IE 400: Principles of Engineering Management
Study Set 2 Solutions

Summer 2021

Question 1.

- x_1, x_2 basic,
  x_3, x_4 nonbasic ($x_3 = 0,\ x_4 = 0$)
  Then,

  $$\begin{aligned} x_1 - x_2 &= 15 \\ x_1 + x_2 &= 9 \end{aligned}$$

  Solving this system, we obtain $x = (12, -3, 0, 0)$. This is basic but not feasible since $x_2 = -3 < 0$.

- x_1, x_3 basic,
  x_2, x_4 nonbasic ($x_2 = 0,\ x_4 = 0$)
  Then,

  $$\begin{aligned} x_1 + 3x_3 &= 15 \\ x_1 &= 9 \end{aligned}$$

  Solving this system, we obtain $x = (9, 0, 2, 0)$. This is a basic feasible solution.

- x_1, x_4 basic,
  x_2, x_3 nonbasic ($x_2 = 0,\ x_3 = 0$)
  Then,

$$x_1 + 5x_4 = 15$$
$$x_1 + x_4 = 9$$

  Solving this system, we obtain $x = (15/2, 0, 0, 3/2)$. This is a basic feasible solution.

- x_2, x_3 basic,
  x_1, x_4 nonbasic ($x_1 = 0,\ x_4 = 0$)
  Then,

$$-x_2 + 3x_3 = 15$$
$$x_2 = 9$$

  Solving this system, we obtain $x = (0, 9, 8, 0)$. This is a basic feasible solution.

- x_2, x_4 basic,
  x_1, x_3 nonbasic ($x_1 = 0,\ x_3 = 0$)
  Then,

$$-x_2 + 5x_4 = 15$$
$$x_2 + x_4 = 9$$

  Solving this system, we obtain $x = (0, 5, 0, 4)$. This is a basic feasible solution.

- x_3, x_4 basic,
  x_1, x_2 nonbasic ($x_1 = 0,\ x_2 = 0$)
  Then,

$$3x_3 + 5x_4 = 15$$
$$x_4 = 9$$

Solving this system, we obtain $x = (0, 0, -10, 9)$. This is basic but not feasible since $x_3 = -10 < 0$.

Question 2.

First convert the LP into standard form:

$$
\begin{aligned}
\max\ & 5x_1 + 3x_2 + x_3 \\
\text{s.t.}\ & x_1 + x_2 + 3x_3 + s_1 = 6 \\
& 5x_1 + 3x_2 + 6x_3 + s_2 = 15 \\
& x_i \geq 0 \ i = 1, 2, 3 \\
& s_1, s_2 \geq 0
\end{aligned}
$$

	z	x_1	x_2	x_3	s_1	s_2	RHS
1	1	-5	-3	-1	0	0	0
s_1	0	1	1	3	1	0	6
s_2	0	5	3	6	0	1	15

x_1 enters. Apply MRT: min$\{6/1, 15/5\}$=3 Hence, s_2 leaves.

	z	x_1	x_2	x_3	s_1	s_2	RHS
1	1	0	0	5	0	1	15
s_1	0	0	2/5	9/5	1	-1/5	3
x_1	0	1	3/5	6/5	0	1/5	3

Since there is no negative row 0 coefficient, the solution is optimal: $x^* = (3, 0, 0)$ and $z^* = 15$.

Since there exists 0 in row 0, there may be alternative optimal solutions: Let x_2 enter the basis. Apply MRT: $\min\{\frac{3}{2/5}, \frac{3}{3/5}\} = 5$. Then x_1 leaves.

	z	x_1	x_2	x_3	s_1	s_2	RHS
1	1	0	0	5	0	1	15
s_1	0	-2/3	0	1	1	-1/3	1
x_2	0	5/3	1	2	0	1/3	5

$x^* = (0,5,0)$ and z^*=15.
(If x_1 enters again, you turn back to the previous tableau!)

Question 3.

Introduce slack variables s_1, s_2 and s_3. Then the LP in the standard form becomes:

$$\begin{aligned} \max\ & 2x_1 + x_2 + 6x_3 - 4x_4 \\ \text{s.t.}\ & x_1 + 2x_2 + 4x_3 - x_4 + s_1 = 6 \\ & 2x_1 + 3x_2 - x_3 + x_4 + s_2 = 12 \\ & x_1 + x_3 + x_4 + s_3 = 6 \\ & x_i \geq 0\ i = 1,2,3,4,5,6,7 \end{aligned}$$

Now construct the simplex tableau for initial basis s_1, s_2 and s_3. Now we will add the required variables to basis one by one. (Be careful! The following are not Simplex iterations to find the optimal solution!)

	z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
1	1	-2	-1	-6	4	0	0	0	0
s_1	0	1	2	4	-1	1	0	0	6
s_2	0	2	3	-1	1	0	1	0	12
s_3	0	1	0	1	1	0	0	1	6

Let x_1 enter and s_1 leave.

	z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
1	1	0	3	2	2	2	0	0	12
x_1	0	1	2	4	-1	1	0	0	6
s_2	0	0	-1	-9	3	-2	1	0	0
s_3	0	0	-2	-3	2	-1	0	1	0

Let x_2 enter and s_2 leave.

	z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
1	1	0	0	-25	11	-4	3	0	12
x_1	0	1	0	-14	5	-3	2	0	6
x_2	0	0	1	9	-3	2	-1	0	0
s_3	0	0	0	15	-4	3	-2	1	0

Let x_4 enter and s_3 leave.

	z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
1	1	0	0	65/4	0	17/4	-5/2	11/4	12
x_1	0	1	0	19/4	0	3/4	-1/2	5/4	6
x_2	0	0	1	-9/4	0	-1/4	1/2	-3/4	0
x_4	0	0	0	-15/4	1	-3/4	1/2	-1/4	0

The Simplex tableau is constructed for basis x_1, x_2 and x_4. This tableau is not optimal because there is a negative coefficient in row zero. Hence, s_2 enters.

To find the variable that leaves the basis, apply MRT: $\min\{0, 0\} = 0$ Hence, x_2 (or x_4) leaves.

Question 4.

The LP in the standard form:

$$\begin{aligned} \max\ & 10x_1 + 8x_2 - 3x_3 + 3x_4 \\ \text{s.t.}\ & 2x_1 + 4x_2 - 0.5x_3 + 0.5x_4 + s_1 = 3 \\ & -2x_1 + 6x_2 - 4.5x_3 + 4.5x_4 + s_2 = 7 \\ & x_1, x_2, x_3, x_4, s_1, s_2 \geq 0 \end{aligned}$$

	z	x_1	x_2	x_3	x_4	s_1	s_2	RHS
z	1	-10	-8	3	-3	0	0	0
s_1	0	2	4	-0.5	0.5	1	0	3
s_2	0	-2	6	-4.5	4.5	0	1	7

x_1 enters, s_1 leaves.

	z	x_1	x_2	x_3	x_4	s_1	s_2	RHS
z	1	0	12	0.5	-0.5	5	0	15
x_1	0	1	2	-0.25	0.25	0.5	0	3/2
s_2	0	0	10	-5	5	1	1	10

x_4 enters, s_2 leaves.

	z	x_1	x_2	x_3	x_4	s_1	s_2	RHS
z	1	0	13	0	0	5.1	0.1	16
x_1	0	1	1.5	0	0	0.45	-0.05	1
x_4	0	0	2	-1	1	0.2	0.2	2

(a) Yes, it is a basic solution. Any tableu solution in proper form will be basic, since it corresponds to a basis B.

(b) Yes, it is a basic feasible solution. All RHS values are nonnegative.

(c) Yes, it is an optimal BFS. All row zero elements are ≥ 0.

Question 5.
Decision Variables:
x_1: Amount of TLs invested (in millions)
x_2: Amount of TLs borrowed (in millions)

Model:

$$
\begin{aligned}
\max\ & -0.1x_1 + 0.5x_2 \\
\text{s.t. } & x_1 \leq 1 \\
& x_2 \leq 0.4x_1 \\
& x_1 - x_2 \leq 0.8 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Now, apply simplex method. The LP in standard form is:

$$
\begin{aligned}
\max\ & -0.1x_1 + 0.5x_2 \\
\text{s.t.}\ & x_1 + s_1 = 1 \\
& x_2 - 0.4x_1 + s_2 = 0 \\
& x_1 - x_2 + s_3 = 0.8 \\
& x_1, x_2, s_1, s_2, s_3 \geq 0
\end{aligned}
$$

	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	0.1	-0.5	0	0	0	0
s_1	0	1	0	1	0	0	1
s_2	0	-0.4	1	0	1	0	0
s_3	0	1	-1	0	0	1	0.8

x_2 enters, s_2 leaves.

	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	-0.1	0	0	0.5	0	0
s_1	0	1	0	1	0	0	1
x_2	0	-0.4	1	0	1	0	0
s_3	0	0.6	0	0	1	1	0.8

x_1 enters, s_1 leaves.

	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	0	0	0.1	0.5	0	0.1
x_1	0	1	0	1	0	0	1
x_2	0	0	1	0.4	1	0	0.4
s_3	0	0	0	-0.6	1	1	0.2

The optimal solution is $x^* = (1, 0.4)$, and $z^* = 0.1$.

Question 6.

First Tableau:

x_1	x_2	e_1	e_2	x_3	x_4	RHS
-2	-3	0	0	-M	-M	0
2	1	-1	0	1	0	4
1	-1	0	-1	0	1	1

x_3 and x_4 are basic variables but their row zero coefficients are not zero so we need to add the first and the second row to row zero after multiplying by M:

x_1	x_2	e_1	e_2	x_3	x_4	RHS
3M-2	-3	-M	-M	0	0	5M
2	1	-1	0	1	0	4
1	-1	0	-1	0	1	1

Then, x_1 enters x_4 leaves.

Second Tableau:

x_1	x_2	s_1	e_1	x_3	x_4	RHS
0	0	0	0	-1	-1	0
1	1	1	0	0	0	3
2	1	0	-1	1	0	4
1	1	0	0	0	1	3

x_3 and x_4 are basic variables but their row zero coefficients are not zero so we need to add the second and the third row to row zero:

x_1	x_2	s_1	e_1	x_3	x_4	RHS
3	2	0	-1	0	0	7
1	1	1	0	0	0	3
2	1	0	-1	1	0	4
1	1	0	0	0	1	3

Then, x_1 enters x_3 leaves.

Third Tableau:

x_1	x_2	x_3	s_1	s_2	e_1	RHS
-2	-1	-1	0	0	0	0
1/3	2/3	0	1	0	1/3	1
2	1	0	0	1	0	3
2/3	1/3	1	0	0	-1/3	1

x_3 is a basic variable but its row zero coefficient is not zero so we need to add the third row to row zero:

x_1	x_2	x_3	s_1	s_2	e_1	RHS
-4/3	-2/3	0	0	0	-1/3	1
1/3	2/3	0	1	0	1/3	1
2	1	0	0	1	0	3
2/3	1/3	1	0	0	-1/3	1

Then, x_1 enters and s_2 leaves (or x_3 leaves).

Question 7.

(a) The tableau is final and there exists a unique optimal solution.

$$A < 0,\ \ B = 0,\ \ C \text{ any}, D \geq 0, E = 1,\ \ F \text{ any}$$

(b) The simplex method determines an unbounded solution from this tableau.

$$A > 0,\ \ B = 0,\ \ C \leq 0,\ \ D \geq 0, E = 1,\ \ F \text{ any}$$

(c) The current bfs is degenerate (not necessarily optimal).

$$A \text{ any},\ \ B = 0,\ \ C \text{ any},\ \ D = 0,\ \ E = 1,\ \ F \text{ any}$$

(d) d) The current solution is optimal, there are alternative optimal solutions but no alternative optimal bfs.

$$A = 0,\ \ B = 0,\ \ C \leq 0,\ \ D \geq 0,\ \ E = 1,\ \ F \text{ any}$$

Question 8.
Convert into standard form and add artificial variable.

$$\begin{aligned} \min\ & -3x_1 + 4x_2 \\ \text{s.t.}\ & x_1 + x_2 + s_1 = 4 \\ & 2x_1 + 3x_2 - e_1 + a_1 = 18 \\ & x_1, x_2, s_1, e_1, a_1 \geq 0 \end{aligned}$$

PHASE 1

	z	x_1	x_2	s_1	e_1	a_1	RHS
1	1	0	0	0	0	-1	0
s_1	0	1	1	1	0	0	4
a_1	0	2	3	0	-1	1	18

Add second row to row 0.

	z	x_1	x_2	s_1	e_1	a_1	RHS
1	1	2	3	0	-1	0	18
s_1	0	1	1	1	0	0	4
a_1	0	2	3	0	-1	1	18

x_2 enters. Apply MRT: $\min\{4/1, 18/2\} = 4$. Then, s_1 leaves the basis.

	z	x_1	x_2	s_1	e_1	a_1	RHS
1	1	-1	0	-3	-1	0	6
x_2	0	1	1	1	0	0	4
a_1	0	-1	0	-3	-1	1	6

The tableau is optimal, but a_1 is still in the basis. Hence, the Phase 1 problem is infeasible.

Question 9.
Convert the following LP to standard form and solve with the Big-M method.

$$\begin{aligned}
\max\ & -x_1 + 3x_2 + x_3 \\
\text{s.t.}\ & -x_1 + x_2 - x_3 \geq -4 \\
& 2x_1 + x_2 + x_3 \geq 10 \\
& x_1 \geq 0 \\
& x_2 \leq 0 \\
& x_3 \text{ free}
\end{aligned}$$

Let $x_3 = x_3' - x_3''$ where $x_3' \geq 0$ and $x_3'' \geq 0$. Let $-x_2 = x_2'$. Introduce slack variable s_1 and excess variable e_2.

$$\begin{aligned}
\max\ & -x_1 - 3x_2' + (x_3' - x_3'') - Ma_2 \\
\text{s.t. } & x_1 + x_2' + (x_3' - x_3'') + s_1 = 4 \\
& 2x_1 - x_2' + (x_3' - x_3'') - e_2 + a_2 = 10 \\
& x_1, x_2', x_3', x_3'', s_1, e_2 \geq 0
\end{aligned}$$

	z	x_1	x_2'	x_3'	x_3''	s_1	e_2	a_2	RHS
	1	1	3	-1	1	0	0	M	0
s_1	0	1	1	1	-1	1	0	0	4
a_1	0	2	-1	1	-1	0	-1	1	10

Multiply the second row with $-M$ and add it to row 0:

	z	x_1	x_2'	x_3'	x_3''	s_1	e_2	a_2	RHS
	1	-2M+1	M+3	-M-1	M+1	0	M	0	-10M
s_1	0	1	1	1	-1	1	0	0	4
a_1	0	2	-1	1	-1	0	-1	1	10

x_1 enters the basis. By minimum ratio test: $\min\{4/1, 10/2\} = 4$, s_1 leaves:

	z	x_1	x_2'	x_3'	x_3''	s_1	e_2	a_2	RHS
	1	0	3M+2	M-2	-M+2	2M-1	M	0	-2M-4
x_1	0	1	1	1	-1	1	0	0	4
a_1	0	0	-3	-1	1	-2	-1	1	2

x_3'' enters the basis. Apply MRT: $\min\{2/1\}$. Then, a_1 leaves:

	z	x_1	x_2'	x_3'	x_3''	s_1	e_2	a_2	RHS
	1	0	8	0	0	3	2	M-2	-8
x_1	0	1	-2	0	0	-1	-1	1	6
x_3''	0	0	-3	-1	1	-2	-1	1	2

No negative coefficient in row 0. The tableau is final. (a_2 is not in the basis so remove the column a_2 from the tableau)

Then,

$$x^* = (x_1, x_2', x_3', x_3'', s_1, e_2) = (6, 0, 0, 2, 0, 0)$$

and $z^* = -8$. Note that $x_2 = -x_2' = 0$, $x_3 = x_3' - x_3'' = 0 - 2 = -2$.

Question 10.

Convert the LP to standard form:

$$\begin{aligned} &\max\ x_1 + 2x_2 + 4x_3 - Ma_1 \\ &\text{s.t.}\ \ -x_1 + x_2 - x_3 - e_1 + a_1 = 2 \\ &\qquad -2x_1 + x_2 + x_3 + s_2 = 1 \\ &\qquad x_1, x_2, x_3, e_1, a_1, s_2 \geq 0 \end{aligned}$$

	z	x_1	x_2	x_3	e_1	a_1	s_2	RHS
z	1	-1	-2	-4	0	M	0	0
a_1	0	-1	1	-1	-1	1	0	2
s_2	0	-2	1	1	0	0	1	1

	z	x_1	x_2	x_3	e_1	a_1	s_2	RHS
z	1	M-1	-M-2	M-4	M	0	0	-2M
a_1	0	-1	1	-1	-1	1	0	2
s_2	0	-2	1	1	0	0	1	1

x_2 enters, s_2 leaves.

	z	x_1	x_2	x_3	e_1	a_1	s_2	RHS
z	1	-M-5	0	2M-2	M	0	M+2	-M+2
a_1	0	1	0	-2	-1	1	-1	1
x_2	0	-2	1	1	0	0	1	1

x_1 enters, a_1 leaves.

	z	x_1	x_2	x_3	e_1	a_1	s_2	RHS
z	1	0	0	-12	-5	M+5	-3	7
x_1	0	1	0	-2	-1	1	-1	1
x_2	0	0	1	-3	-2	2	-1	3

x_3 enters, but its coeffients on the constraints are negative. Then, there is no leaving variable. Hence, the problem is **unbounded**.

Question 11.

(a) The tableau is final and there exists a unique optimal solution.
A > 0, B any, C any, D any, E ≥ 0

(b) The simplex method determines an unbounded solution from this tableau.
A < 0, B any, C ≤ 0, D ≤ 0, E ≥ 0

(c) The current bfs is degenerate (not necessarily optimal).
A any, B any, C any, D any, E $= 0$

(d) The current solution is optimal, there are alternative optimal solutions but no alternative optimal bfs.
A $= 0$, B any, C ≤ 0, D ≤ 0, E ≥ 0

Question 12.

Two-Phase Method:

The phase-I problem is the following:

$$
\begin{array}{llllllll}
\min & a_1 & + a_2 & & & & & \\
\text{s.t.} & & & & & & & \\
& 2x_1 & + x_2 & + 3x_3 & & + a_1 & & = 60 \\
& 3x_1 & + 3x_2 & + 5x_3 & - e_1 & & + a_2 & = 120 \\
& x_1, & x_2, & x_3, & e_1, & a_1, & a_2 & \geq 0
\end{array}
$$

Now we need to solve phase-I LP:

Tableau 0:

z	x_1	x_2	x_3	e_1	a_1	a_2	RHS
1	0	0	0	0	-1	-1	0
0	2	1	3	0	1	0	60
0	3	3	5	-1	0	1	120

The simplex tableau above is not in proper format since the row zero coefficients of basic variables a_1 and a_2 are not zero. We add the first and second row to row zero to make these coefficients zero:

Tableau 0:

z	x_1	x_2	x_3	e_1	a_1	a_2	RHS	
1	5	4	8	-1	0	0	180	
0	2	1	3	0	1	0	60	60/3
0	3	3	5	-1	0	1	120	120/5

Now we can start simplex iterations: since it is a minimization problem we are looking for positive coefficients to decide on entering variables.

x_3 enters and a_1 leaves:

Tableau 1:

z	x_1	x_2	x_3	e_1	a_1	a_2	RHS	
1	-1/3	4/3	0	-1	-8/3	0	20	
0	2/3	1/3	1	0	1/3	0	20	20/(1/3)
0	-1/3	4/3	0	-1	-5/3	1	20	20/(4/3)

x_2 enters and a_2 leaves:

Tableau 2:

z	x_1	x_2	x_3	e_1	a_1	a_2	RHS
1	0	0	0	0	-1	-1	0
0	3/4	0	1	1/4	3/4	-1/4	15
0	-1/4	1	0	-3/4	-5/4	3/4	15

Both a_1 and a_2 left the basis so we can delete their columns from the tableau and move onto phase-II. We write the original objective function coefficients to the tableau:

Tableau 0:

z	x_1	x_2	x_3	e_1	RHS
1	-3	-2	-4	0	0
0	3/4	0	1	1/4	15
0	-1/4	1	0	-3/4	15

Again, notice that this tableau is not in proper format, we need to make row zero coefficients of basic variables x_2 and x_3 zero:

Tableau 0:

z	x_1	x_2	x_3	e_1	RHS
1	-1/2	0	0	-1/2	90
0	3/4	0	1	1/4	15
0	-1/4	1	0	-3/4	15

Stop. The tableau is optimal since there is no positive entry in row zero.

The optimal solution is $x_1^*=0$, $x_2^*=x_3^*=15$ with objective value 90.

Now let's solve the same problem with Big-M method.

<u>Big-M method:</u>

The LP is the following:

$$
\begin{array}{llllllll}
\min & 3x_1 & + 2x_2 & + 4x_3 & + Ma_1 & + Ma_2 & & \\
\text{s.t.} & & & & & & & \\
& 2x_1 & + x_2 & + 3x_3 & & + a_1 & & = 60 \\
& 3x_1 & + 3x_2 & + 5x_3 & - e_1 & & + a_2 & = 120 \\
& x_1, & x_2, & x_3, & e_1, & a_1, & a_2 & \geq 0
\end{array}
$$

Tableau 0:

z	x_1	x_2	x_3	e_1	a_1	a_2	RHS
1	-3	-2	-4	0	-M	-M	0
0	2	1	3	0	1	0	60
0	3	3	5	-1	0	1	120

Let's put the tableau in the proper format first:

Tableau 0:

z	x_1	x_2	x_3	e_1	a_1	a_2	RHS	
1	5M-3	4M-2	8M-4	-M	0	0	180M	
0	2	1	3	0	1	0	60	60/3
0	3	3	5	-1	0	1	120	120/5

x_3 enters and a_1 leaves:

Tableau 1:

z	x_1	x_2	x_3	e_1	a_1	a_2	RHS	
1	-(M+1)/3	(4M-2)/3	0	-M	(4-8M)/3	0	80 +20M	
0	2/3	1/3	1	0	1/3	0	20	20/(1/3)
0	-1/3	4/3	0	-1	-5/3	1	20	20/(4/3)

x_2 enters and a_2 leaves:

Tableau 2:

z	x_1	x_2	x_3	e_1	a_1	a_2	RHS
1	-1/2	0	0	-1/2	1/2-M	1/2-M	90
0	3/4	0	1	1/4	3/4	-1/4	15
0	-1/4	1	0	-3/4	-5/4	3/4	15

Stop. The tableau is optimal since there is no positive entry in row zero.

The optimal solution is $x_1^*=0$, $x_2^*=x_3^*=15$ with objective value 90.