IE 400: Principles of Engineering Management
Study Set 4 Solutions

Fall 2021

Question 1.

First calculate the ratios:

x_1 : $9/6$

x_2 : $13/3$

x_3 : $10/2$

x_4 : $8/4$

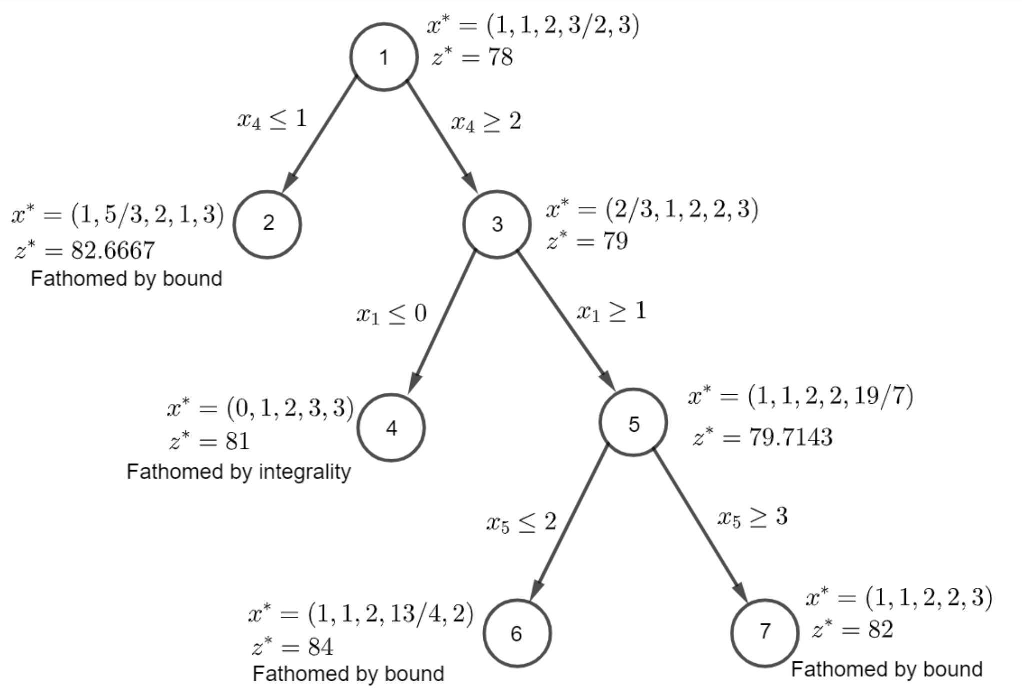
x_5 : $8/7$

For the **minimization problem**, sort the ratios from smallest to largest:

$$x_5 < x_1 < x_4 < x_2 < x_3$$

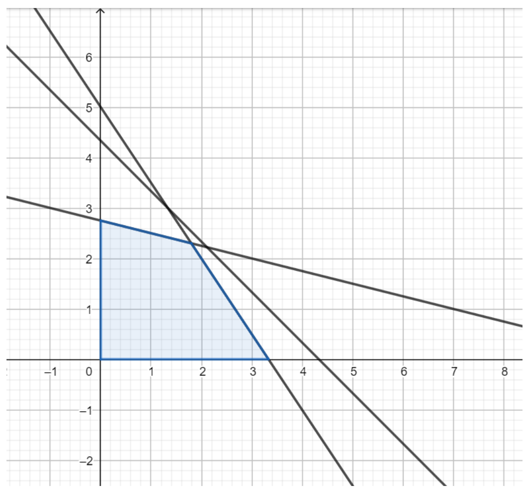
Solve the LP relaxation of the knapsack model according to the above priority list (x_5 is the most preferable and x_3 is the least preferable one). The solution to the LP relaxation of the given model is displayed at node 1. Since this is not an integer solution, we branch from node 1, and add new constraints as given in the Branch and Bound tree below. Note that at each node we solve the LP relaxation of the IP model along with the newly added constraints.
Note that node 2 is fathomed by bound, because we have found an integer solution at node 4 that has a better objective value than the one at node 2.

(If we branch node 2, then the resulting nodes would have an objective value $\geq$ 82.6667 which will always result in a solution worse than node 4)

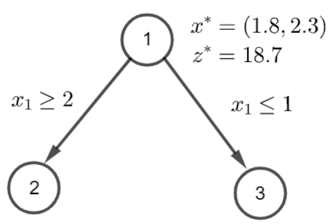


The optimal solution is $x^* = (0, 1, 2, 3, 3)$.

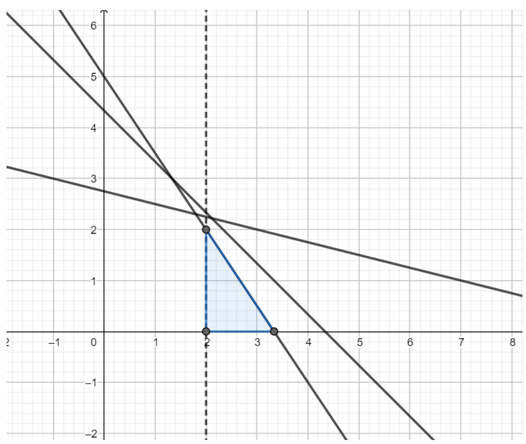
Question 2.
Solve the LP relaxation of the original model using graphical method:



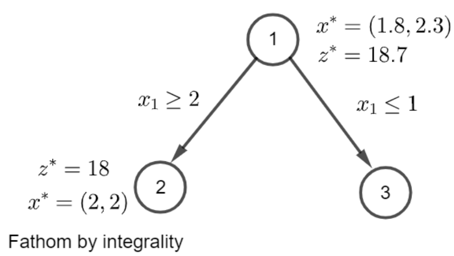
For c=(4,5), we obtain $x^* = (1.8, 2.3)$. This is not an integer solution, so start branching:



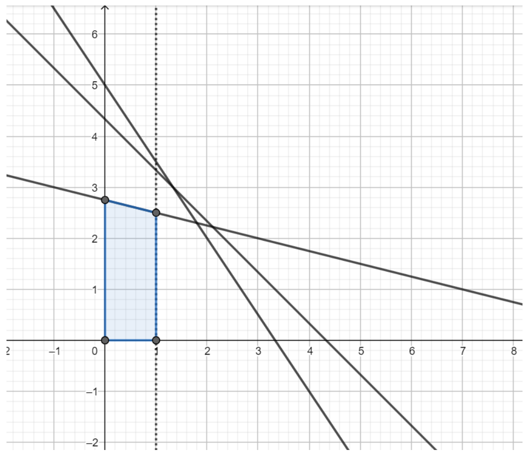
Now move to node 2. Here the constraint $x_1 \geq 2$ is added to the model:



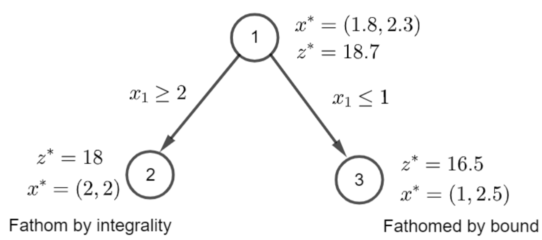
Solve the new model with graphical method and update the branch and bound tree:



We obtained an integer solution at node 2. Hence, it is fathomed by integrality. Now move to the third node. Add the constraint $x_1 \leq 1$ to the original model:



Solving via the graphical method we obtain:



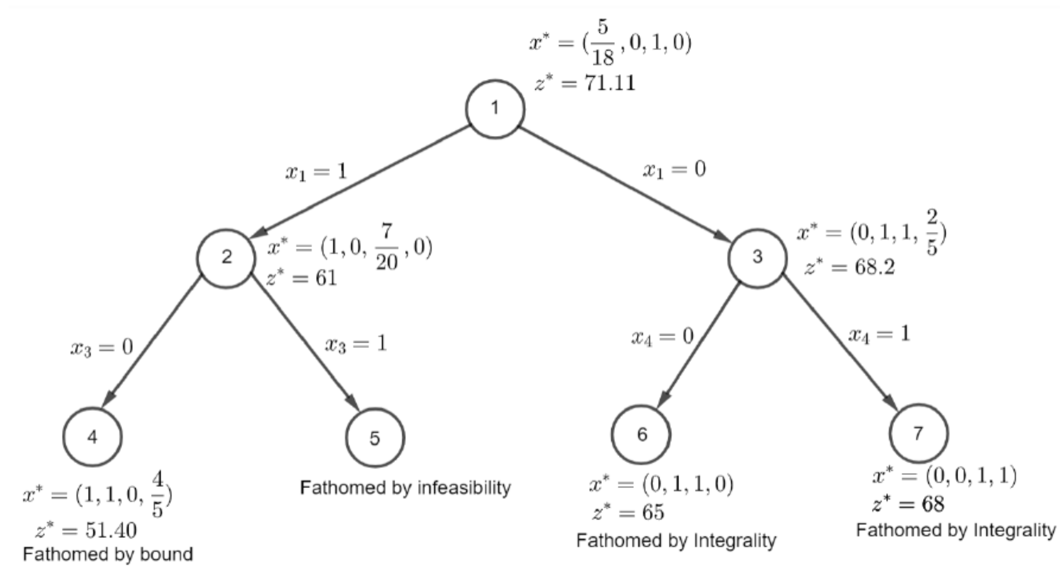
Node 3 is fathomed by bound, because we have already obtained an integer solution at node 2 that has a better objective value (for maximization) than the model at node 3 . If we branch the third node, the objective value will be $\leq$ 16.5 !

We cannot make further branches. Then, the optimal solution is $x^* = (2, 2)$ and $z^* = 18$.

Question 3.

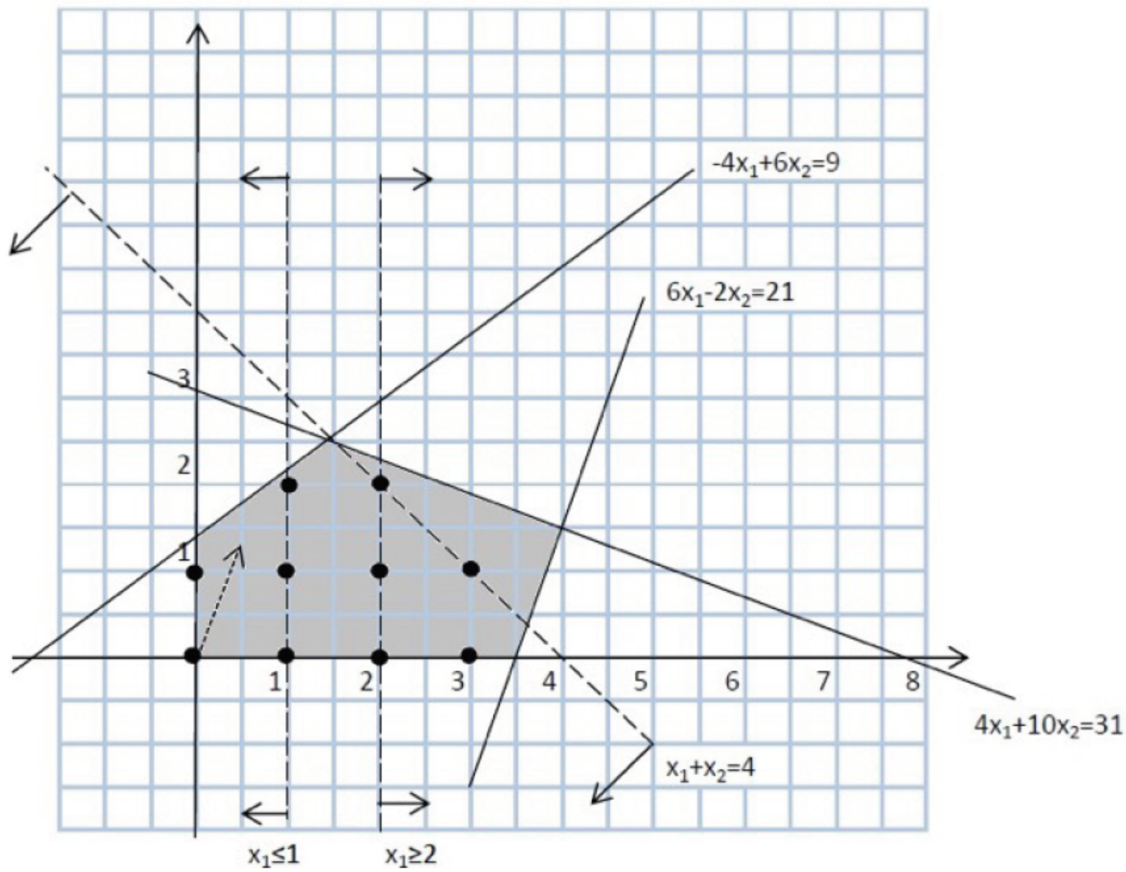
Order of the variables:

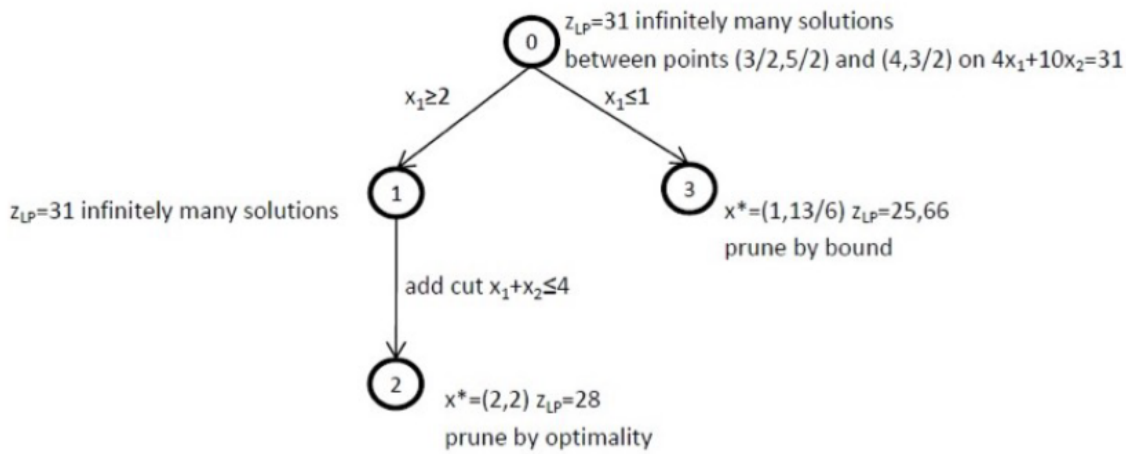
$x_3 > x_1 > x_2 > x_4$



Optimal Solution is: $x^* = (0, 0, 1, 1)$.

Question 4.

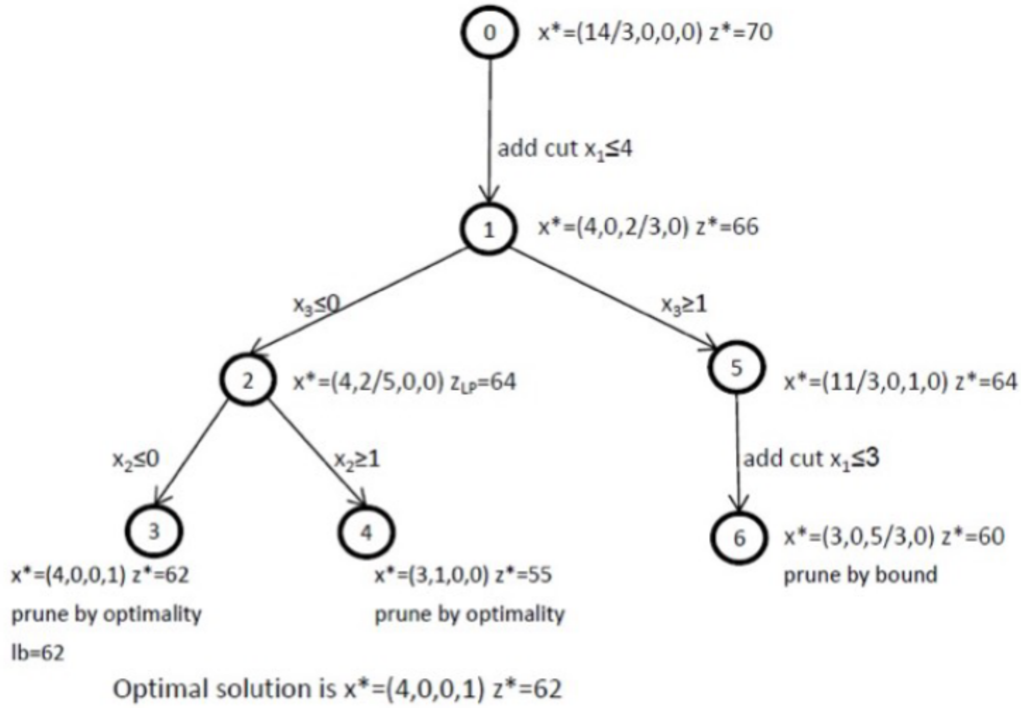




0
z_{LP}=31 infinitely many solutions
between points (3/2,5/2) and (4,3/2) on $4x_1+10x_2=31$
$x_1 \geq 2$
$x_1 \leq 1$
1
z_{LP}=31 infinitely many solutions
3
$x^*=(1,13/6)$ z_{LP}=25,66
prune by bound
add cut $x_1+x_2 \leq 4$
2
$x^*=(2,2)$ z_{LP}=28
prune by optimality

Question 5.

Ratios $x_1 > x_3 > x_2 > x_4$. $(5 > 3 > 2 > 1)$.



Question 6.

(a) Since x_6, $x_7 \in \{0, 1\}$, the node will be fathomed by infeasibility if

$$a_1 + a_2 < 15$$

(b) To have integer solution we must have:

$$a_1 + a_2 = 15 \text{ or } a_1 = 15 \text{ or } a_2 = 15$$

(c) Since current incumbet is 34, the problem is minimization we fathom by bound if we get an objective greater than or equal to 34.

$$c_1 x_6 + c_2 x_7 + 15 \geq 34 \rightarrow c_1 x_6 + c_2 x_7 \geq 19$$

This is like a knapsack problem:

$$\begin{aligned} &\min\ c_1 x_6 + c_2 x_7 \\ &\text{s.t. } a_1 x_6 + a_2 x_7 \geq 15 \end{aligned}$$

Case i: $\frac{c_1}{a_1} \leq \frac{c_2}{a_2}$

- if $a_1 \geq 15$ and $c_1 \frac{15}{a_1} \geq 19$, then the node will be fathomed by bound.
- if $a_1 < 15$ and $c_1 + c_2 \frac{15 - a_1}{a_2} \geq 19$, then the node will be fathomed by bound.

Case ii: $\frac{c_2}{a_2} \leq \frac{c_1}{a_1}$

Similar as case i.

(d) If none of the conditions from a, b, c is satisfied, the node is not fathomed.

Question 7.

(a) Fathomed by integrality. Once we try solve the LP relaxation to the given problem, we see that the optimal value found is $(0, 1, 1)$ with an objective value of -8. No need to keep on branching from this node as our solution is made up of integer valued x_1 as well as binary x_2 and x_3.

(b) Fathomed by infeasibility. Once we try solve the LP relaxation to the given problem, we see that the constraint can not be satisfied when x_2 and x_3 are less than or equal to 1.

(c) Not fathomed, branching should continue. Once we try solve the LP relaxation to the given problem, we see that the optimal solution is $(0, 0.5, 1)$ with an objective value of -10.5 which will branch into two child nodes: $(0, 0, 1)$ and $(0, 1, 1)$.

(d) Fathomed by integrality. Once we solve the LP relaxation to the given problem, we see that the optimal solution is $(2, 0, 1)$ with an objective value of -3. No need to keep on branching from this node as our solution is made up of integer valued x_1 as well as binary x_2 and x_3.