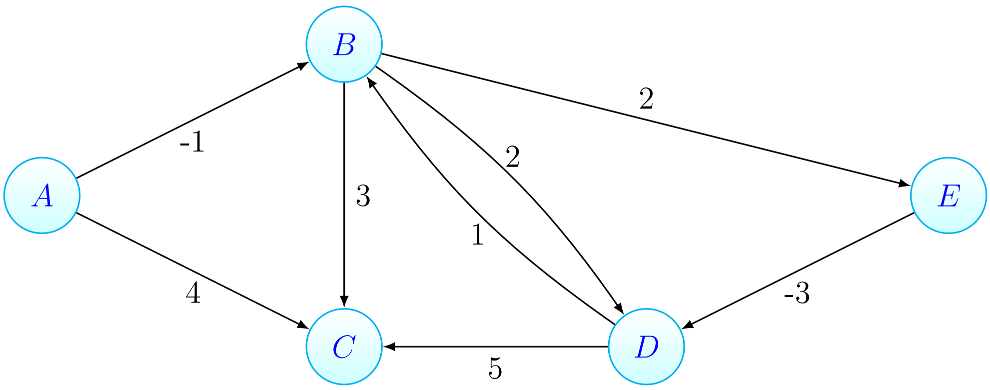
IE 400: Principles of Engineering Management
Study Set 5 Solutions

Fall 2021

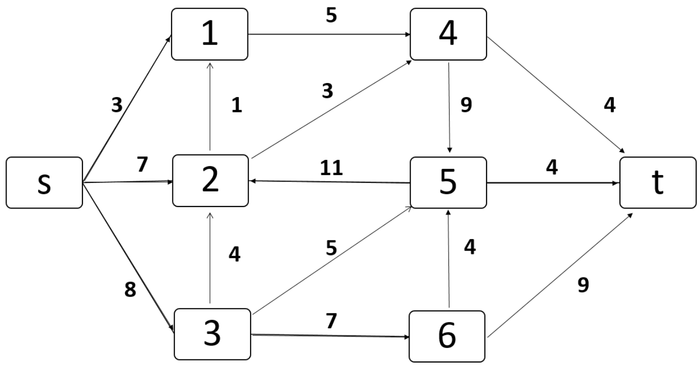
Question 1. Use Bellman Ford algorithm to find the shortest path of the following graph where A is the source node.



$$V_{[k]}^{(t)} = \min\{V_{[i]}^{t-1} + c_{ik}\}$$

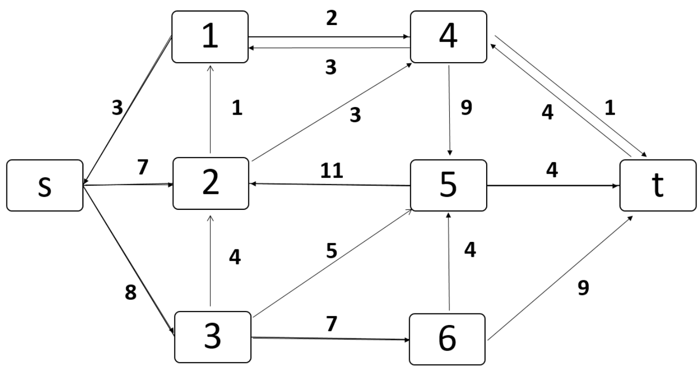
t	$V_{(A)}^t$	$V_{(B)}^t$	$V_{(C)}^t$	$V_{(D)}^t$	$V_{(E)}^t$	d(A)	d(B)	d(C)	d(D)	d(E)
0	0	$+\infty$	$+\infty$	$+\infty$	$+\infty$					
1	0	-1	4	$+\infty$	$+\infty$		A	A		
2	0	-1	2	1	1			B	B	B
3	0	-1	2	-2	1				E	

Question 2. Consider costs as capacities. Using Ford - Fulkerson algorithm find max-flow.



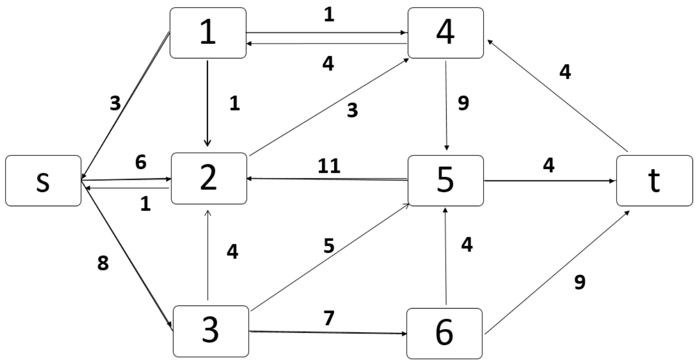
Step 1: Choose path s - 1 - 4 - t

min{3, 5, 4}= 3



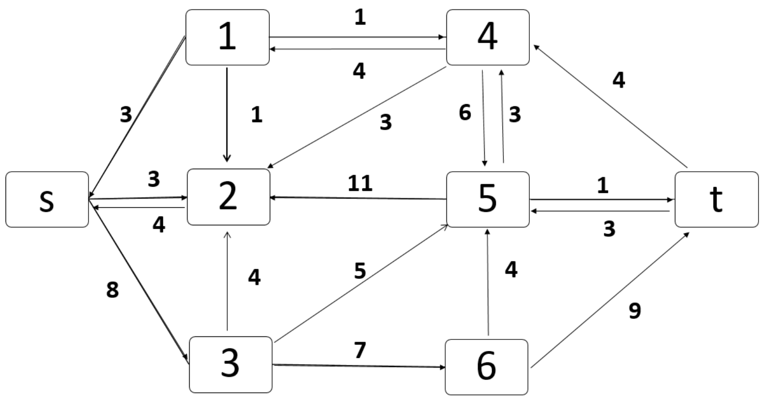
Step 2: Choose path s - 2 - 1 - 4 - t

min{7, 1, 2, 1}= 1



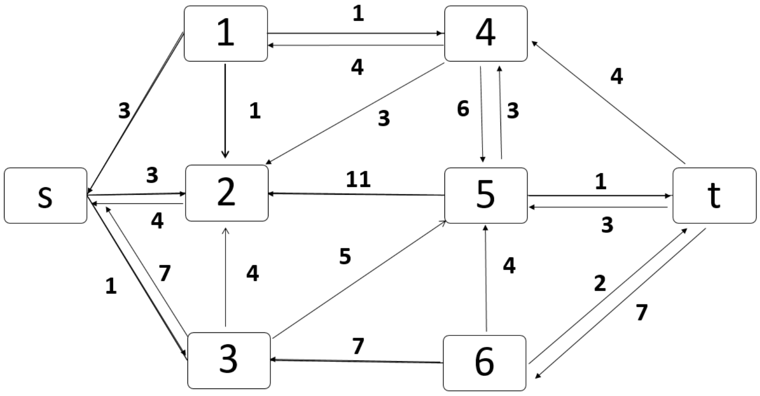
Step 3: Choose path s - 2 - 4 - 5 - t

min{6, 3, 9, 4}= 3



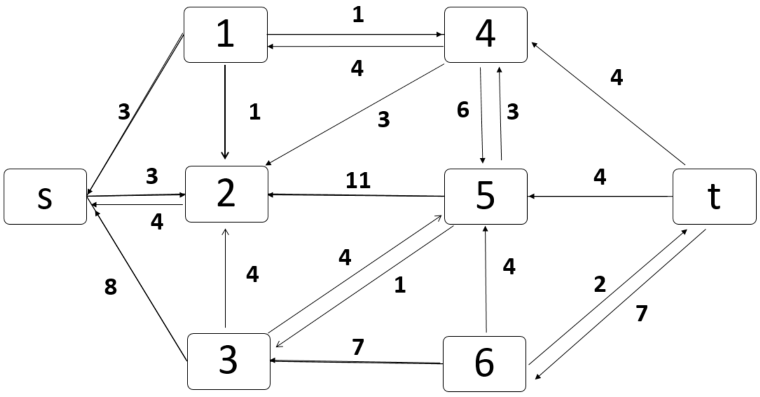
Step 4: Choose path s - 3 - 6 - t

min{8, 7, 9}= 7



Step 5: Choose path s - 3 - 5 - t

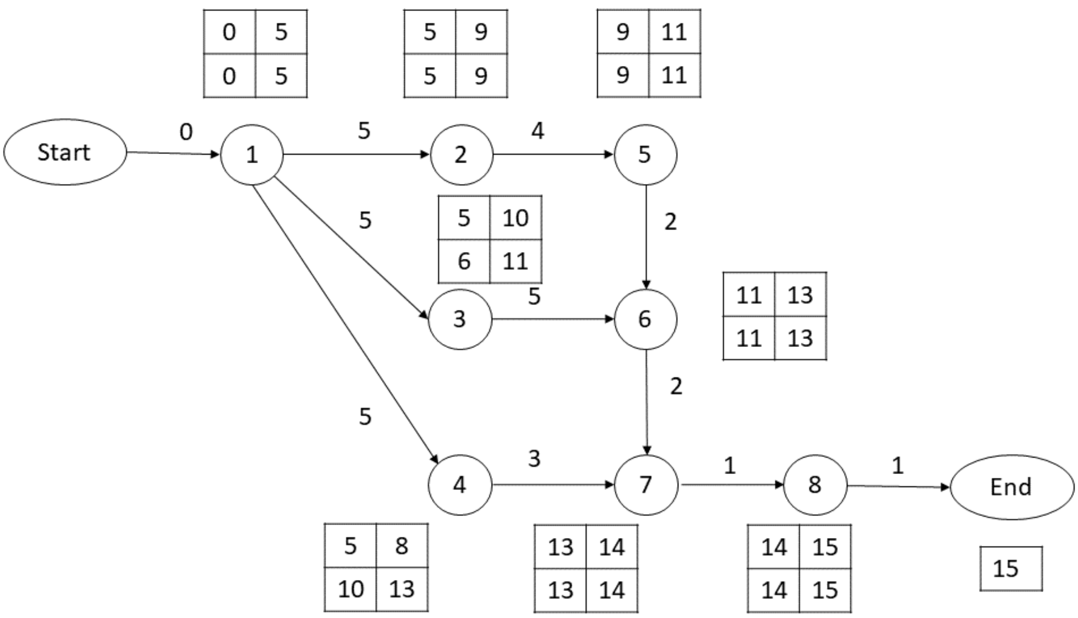
min$\{1, 5, 1\}$= 1



There is no other path from s to t.
Since max-flow = min-cut in optimal solution. The minimum cost to block flood's way to city is 15.

Question 3.

Activity	Predecessors	Duration
1 (Design)	-	5
2 (make A)	1	4
3 (make B)	1	5
4 (make C)	1	3
5 (test A)	2	2
6 (assembly A&B)	3, 5	2
7 (attach C)	6, 4	1
8 (testing)	7	1



Critical path: 1 - 2 - 5 - 6 - 7 - 8

Duration: 15

Activity	1	2	3	4	5	6	7	8
Slack	0	0	1	5	0	0	0	0

For the LP define decision variable x_j: start time of activity $j, j = 1, ..., 10, \text{start, end}$

$$
\begin{aligned}
\min \quad & x_{end} \\
\\
\text{s.t.} \quad & x_1 \geq x_{start} \\
& x_2 \geq x_1 + 5 \\
& x_3 \geq x_1 + 5 \\
& x_4 \geq x_1 + 5 \\
& x_5 \geq x_2 + 4 \\
& x_6 \geq x_3 + 5 \\
& x_6 \geq x_5 + 2 \\
& x_7 \geq x_6 + 2 \\
& x_7 \geq x_4 + 3 \\
& x_8 \geq x_7 + 1 \\
& x_{end} \geq x_8 + 1 \\
& x_{start} = 0
\end{aligned}
$$

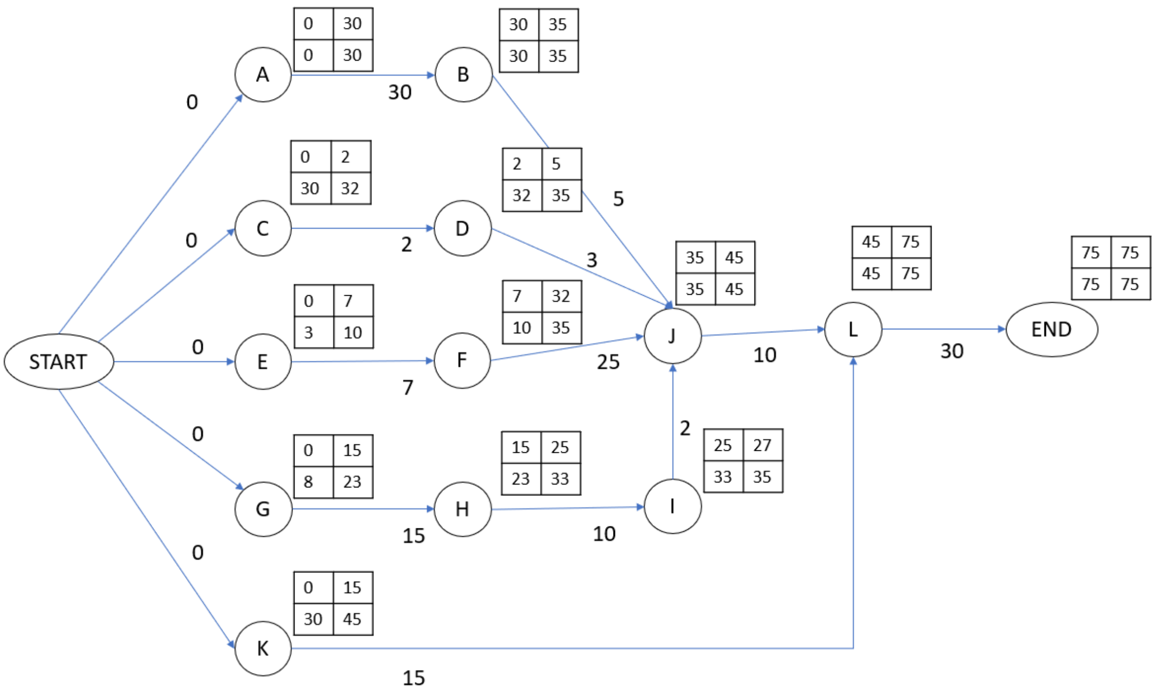
The tight constraints of the optimal solution for this LP, which hold at equality, will give you the critical path.

Question 4.

ES: Earliest Start, EF: Earliest Finish, LS: Latest Start, LF: Latest Finish

ES	EF
LS	LF

(a)-(b)



Activity	Slack
A	0
B	0
C	30
D	30
E	3
F	3
G	8
H	8
I	8
J	0
K	30
L	0

(c) Since the slack for cutting the onions and mushrooms is 3, the dinner will be delayed 6-3=3 minutes.
If the cutting time is 2 minutes, slack is 10 (latest finish time) – 2 (task duration) = 8 minutes. Since the slack time is more than the delay time, it does not delay the dinner.

(d) Let N be the set of activities and A be the set of arc in the project network.

Parameters:
a_i : Duration of activity i, for all i $\in N$
d : Project duration (75)

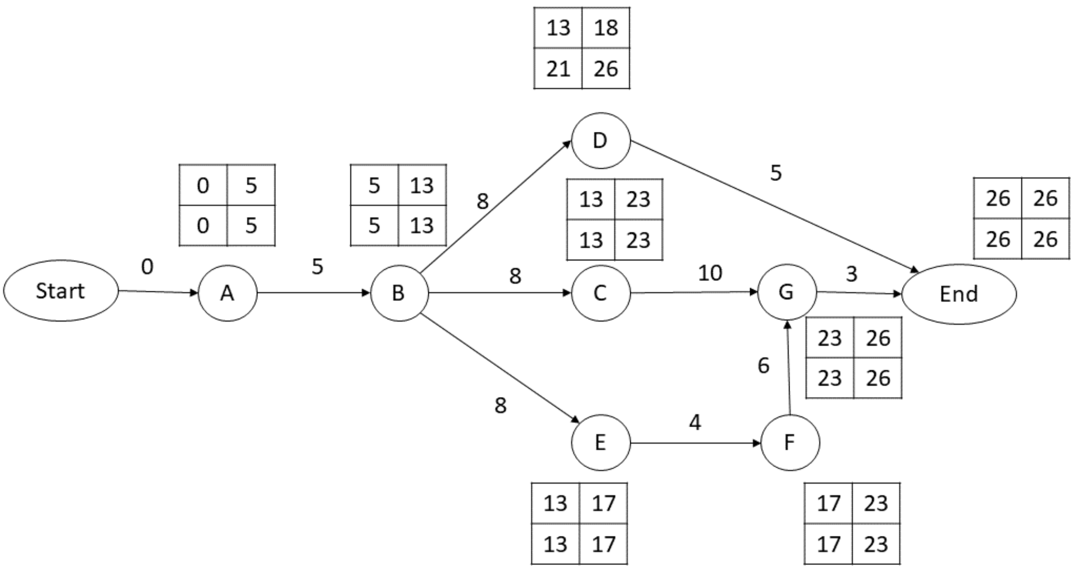
Decision Variables:
x_i : Start time of activity i, for all $i \in N$

Model:

$$\begin{aligned}
\max \ & \sum_{i \in N} x_i \\
\text{s.t.} \ & x_j \geq x_i + a_i && (i,j) \in A \\
& x_{end} \leq d \\
& x_i \geq 0 && i \in N \\
& x_{start} = 0
\end{aligned}$$

Question 5.

(a)



Critical path: A - B - C - G and A - B - E - F - G

Duration: 26

(b) D can be delayed for 8 days.

(c) Decision Variables:

x_j: start time of activity j

y_j: number of days duration of activity j is reduced

Parameters:

c_j: cost of reduction

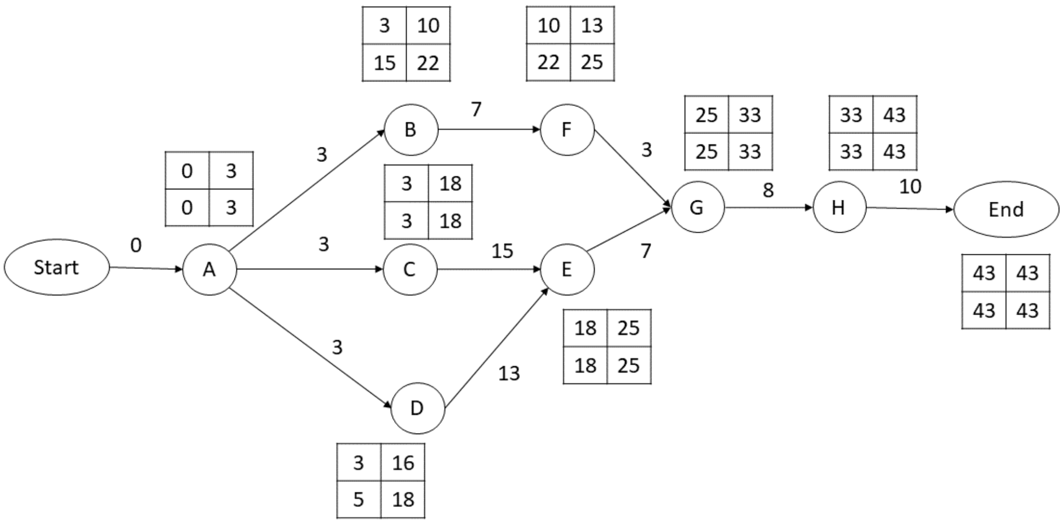
r_j: max reduction

d_j: duration of activity

$$
\begin{array}{ll}
\min & \sum_j (c_j y_j) \\
\\
\text{subject to} & y_j \leq r_j \\
& x_j \geq x_i + d_i - y_i \quad \forall (i,j) \in A \\
& x_{start} = 0 \\
& x_{end} \leq 20 \\
& y_j \geq 0 \qquad\qquad \forall j
\end{array}
$$

Question 6.

(a)



Activity	A	B	C	D	E	F	G	H
Slack	0	12	0	2	0	12	0	0

Critical path: A - C - E - G - H

(b) N: set of activities

Decision Variables:

x_i: start time of activity i, $i \in N$

Model:

$$
\begin{array}{ll}
\min & \sum_{i \in N} x_i \\
\\
\text{s.t.} & x_{end} \leq 43 \\
& x_{start} = 0 \\
& x_A \geq x_{start} \\
& x_B \geq x_A + 3 \\
& x_C \geq x_A + 3 \\
& x_D \geq x_A + 3 \\
& x_E \geq x_C + 15 \\
& x_E \geq x_D + 13 \\
& x_F \geq x_B + 7 \\
& x_G \geq x_F + 3 \\
& x_G \geq x_F + 7 \\
& x_H \geq x_G + 8 \\
& x_{end} \geq x_H + 10 \\
& x_i \geq 0 \ \forall \ i \in N
\end{array}
$$

(c) No, the critical path does not change. The increase in the path A - B - F - G - H is $4 + 7 = 11$, which is smaller than the slack of activities B and F.

(d) **Additional Decision Variable:**

y_i: number of days that the duration of activity i is reduced, $i \in N$

Model:

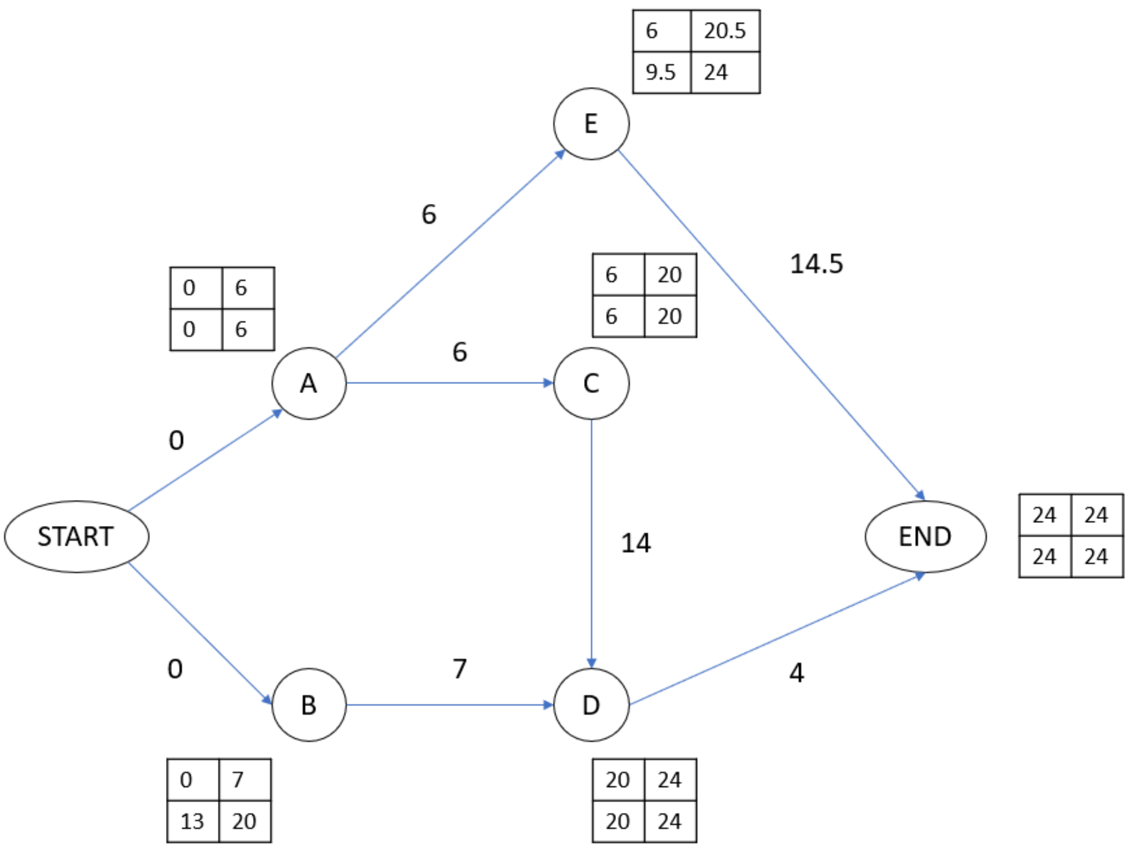
$$\min \quad 10y_A + 21y_B + 15y_C + 23y_D + 12y_E + 14y_F + 13y_6 + 20y_H$$

$$
\begin{aligned}
\text{s.t.} \quad & x_A \geq x_{start} \\
& x_B \geq x_A + 3 - y_A \\
& x_C \geq x_A + 3 - y_A \\
& x_D \geq x_A + 3 - y_A \\
& x_E \geq x_C + 15 - y_C \\
& x_E \geq x_D + 13 - y_D \\
& x_F \geq x_B + 7 - y_B \\
& x_G \geq x_F + 3 - y_F \\
& x_G \geq x_F + 7 - y_E \\
& x_H \geq x_G + 8 - y_G \\
& x_{end} \geq x_H + 10 - y_H \\
& x_{end} \leq 40 \\
& y_A \leq 1 \\
& y_B \leq 2 \\
& y_C \leq 2 \\
& y_D \leq 1 \\
& y_E \leq 2 \\
& y_F \leq 1 \\
& y_G \leq 1 \\
& y_H \leq 3 \\
& y_i \geq 0, \text{integer } \forall i \in N \\
& x_i \geq 0 \ \forall \ i \in N
\end{aligned}
$$

Question 7.

(a)

Activity	Mean	Variance
A	6	4/36
B	7	196/36
C	14	256/36
D	4	4/36
E	14.5	169/36



The critical path is A-C-D.

(b) Note that

$$\mathbb{P}(18 \leq t \leq 26) = \mathbb{P}(t \leq 26) - \mathbb{P}(t \leq 18)$$

We have, $\sigma_p = 2.708$

Note that $Z_{18} = \frac{18-24}{2.708} = -2.2156$

Then, $\mathbb{P}(t \leq 18) = \mathbb{P}(Z \leq -2.2156) = 0.013$

$Z_{26} = \frac{26-24}{2.708} = 0.739$

$\mathbb{P}(t \leq 26) = \mathbb{P}(Z \leq 0.739) = 0.770$

Hence,

$$\mathbb{P}(18 \leq t \leq 26) = \mathbb{P}(t \leq 26) - \mathbb{P}(t \leq 18) = 0.757$$

Question 8. The model is in the following form:

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} \alpha_j x_j \leq b_1 \\ & \sum_{j=1}^{n} d_j x_j \leq b_2 \\ & x_j \geq 0 \qquad\qquad\qquad\qquad j = 1, \ldots, n \end{aligned}$$

x_j : Decision at stage j, $j = 1, \ldots, n$ (amount of type i items taken)
States: R_1, R_2 (Amount of resources available)
$f_j(R_1, R_2)$: Maximum value of the objective function considering stages (items) $j, j+1, \ldots, n$ with available resources R_1 and R_2.

So the **recursion function** is as follows:

$$f_j(R_1, R_2) = \max_{\substack{\alpha_j x_j \leq R_1 \\ d_j x_j \leq R_2 \\ x_j \geq 0}} \{c_j x_j + f_{j+1}(R_1 - a_j x_j, R_2 - d_j x_j)\} j = 1, \ldots, n$$

Base Case: $f_{n+1}(k_1, k_2) = 0$ for all k_1, k_2
Optimal Solution Value: $f_1(b_1, b_2)$

Then, for the LP given in the question;

Base Case: $f_3(430 - 2x_1 - x_2, 230 - x_2) = 0$

Optimal Solution Value: $f_1(430, 230)$

$$
\begin{aligned}
f_2(430 - 2x_1, 230) &= \max_{\substack{x_2 \le 430 - 2x_1 \\ x_2 \le 230 \\ x_2 \ge 0}} \{5x_2 + \underbrace{f_3(430 - 2x_1 - x_2, 230 - x_2)}_{0}\} \\
&= \max_{\substack{x_2 \le 430 - 2x_1 \\ x_2 \le 230 \\ x_2 \ge 0}} \{5x_2\} \\
&= \min\{5(430 - 2x_1), 5 \cdot 230\} \\
&= \begin{cases} -10x_1 + 2150 & \text{if } 100 \le x_1 \le 215 \\ 1150 & \text{if } 0 \le x_1 \le 100 \end{cases}
\end{aligned}
$$

The minimization in the third equality follows as x_2 cannot be greater than the minimum of $430 - 2x_1$ and 230.

$$
\begin{aligned}
f_1(430, 230) &= \max_{\substack{2x_1 \le 430 \\ x_1 \ge 0}} \{2x_1 + f_2(430 - 2x_1, 230)\} \\
&= \max_{x_1} \begin{cases} 2x_1 + 1150 & \text{if } 0 \le x_1 \le 100 \\ -8x_1 + 2150 & \text{if } 100 \le x_1 \le 215 \end{cases} \\
&= 1350
\end{aligned}
$$

(You can also graphically verify that $x_1^* = 100$ and $x_2^* = 230$. Therefore, $f_1(430, 230) = 2x_1^* + 5x_2^*$.)

Question 9.
State: s (Capacity available in the beginning of a year)
Stage: t (Years $1, \ldots, T$)
Decision: x (Change in the capacity)

$f_t(s)$: the minimum cost of meeting power requirements from year t to T having capacity s in the beginning of year t.

Base Case: $f_{T+1}(s) = 0$ for all $s \geq 0$.
Optimal Solution Value: $f_1(100)$
Recursion Function: $f_t(s) = \min_{s+x \geq r_t}\{c_t(x) + m_t(s+x) + f_{t+1}(0.9(s+x))\}$

Question 10.
b_i: the minimum number of workforce needed for year i
x_i: the number of workers in year i
Salaries: $\$30000x_i$
Hiring cost: $\$10000(x_i - \lfloor 0.9x_{i-1} \rfloor)^+$
Firing cost: $\$20000(\lfloor 0.9x_{i-1} \rfloor - x_i)^+$
Stages: Years $i = 1, \ldots, 5$
States: Number of workers in the previous year
$f_i(x_{i-1})$: minimal cost of maintaining a workforce from year i to 5 with workforce (x_{i-1}) in the previous year

Base Case: $f_6(x_5) = 0$
Optimal Solution Value: $f_1(20)$ (Since we started with 20 workers)
Recursion Function:

$$f_i(x_{i-1}) = \min_{b_i \leq x_i \leq \max\{b_1, \ldots, b_n\}} \{f_{i+1}(x_i) + 30000x_i + \$10000(x_i - \lfloor 0.9x_{i-1} \rfloor)^+ + \$20000(\lfloor 0.9x_{i-1} \rfloor - x_i)^+\}$$

for $i = 1, \ldots, 5$