IE 400: Principles of Engineering Management
In-Class Example Solutions

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1 Work Scheduling (page:13)

The mathematical formulation of the problem is given as follows:

Decision Variables

x_i : number of workers who are NOT working on day i, i = 1,..., 7 (1: Mon, 7: Sun)

Objective is to minimize total number of of workers:

$$\begin{aligned}
\min \quad & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 && (1)\\
s.t \quad & x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \geq 8 && (2)\\
& x_1 + x_3 + x_4 + x_5 + x_6 + x_7 \geq 6 && (3)\\
& x_1 + x_2 + x_4 + x_5 + x_6 + x_7 \geq 7 && (4)\\
& x_1 + x_2 + x_3 + x_5 + x_6 + x_7 \geq 6 && (5)\\
& x_1 + x_2 + x_3 + x_4 + x_6 + x_7 \geq 9 && (6)\\
& x_1 + x_2 + x_3 + x_4 + x_5 + x_7 \geq 11 && (7)\\
& x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \geq 9 && (8)\\
& x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0 && (9)\\
& x_i = integer \qquad \forall i = 1, \ldots, 7 && (10)
\end{aligned}$$

2 The 0-1 Knapsack Problem (page:19)

The mathematical formulation of the problem is given as follows:

Decision Variables

$$x_i = \begin{cases} 1, & \text{if item i is packed into the suitcase} \\ 0, & \text{otherwise} \end{cases}$$

Objective is to maximize total utility:

$$
\begin{aligned}
&\max \quad && \sum_{i=1}^{n} c_i x_i && && (11)\\
&s.t \quad && \sum_{i=1}^{n} a_i x_i \leq b && && (12)\\
& && x_i \in \{0,1\} && \forall i = 1,\ldots,n && (13)
\end{aligned}
$$

3 Workforce Scheduling (page:22)

The mathematical formulation of the problem is given as follows:

Decision Variables

w_t : number of workers working during time period t, t = 0,...,4

f_t : number of workers fired at the beginning of time period t, t = 1,...,4

h_t : number of workers hired at the beginning of time period t, t = 1,...,4

$\imath_t$: number of pairs of shoes in the inventory at the end of time period t, t = 0,...,4

p_t : number of pairs of shoes produced during time period t, t = 1,...,4

Objective is to find a work schedule that meets the total demand at a minimum cost:

$$
\begin{aligned}
&\min \quad && \sum_{t=1}^{4} 5p_t + 30\imath_t + 1500w_t + 2000f_t + 1600h_t && && (14)\\
&s.t \quad && w_0 \geq 3 && && (15)\\
& && \imath_0 = 50 && && (16)\\
& && \imath_{t-1} + p_t = d_t + \imath_t && \forall t = 1,\ldots,4 && (17)\\
& && w_t = w_{t-1} + h_t - f_t && \forall t = 1,\ldots,4 && (18)\\
& && p_t - 40w_t \leq 0 && \forall t = 1,\ldots,4 && (19)\\
& && w_t, f_t, h_t, f_t, \imath_t, p_t \geq 0 && \forall t = 1,\ldots,4 && (20)\\
& && w_t, f_t, h_t, f_t, \imath_t, p_t = integer && \forall t = 1,\ldots,4 && (21)
\end{aligned}
$$

4 Modeling Fixed Costs (page:68)

The mathematical formulation of the problem is given as follows:

Decision Variables

$$y_i = \begin{cases} 1, & \text{if a potential warehouse i is opened i=1,...,m} \\ 0, & \text{otherwise} \end{cases}$$

x_{ij} = Number of units supply from warehouse i to client j, i=1,...,m, j=1,...,n

Objective is order to minimize total fixed costs and transportation costs while meeting the demand and capacity constraints:

$$\min \quad \sum_{i=1}^{m} f_i y_i + \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \tag{22}$$

$$s.t \quad \sum_{i=1}^{m} x_{ij} \geq d_j \qquad \forall j = 1, \ldots, n \tag{23}$$

$$\sum_{j=1}^{n} x_{ij} \leq a_i y_i \qquad \forall i = 1, \ldots, m \tag{24}$$

$$y_i \in \{0, 1\} \qquad \forall i = 1, \ldots, m \tag{25}$$

$$x_{ij} \geq 0 \qquad \forall i = 1, \ldots, m, \forall j = 1, \ldots, n \tag{26}$$

5 The Set Covering Problem (page:70)

The mathematical formulation of the problem is given as follows:

Decision Variables

$$x_i = \begin{cases} 1, & \text{if a hospital is built in potential location i, i=1,2,3,4} \\ 0, & \text{otherwise} \end{cases}$$

Objective is order to minimize number of hospitals that should be opened:

$$
\begin{aligned}
\min \quad & x_1 + x_2 + x_3 + x_4 && && (27)\\
s.t \quad & x_1 + x_3 \geq 1 && && (28)\\
& x_2 + x_3 \geq 1 && && (29)\\
& x_2 + x_4 \geq 1 && && (30)\\
& x_1 + x_4 \geq 1 && && (31)\\
& x_3 + x_4 \geq 1 && && (32)\\
& x_i \in \{0, 1\} && \forall i = 1, \ldots, 4 && (33)
\end{aligned}
$$

6 Location Problems (Cover, Center, Median) (page:71-77)

Ankara municipality is providing service to 6 regions. The travel times (in minutes) between these regions, say t_{ij} values are:

	R1	R2	R3	R4	R5	R6
R1	0	10	20	30	30	20
R2	10	0	25	35	20	10
R3	20	25	0	15	30	20
R4	30	35	15	0	15	25
R5	30	20	30	15	0	14
R6	20	10	20	25	14	0

(a) **Cover:** The municipality wants to build fire stations in some of these regions. Find the minimum number of fire stations required if the municipality wants to ensure that there is a fire station within 15 minute drive time from each region.

Decision Variables: The coverage drive time $T^c = 15$ is the maximum drive time between a fire station and a region that it can serve. Furthermore, in this formulation we do not need to explicitly use the travel times. Instead, we can use the following indicator variable.

$$
a_{ij} = \begin{cases} 1, & \text{if } t_{ij} < 15 \\ 0, & \text{otherwise} \end{cases} \tag{34}
$$

Let's define the decision variable.

$$
x_j = \begin{cases} 1, & \text{if fire station located at j} \\ 0, & \text{otherwise} \end{cases} \tag{35}
$$

IP Model:

$$\min \quad \sum_{j=1}^{6} x_j \tag{36}$$

$$s.t \quad \sum_{j=1}^{6} a_{ij} x_j \geq 1 \qquad \forall i = 1, \ldots, 6 \tag{37}$$

$$x_j \in \{0,1\} \qquad \forall j = 1, \ldots, 6 \tag{38}$$

(b) **Median:** The municipality now wants to build 2 post offices. There is going to be a back and forth trip each day from each post office to each one of the regions assigned to this post office. Formulate a model which will minimize the total trip time.

Decision Variables:

$$y_{ij} = \begin{cases} 1, & \text{if region i assigned to post office located at j} \\ 0, & \text{otherwise} \end{cases} \tag{39}$$

$$x_j = \begin{cases} 1, & \text{if post office located at j} \\ 0, & \text{otherwise} \end{cases} \tag{40}$$

IP Model:

$$\min \quad \sum_{j=1}^{6} \sum_{i=1}^{6} 2 t_{ij} y_{ij} \tag{41}$$

$$s.t \quad \sum_{j=1}^{6} y_{ij} = 1 \qquad \forall i = 1, \ldots, 6 \tag{42}$$

$$y_{ij} \leq x_j \qquad \forall i, j = 1, \ldots, 6 \tag{43}$$

$$\sum_{j=1}^{6} x_j = 2 \tag{44}$$

$$x_j \in \{0,1\} \qquad \forall j = 1, \ldots, 6 \tag{45}$$

$$y_{ij} \in \{0,1\} \qquad \forall i, j = 1, \ldots, 6 \tag{46}$$

(c) What if additional to part (b), you have the restriction that a post office cannot provide service to more than 4 regions including itself?

Add the following constraint to above model:

$$\sum_{i=1}^{6} y_{ij} \leq 4 \quad \forall j = 1, \ldots, 6$$

(d) **Center:** The municipality wants to build 2 fire stations so that the time to reach the farthest region is kept at minimum.
Decision Variables:

$$y_{ij} = \begin{cases} 1, & \text{if region i assigned to post office located at j} \\ 0, & \text{otherwise} \end{cases} \tag{47}$$

$$x_j = \begin{cases} 1, & \text{if post office located at j} \\ 0, & \text{otherwise} \end{cases} \tag{48}$$

Let's define an additional decision variable T that represents the maximum travel time between any region and its servicing fire station.

IP Model:

$$\begin{aligned}
\text{min} \quad & T && && (49)\\
s.t \quad & \sum_{j=1}^{6} y_{ij} = 1 && \forall i = 1, \dots, 6 && (50)\\
& y_{ij} \leq x_j && \forall i, j = 1, \dots, 6 && (51)\\
& \sum_{j=1}^{6} x_j = 2 && && (52)\\
& \sum_{j=1}^{6} t_{ij} y_{ij} \leq T && && (53)\\
& x_j \in \{0, 1\} && \forall j = 1, \dots, 6 && (54)\\
& y_{ij} \in \{0, 1\} && \forall i, j = 1, \dots, 6 && (55)
\end{aligned}$$

(e) Consider the median problem in part (b). Assume that in addition to the back and forth trips between the post offices and the assigned regions, there will be a trip between the two post offices. The municipality wants to locate the offices in such a way that the total trip time is minimized.
Decision Variables:
Let's define an additional decision variable

$$z_{ij} = \begin{cases} 1, & \text{if post office located at both i and j } (j \neq i) \\ 0, & \text{otherwise} \end{cases} \tag{56}$$

IP Model:

$$\min \quad \sum_{j=1}^{6}\sum_{i=1}^{6} 2t_{ij}y_{ij} + \sum_{j=1,j\neq i}^{6}\sum_{i=1}^{6} 2t_{ij}z_{ij} \tag{57}$$

$$s.t \quad \sum_{j=1}^{6} y_{ij} = 1 \qquad \forall i = 1,\ldots,6 \tag{58}$$

$$y_{ij} \leq x_j \qquad \forall i,j = 1,\ldots,6 \tag{59}$$

$$\sum_{j=1}^{6} x_j = 2 \tag{60}$$

$$z_{ij} \geq x_i + x_j - 1 \qquad \forall i,j = 1,\ldots,6, j \neq i \tag{61}$$

$$x_j \in \{0,1\} \qquad \forall j = 1,\ldots,6 \tag{62}$$

$$y_{ij} \in \{0,1\} \qquad \forall i,j = 1,\ldots,6 \tag{63}$$

7 Blending problem revisited with binary variables (page:83-85)

A company produces animal feed mixture. The mix contains 2 active ingredients and a filler material. 1 kg of feed mix must contain a minimum quantity of each of the following 4 nutrients:

Nutrients	A	B	C	D
Required amount (in grams) in 1 kg of mix	90	50	20	2

The ingredients have the following nutrient values and costs:

	A	B	C	D	Cost/kg
Ingredient1 (gram/kg)	100	80	40	10	\$40
Ingredient2 (gram/kg)	200	150	20	N/A	\$60

(a) What should be the amount of active ingredients and filler material in 1 kg of feed mix in order to satisfy the nutrient requirements at minimum total cost?
Decision Variables:
x_1 = amount (kg) of ingredient 1 in one kg of feed mix
x_2 = amount (kg) of ingredient 2 in one kg of feed mix
x_3 = amount (kg) of filler in one kg of feed mix
IP Model:

$$
\begin{aligned}
\min \quad & 40x_1 + 60x_2 && (64)\\
s.t \quad & x_1 + x_2 + x_3 = 1 && (65)\\
& 100x1 + 200x_2 \geq 90 && (66)\\
& 80x_1 + 150x_2 \geq 50 && (67)\\
& 40x_1 + 20x_2 \geq 20 && (68)\\
& 10x_1 \geq 2 && (69)\\
& x_1, x_2, x_3 \geq 0 && (70)
\end{aligned}
$$

(b) Suppose now that we have the following additional limitations

- If we use any of ingredient 2, we incur a fixed cost of \$15 (in addition to the \$60 unit cost).
- It is enough to only satisfy 3 of the nutrition requirements rather than 4.

Decision Variables (Additional to those for Part a):

$$
y = \begin{cases} 1, & \text{if } x_2 \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{71}
$$

$$
z_i = \begin{cases} 1, & \text{if nutrient constraint i is satisfied} \\ 0, & \text{otherwise} \end{cases} \tag{72}
$$

IP Model:

$$
\begin{array}{lll}
\min & 40x_1 + 60x_2 + 15y & (73)\\
s.t & x_1 + x_2 + x_3 = 1 & (74)\\
& 100x_1 + 200x_2 \geq 90z_1 & (75)\\
& 80x_1 + 150x_2 \geq 50z_2 & (76)\\
& 40x_1 + 20x_2 \geq 20z_3 & (77)\\
& 10x_1 \geq 2z_4 & (78)\\
& x_2 \leq y & (79)\\
& z_1 + z_2 + z_3 + z_4 \geq 3 & (80)\\
& x_1, x_2, x_3 \geq 0 & (81)\\
& y \in \{0, 1\} & (82)\\
& z_i \in \{0, 1\} \qquad \forall i = 1, \ldots, 4 & (83)
\end{array}
$$