 \tag{1}$$
$$\text{or } y_{11,4} + y_{12,4} + y_{13,4} + y_{14,4} \leq 3 \tag{2}$$

In order to linearize the constraint, add the following decision variable:

$$w = \begin{cases} 0 & \text{if constraint 1 holds} \\ 1 & \text{if constraint 2 holds} \end{cases}$$

Then add the following constraints to the model:

$$\begin{aligned} &- y_{5,4} - y_{6,4} - y_{7,4} + 2 \leq Mw \\ &y_{11,4} + y_{12,4} + y_{13,4} + y_{14,4} - 3 \leq M(1 - w) \\ &w \in \{0, 1\} \end{aligned}$$

where M is a sufficiently large number i.e. we can take M=2. Then the constraints can be arranged as,

$$
\begin{aligned}
&y_{5,4} + y_{6,4} + y_{7,4} \geq 2 - 2w \\
&y_{11,4} + y_{12,4} + y_{13,4} + y_{14,4} \leq 5 - 2w \\
&w \in \{0, 1\}
\end{aligned}
$$

Question 6.

Decision Variables:

$$y_i = \begin{cases} 1 & \text{if constraint i is satisfied} \\ 0 & \text{o.w} \end{cases}$$

$$z_1 = \begin{cases} 1 & \text{if at least one of 7,8,9 is satisfied} \\ 0 & \text{o.w} \end{cases}$$

$$z_2 = \begin{cases} 1 & \text{if at least two of 10,11,12,13 are satisfied} \\ 0 & \text{o.w} \end{cases}$$

Model:

$$
\begin{aligned}
&\min\ 0 \\
&\text{s.t.}\ \sum_{j=1}^{n} a_{ij}x_j \le b_i + M(1-y_i) && i=1,\ldots,20 && (1)\\
&\sum_{j=1}^{n} a_{ij}x_j \ge b_i - My_i && i=1,\ldots,20 && (2)\\
&\sum_{i=1}^{20} y_i \ge 10 && && (3)\\
&y_1 \le y_4 && && (4)\\
&y_2 + y_3 \le 1 && && (5)\\
&y_5 = y_6 && && (6)\\
&y_7 + y_8 + y_9 \ge z_1 && && (7)\\
&y_{10} + y_{11} + y_{12} + y_{13} \ge 2z_2 && && (8)\\
&z_1 + z_2 \ge 1 && && (9)\\
&z_1, z_2 \in \{0,1\} && && (10)\\
&y_i \in \{0,1\} && i=1,\ldots,20 && (11)
\end{aligned}
$$

where M is a sufficiently large number. Observe that we minimize zero, which means that the model is a feasibility problem. If a feasible solution is found for the above model, the objective will take value 0.

Let's take a closer look at constraints (1) and (2):

- If $y_i = 1$:

  Constraint (1) becomes $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ which means that the constraint i is satisfies.
  Constraint (2) becomes $\sum_{j=1}^{n} a_{ij}x_j \geq b_i - M$. This makes the constraint redundant as $-M$ is a really small number.

- If $y_i = 0$:

  Constraint (1) becomes $\sum_{j=1}^{n} a_{ij}x_j \leq b_i + M$ which makes the constraint redundant as M is a really large number.
  Constraint (2) becomes $\sum_{j=1}^{n} a_{ij}x_j \geq b_i$. This means constraint is not satisfied.

(3): Point x should satisfy at least 10 of the 20 constraints.
(4): If point x satisfies constraint 1, then it should also satisfy constraint 4.
(5): If point x satisfies constraint 2, then it should not satisfy constraint 3.
(6): If point x satisfies either both of the constraints 5 and 6 or neither of them.
(7)-(8)-(9): Point x should satisfy either at least one of the constraints 7,8,9 or at least two of constraints 10,11,12,13.

Question 7.

Decision Variables:

x_i : Number of workers employed on line i $\;i = 1, 2$

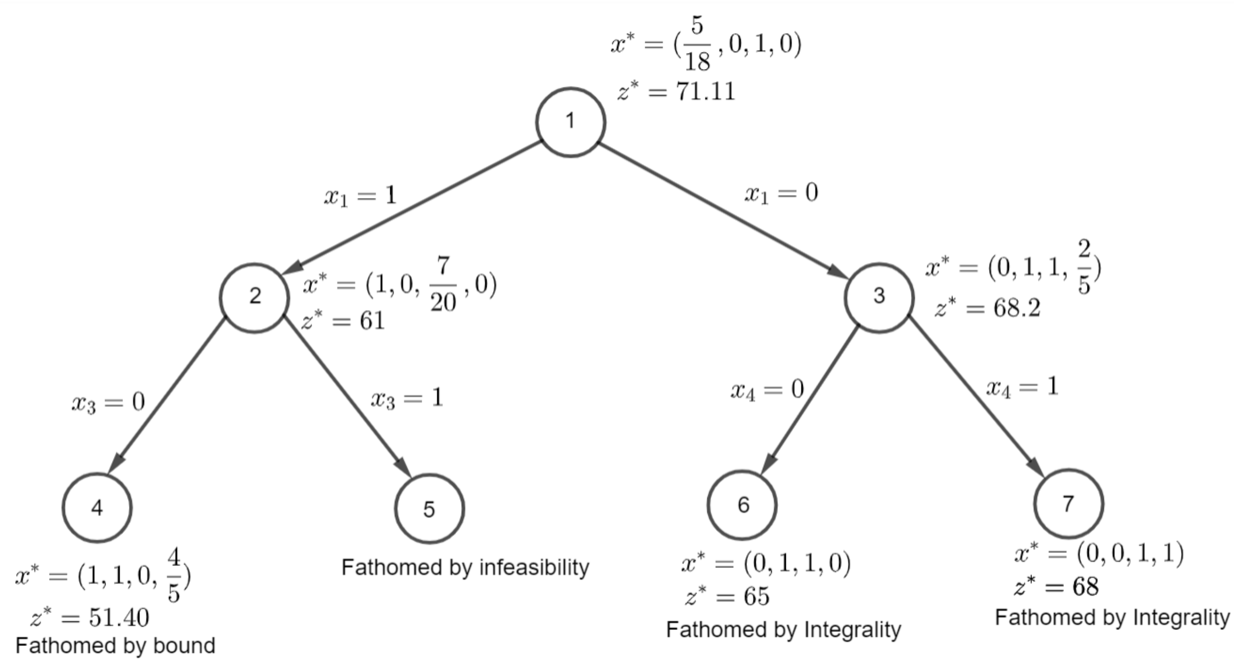
$$y_i = \begin{cases} 1 & \text{if line i is used} \;\; i = 1, 2 \\ 0 & \text{o.w} \end{cases}$$

Model:

$$\begin{aligned}
\text{min } & 1000y_1 + 2000y_2 + 500x_1 + 900x_2 \\
\text{s.t. } & 20x_1 + 50x_2 \geq 120 \\
& 30x_1 + 35x_2 \geq 150 \\
& 40x_1 + 45x_2 \geq 200 \\
& x_1 \leq 7y_1 \\
& x_2 \leq 7y_2 \\
& x_1, x_2 \geq 0 \\
& y_1, y_2 \in \{0, 1\}
\end{aligned}$$

Question 8. Order of the variables:

$$x_3 > x_1 > x_2 > x_4$$



Optimal Solution is: $x^* = (0, 0, 1, 1)$.