

IE 400: Principles of Engineering Management

Study Set 3 Solutions

Spring 2022

Question 1.

•

x_1, x_2 basic,

x_3, x_4 nonbasic ($x_3 = 0, x_4 = 0$)

Then,

$$x_1 - x_2 = 15$$

$$x_1 + x_2 = 9$$

Solving this system, we obtain $x = (12, -3, 0, 0)$. This is basic but not feasible since $x_2 = -3 < 0$.

•

x_1, x_3 basic,

x_2, x_4 nonbasic ($x_2 = 0, x_4 = 0$)

Then,

$$x_1 + 3x_3 = 15$$

$$x_1 = 9$$

Solving this system, we obtain $x = (9, 0, 2, 0)$. This is a basic feasible solution.

•

x_1, x_4 basic,

x_2, x_3 nonbasic ($x_2 = 0, x_3 = 0$)

Then,

$$x_1 + 5x_4 = 15$$

$$x_1 + x_4 = 9$$

Solving this system, we obtain $x = (15/2, 0, 0, 3/2)$. This is a basic feasible solution.

•

x_2, x_3 basic,

x_1, x_4 nonbasic ($x_1 = 0, x_4 = 0$)

Then,

$$-x_2 + 3x_3 = 15$$

$$x_2 = 9$$

Solving this system, we obtain $x = (0, 9, 8, 0)$. This is a basic feasible solution.

•

x_2, x_4 basic,

x_1, x_3 nonbasic ($x_1 = 0, x_3 = 0$)

Then,

$$-x_2 + 5x_4 = 15$$

$$x_2 + x_4 = 9$$

Solving this system, we obtain $x = (0, 5, 0, 4)$. This is a basic feasible solution.

•

x_3, x_4 basic,

x_1, x_2 nonbasic ($x_1 = 0, x_2 = 0$)

Then,

$$3x_3 + 5x_4 = 15$$

$$x_4 = 9$$

Solving this system, we obtain $x = (0, 0, -10, 9)$. This is basic but not feasible since $x_3 = -10 < 0$.

Question 2.

First convert the LP into standard form:

$$\begin{aligned} \max \quad & 5x_1 + 3x_2 + x_3 \\ \text{s.t.} \quad & x_1 + x_2 + 3x_3 + s_1 = 6 \\ & 5x_1 + 3x_2 + 6x_3 + s_2 = 15 \\ & x_i \geq 0 \quad i = 1, 2, 3 \\ & s_1, s_2 \geq 0 \end{aligned}$$

	z	x_1	x_2	x_3	s_1	s_2	RHS
1	1	-5	-3	-1	0	0	0
s_1	0	1	1	3	1	0	6
s_2	0	5	3	6	0	1	15

x_1 enters. Apply MRT: $\min\{6/1, 15/5\}=3$ Hence, s_2 leaves.

	z	x_1	x_2	x_3	s_1	s_2	RHS
1	1	0	0	5	0	1	15
s_1	0	0	$2/5$	$9/5$	1	$-1/5$	3
x_1	0	1	$3/5$	$6/5$	0	$1/5$	3

Since there is no negative row 0 coefficient, the solution is optimal: $x^* = (3, 0, 0)$ and $z^* = 15$.

Since there exists 0 in row 0, there may be alternative optimal solutions: Let x_2 enter the basis. Apply MRT: $\min\{\frac{3}{2/5}, \frac{3}{3/5}\} = 5$. Then x_1 leaves.

	z	x_1	x_2	x_3	s_1	s_2	RHS
1	1	0	0	5	0	1	15
s_1	0	$-2/3$	0	1	1	$-1/3$	1
x_2	0	$5/3$	1	2	0	$1/3$	5

$x^* = (0, 5, 0)$ and $z^*=15$.

(If x_1 enters again, you turn back to the previous tableau!)

Question 3.

The LP in the standard form:

$$\begin{aligned} \max \quad & 10x_1 + 8x_2 - 3x_3 + 3x_4 \\ \text{s.t.} \quad & 2x_1 + 4x_2 - 0.5x_3 + 0.5x_4 + s_1 = 3 \\ & -2x_1 + 6x_2 - 4.5x_3 + 4.5x_4 + s_2 = 7 \\ & x_1, x_2, x_3, x_4, s_1, s_2 \geq 0 \end{aligned}$$

	z	x_1	x_2	x_3	x_4	s_1	s_2	RHS
z	1	-10	-8	3	-3	0	0	0
s_1	0	2	4	-0.5	0.5	1	0	3
s_2	0	-2	6	-4.5	4.5	0	1	7

x_1 enters, s_1 leaves.

	z	x_1	x_2	x_3	x_4	s_1	s_2	RHS
z	1	0	12	0.5	-0.5	5	0	15
x_1	0	1	2	-0.25	0.25	0.5	0	3/2
s_2	0	0	10	-5	5	1	1	10

x_4 enters, s_2 leaves.

	z	x_1	x_2	x_3	x_4	s_1	s_2	RHS
z	1	0	13	0	0	5.1	0.1	16
x_1	0	1	1.5	0	0	0.45	-0.05	1
x_4	0	0	2	-1	1	0.2	0.2	2

- (a) Yes, it is a basic solution. Any tableau solution in proper form will be basic, since it corresponds to a basis B.
- (b) Yes, it is a basic feasible solution. All RHS values are nonnegative.
- (c) Yes, it is an optimal BFS. All row zero elements are ≥ 0 .

Question 4.

Decision Variables:

x_1 : Amount of TLs invested (in millions)

x_2 : Amount of TLs borrowed (in millions)

Model:

$$\begin{aligned} \max \quad & -0.1x_1 + 0.5x_2 \\ \text{s.t.} \quad & x_1 \leq 1 \\ & x_2 \leq 0.4x_1 \\ & x_1 - x_2 \leq 0.8 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Now, apply simplex method. The LP in standard form is:

$$\begin{aligned}
 \max \quad & -0.1x_1 + 0.5x_2 \\
 \text{s.t.} \quad & x_1 + s_1 = 1 \\
 & x_2 - 0.4x_1 + s_2 = 0 \\
 & x_1 - x_2 + s_3 = 0.8 \\
 & x_1, x_2, s_1, s_2, s_3 \geq 0
 \end{aligned}$$

	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	0.1	-0.5	0	0	0	0
s_1	0	1	0	1	0	0	1
s_2	0	-0.4	1	0	1	0	0
s_3	0	1	-1	0	0	1	0.8

x_2 enters, s_2 leaves.

	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	-0.1	0	0	0.5	0	0
s_1	0	1	0	1	0	0	1
x_2	0	-0.4	1	0	1	0	0
s_3	0	0.6	0	0	1	1	0.8

x_1 enters, s_1 leaves.

	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	0	0	0.1	0.5	0	0.1
x_1	0	1	0	1	0	0	1
x_2	0	0	1	0.4	1	0	0.4
s_3	0	0	0	-0.6	1	1	0.2

The optimal solution is $x^* = (1, 0.4)$, and $z^* = 0.1$.

Question 5.

- (a) The tableau is final and there exists a unique optimal solution.

$$A < 0, B = 0, C \text{ any}, D \geq 0, E = 1, F \text{ any}$$

- (b) The simplex method determines an unbounded solution from this tableau.

$$A > 0, B = 0, C \leq 0, D \geq 0, E = 1, F \text{ any}$$

- (c) The current bfs is degenerate (not necessarily optimal).

$$A \text{ any}, B = 0, C \text{ any}, D = 0, E = 1, F \text{ any}$$

- (d) d) The current solution is optimal, there are alternative optimal solutions but no alternative optimal bfs.

$$A = 0, B = 0, C \leq 0, D \geq 0, E = 1, F \text{ any}$$

Question 6.

Convert into standard form and add artificial variable.

$$\begin{aligned}
 \min \quad & -3x_1 + 4x_2 \\
 \text{s.t.} \quad & x_1 + x_2 + s_1 = 4 \\
 & 2x_1 + 3x_2 - e_1 + a_1 = 18 \\
 & x_1, x_2, s_1, e_1, a_1 \geq 0
 \end{aligned}$$

PHASE 1

	z	x_1	x_2	s_1	e_1	a_1	RHS
1	1	0	0	0	0	-1	0
s_1	0	1	1	1	0	0	4
a_1	0	2	3	0	-1	1	18

Add second row to row 0.

	z	x_1	x_2	s_1	e_1	a_1	RHS
1	1	2	3	0	-1	0	18
s_1	0	1	1	1	0	0	4
a_1	0	2	3	0	-1	1	18

x_2 enters. Apply MRT: $\min\{4/1, 18/2\} = 4$. Then, s_1 leaves the basis.

	z	x_1	x_2	s_1	e_1	a_1	RHS
1	1	-1	0	-3	-1	0	6
x_2	0	1	1	1	0	0	4
a_1	0	-1	0	-3	-1	1	6

The tableau is optimal, but a_1 is still in the basis. Hence, the Phase 1 problem is infeasible.

Question 7.

Convert the following LP to standard form and solve with the Big-M method.

$$\begin{aligned}
 \max \quad & -x_1 + 3x_2 + x_3 \\
 \text{s.t.} \quad & -x_1 + x_2 - x_3 \geq -4 \\
 & 2x_1 + x_2 + x_3 \geq 10 \\
 & x_1 \geq 0 \\
 & x_2 \leq 0 \\
 & x_3 \text{ free}
 \end{aligned}$$

Let $x_3 = x'_3 - x''_3$ where $x'_3 \geq 0$ and $x''_3 \geq 0$. Let $-x_2 = x'_2$. Introduce slack variable s_1 and excess variable e_2 .

$$\begin{aligned}
 \max \quad & -x_1 - 3x'_2 + (x'_3 - x''_3) - Ma_2 \\
 \text{s.t.} \quad & x_1 + x'_2 + (x'_3 - x''_3) + s_1 = 4 \\
 & 2x_1 - x'_2 + (x'_3 - x''_3) - e_2 + a_2 = 10 \\
 & x_1, x'_2, x'_3, x''_3, s_1, e_2 \geq 0
 \end{aligned}$$

	z	x_1	x'_2	x'_3	x''_3	s_1	e_2	a_2	RHS
	1	1	3	-1	1	0	0	M	0
s_1	0	1	1	1	-1	1	0	0	4
a_1	0	2	-1	1	-1	0	-1	1	10

Multiply the second row with $-M$ and add it to row 0:

	z	x_1	x'_2	x'_3	x''_3	s_1	e_2	a_2	RHS
	1	-2M+1	M+3	-M-1	M+1	0	M	0	-10M
s_1	0	1	1	1	-1	1	0	0	4
a_1	0	2	-1	1	-1	0	-1	1	10

x_1 enters the basis. By minimum ratio test: $\min\{4/1, 10/2\} = 4$, s_1 leaves:

	z	x_1	x'_2	x'_3	x''_3	s_1	e_2	a_2	RHS
	1	0	3M+2	M-2	-M+2	2M-1	M	0	-2M-4
x_1	0	1	1	1	-1	1	0	0	4
a_1	0	0	-3	-1	1	-2	-1	1	2

x''_3 enters the basis. Apply MRT: $\min\{2/1\}$. Then, a_1 leaves:

	z	x_1	x'_2	x'_3	x''_3	s_1	e_2	a_2	RHS
	1	0	8	0	0	3	2	M-2	-8
x_1	0	1	-2	0	0	-1	-1	1	6
x''_3	0	0	-3	-1	1	-2	-1	1	2

No negative coefficient in row 0. The tableau is final. (a_2 is not in the basis so remove the column a_2 from the tableau)

Then,

$$x^* = (x_1, x'_2, x'_3, x''_3, s_1, e_2) = (6, 0, 0, 2, 0, 0)$$

and $z^* = -8$. Note that $x_2 = -x'_2 = 0$, $x_3 = x'_3 - x''_3 = 0 - 2 = -2$.