IE400 – Principles of Engineering Management

Graphical Solution of two-variable LP Problems

Graphical Solution of 2-variable LP Problems

Example 1.a.)

max $\quad x_1 + 3x_2$

s.t.

$$
\begin{aligned}
x_1 + x_2 &\leq 6 \quad (1)\\
-x_1 + 2x_2 &\leq 8 \quad (2)\\
x_1, x_2 &\geq 0 \quad (3), (4)
\end{aligned}
$$

Graphical Solution of 2-variable LP Problems

Example 1.a.)

max $\quad x_1 + 3x_2$

s.t.

$$x_1 + x_2 \leq 6 \quad (1)$$

$$-x_1 + 2x_2 \leq 8 \quad (2)$$

$$x_1, x_2 \geq 0 \quad (3), (4)$$

General Form:

max $\quad \mathbf{cx}$

s.t.

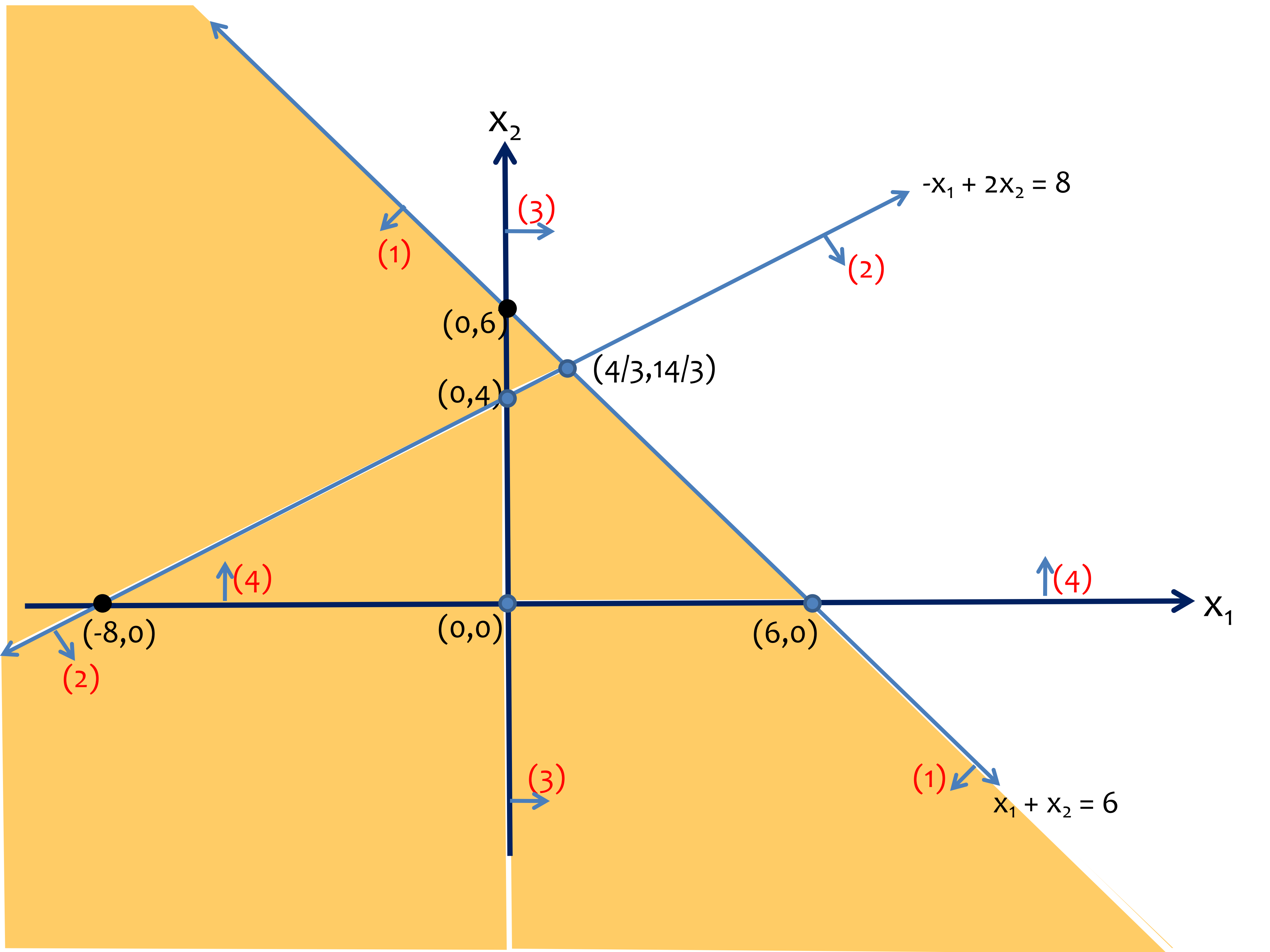
$$\mathbf{Ax} \leq \mathbf{b}$$

$$\mathbf{x} \geq 0$$

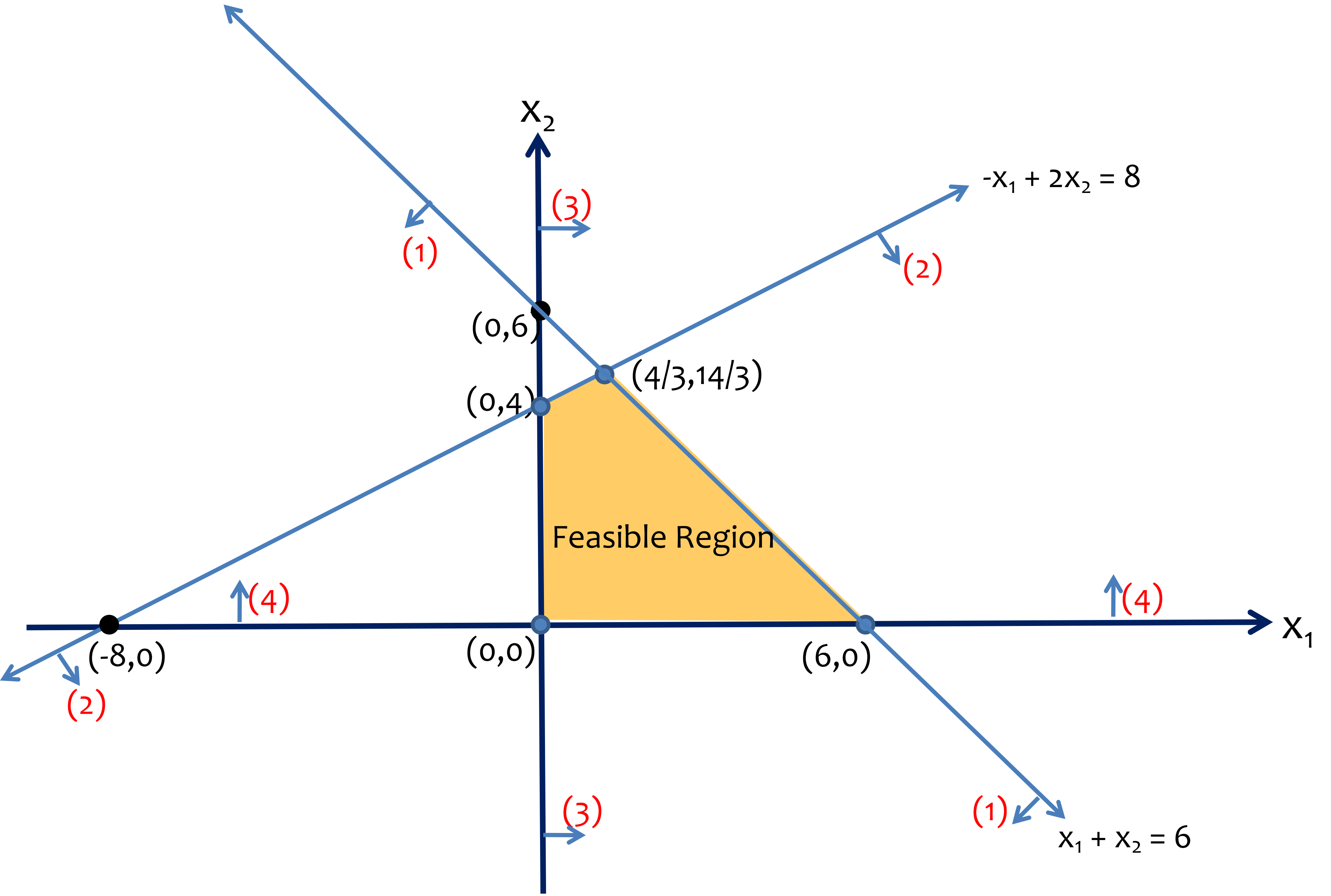
where,

$\mathbf{c} = [1 \;\; 3] \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

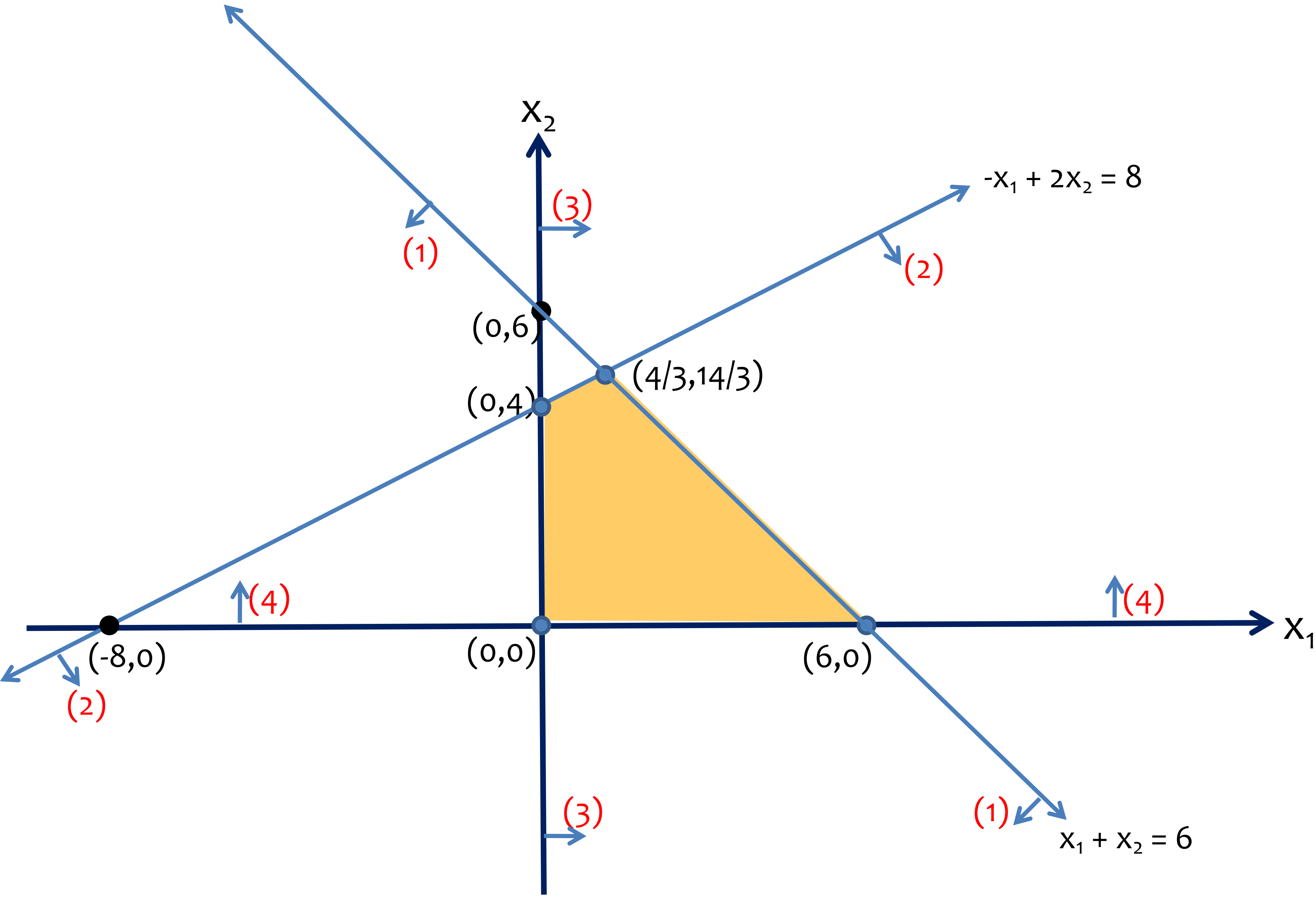
$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$



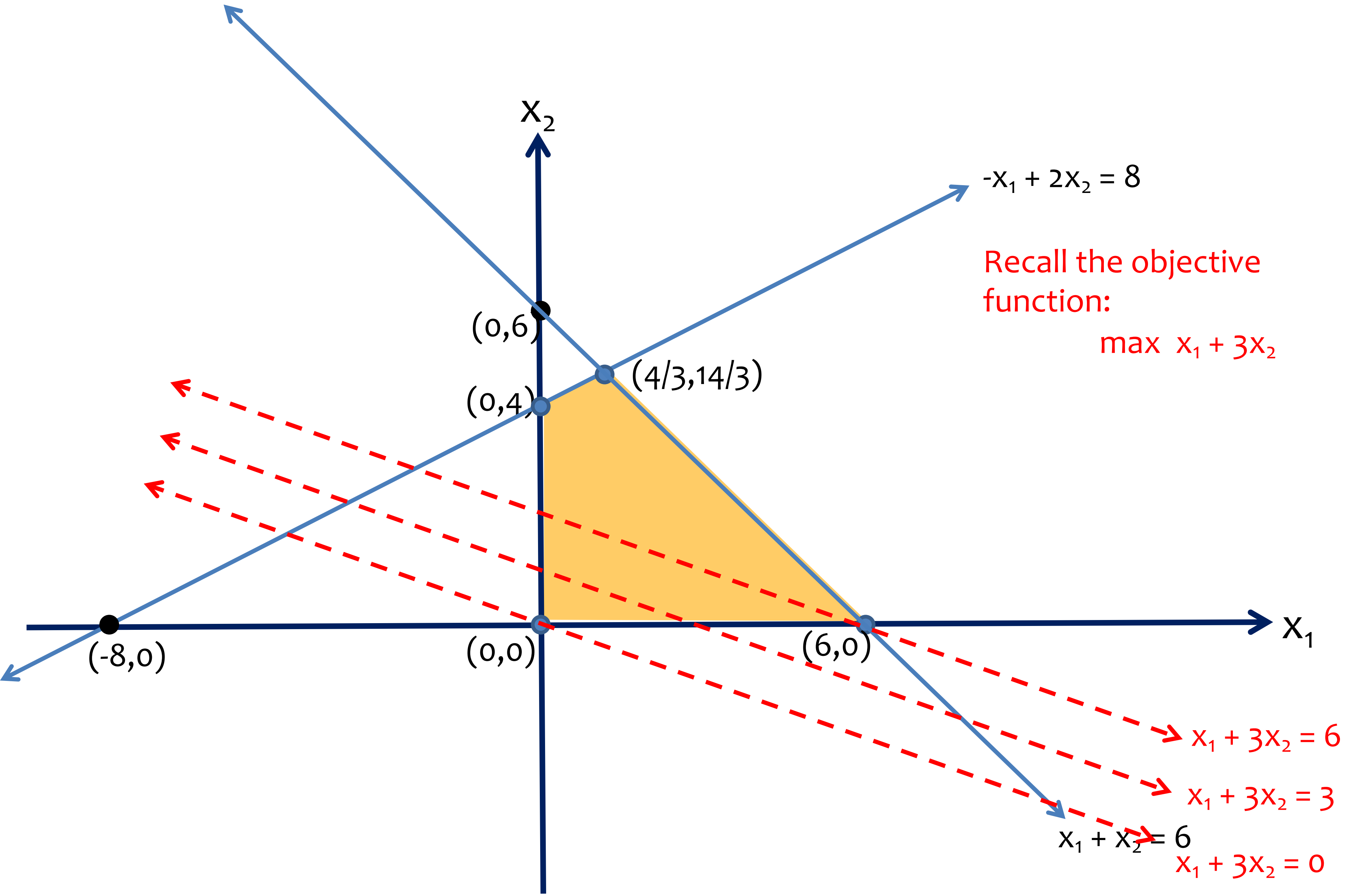
x_2
x_1
$-x_1 + 2x_2 = 8$
$x_1 + x_2 = 6$
(1)
(2)
(3)
(4)
(0,6)
(4/3,14/3)
(0,4)
(-8,0)
(0,0)
(6,0)



x_2
$-x_1 + 2x_2 = 8$
(3)
(1)
(2)
(0,6)
(4/3,14/3)
(0,4)
Feasible Region
(4)
(4)
x_1
(-8,0)
(0,0)
(6,0)
(2)
(3)
(1)
$x_1 + x_2 = 6$



x_2
x_1
$-x_1 + 2x_2 = 8$
$x_1 + x_2 = 6$
(0,6)
(0,4)
(4/3,14/3)
(-8,0)
(0,0)
(6,0)
(1)
(2)
(3)
(4)



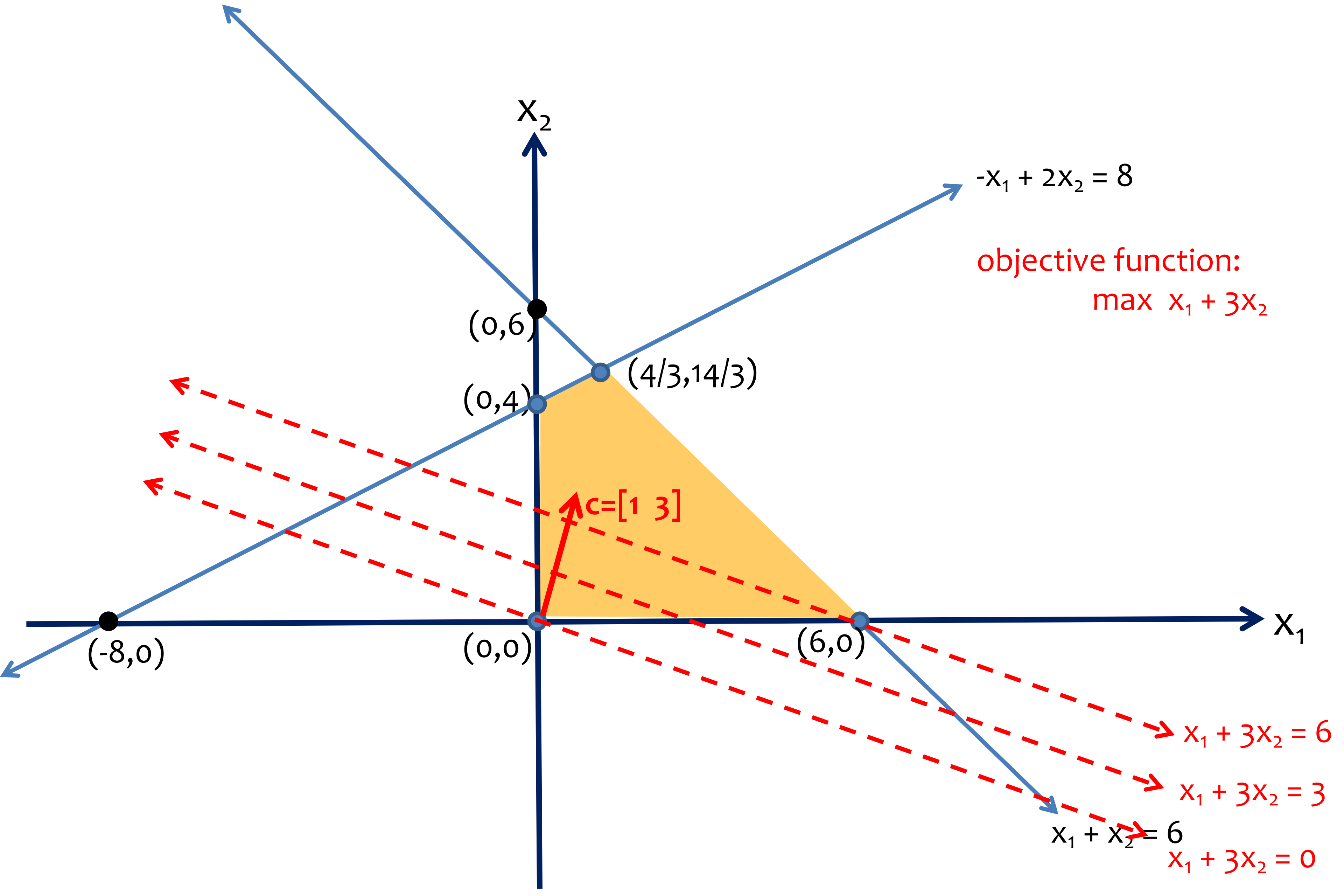
x_2
$-x_1 + 2x_2 = 8$
Recall the objective function:
$\max\ x_1 + 3x_2$
(0,6)
(4/3,14/3)
(0,4)
x_1
(-8,0)
(0,0)
(6,0)
$x_1 + 3x_2 = 6$
$x_1 + 3x_2 = 3$
$x_1 + x_2 = 6$
$x_1 + 3x_2 = 0$

Graphical Solution of 2-variable LP Problems

- A line on which all points have the same z-value ($z = x_1 + 3x_2$) is called :
  - **Isoprofit line** for maximization problems,
  - **Isocost line** for minimization problems.

Graphical Solution of 2-variable LP Problems

- Consider the coefficients of x_1 and x_2 in the objective function and let $\mathbf{c}$=[1 3].



x_2
x_1
$-x_1 + 2x_2 = 8$
objective function:
max $x_1 + 3x_2$
(0,6)
(4/3,14/3)
(0,4)
c=[1 3]
(-8,0)
(0,0)
(6,0)
$x_1 + 3x_2 = 6$
$x_1 + 3x_2 = 3$
$x_1 + x_2 = 6$
$x_1 + 3x_2 = 0$

Graphical Solution of 2-variable LP Problems

- Note that as we move the isoprofit lines in the direction of **c**, the total profit increases!

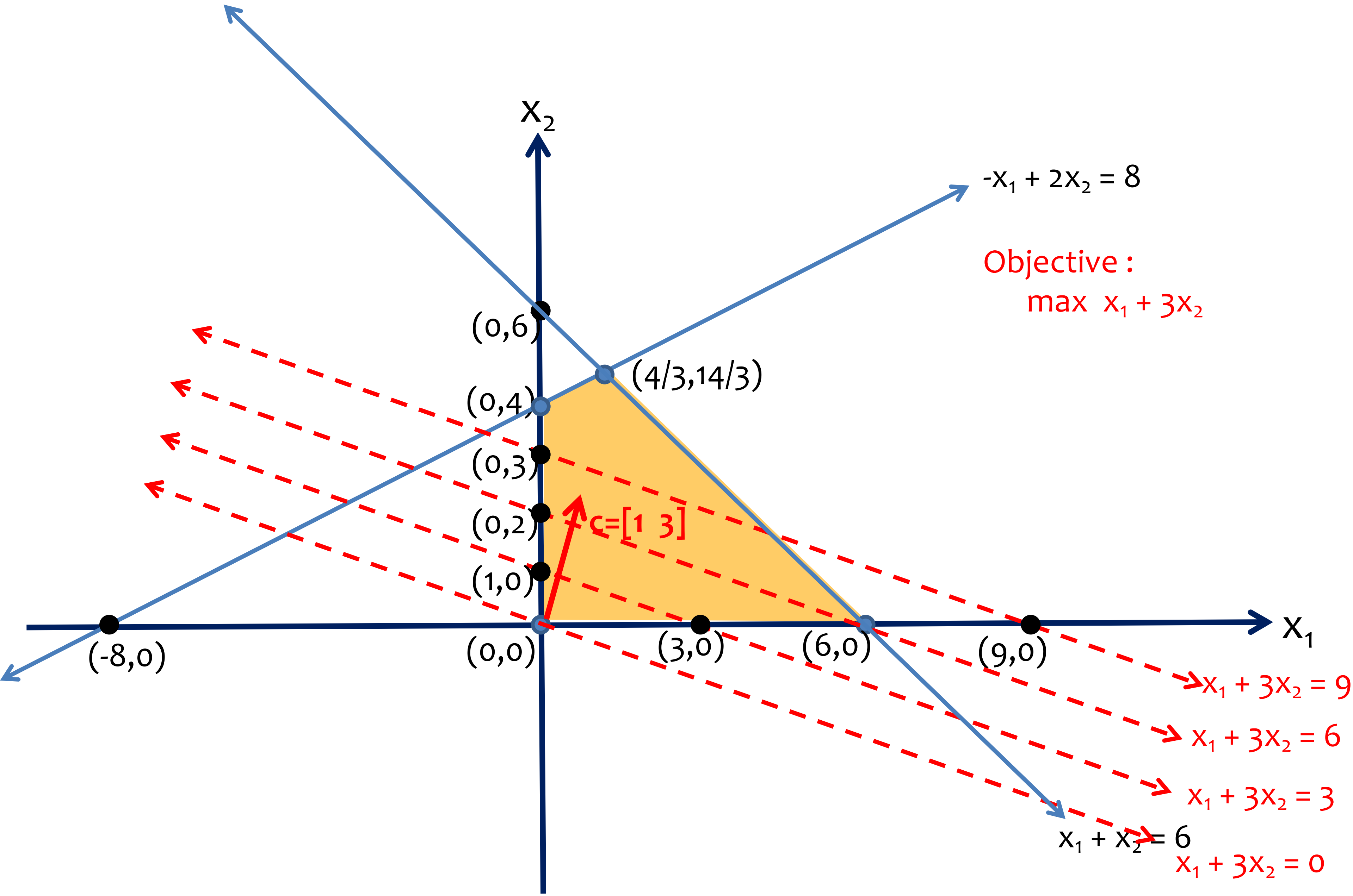
Graphical Solution of 2-variable LP Problems

- For a **maximization problem**, the vector **c**, given by the coefficients of x_1 and x_2 in the objective function, is called the **improving direction**.

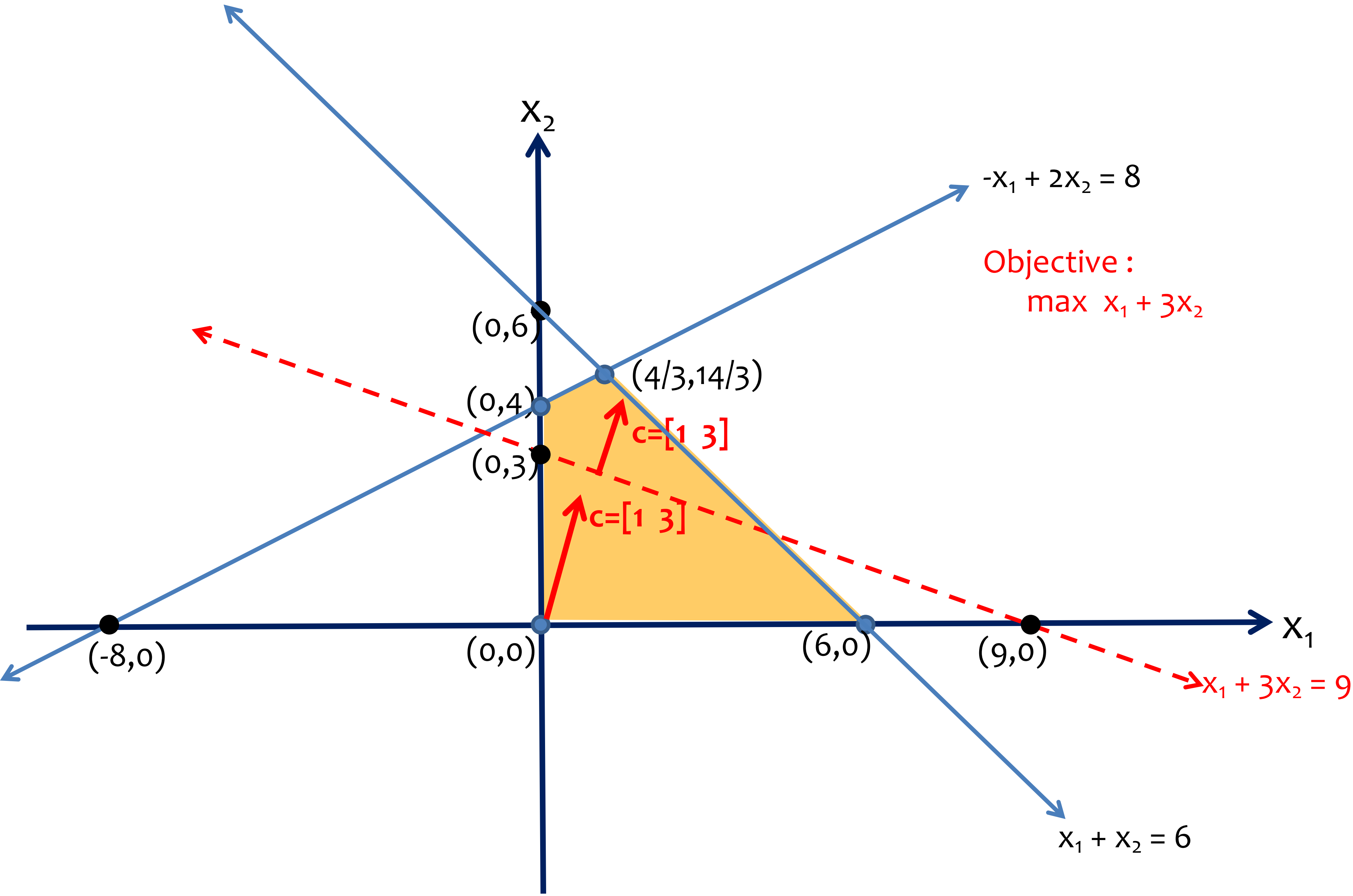
- On the other hand, **-c is the improving direction for minimization problems**!

Graphical Solution of 2-variable LP Problems

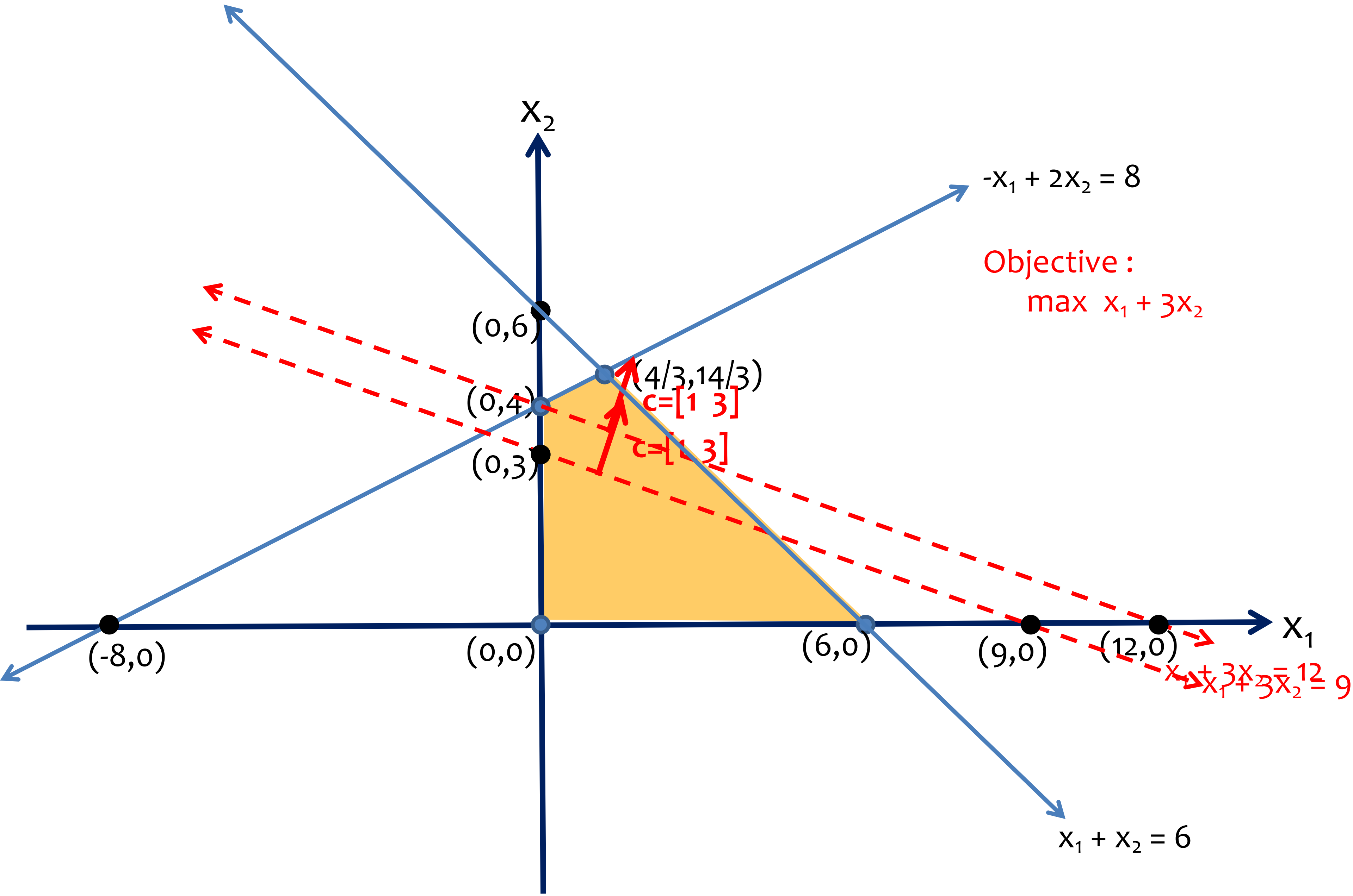
- To solve an LP problem in two variables, we should try to push isoprofit (isocost) lines in the improving direction as much as possible while still staying in the feasible region.



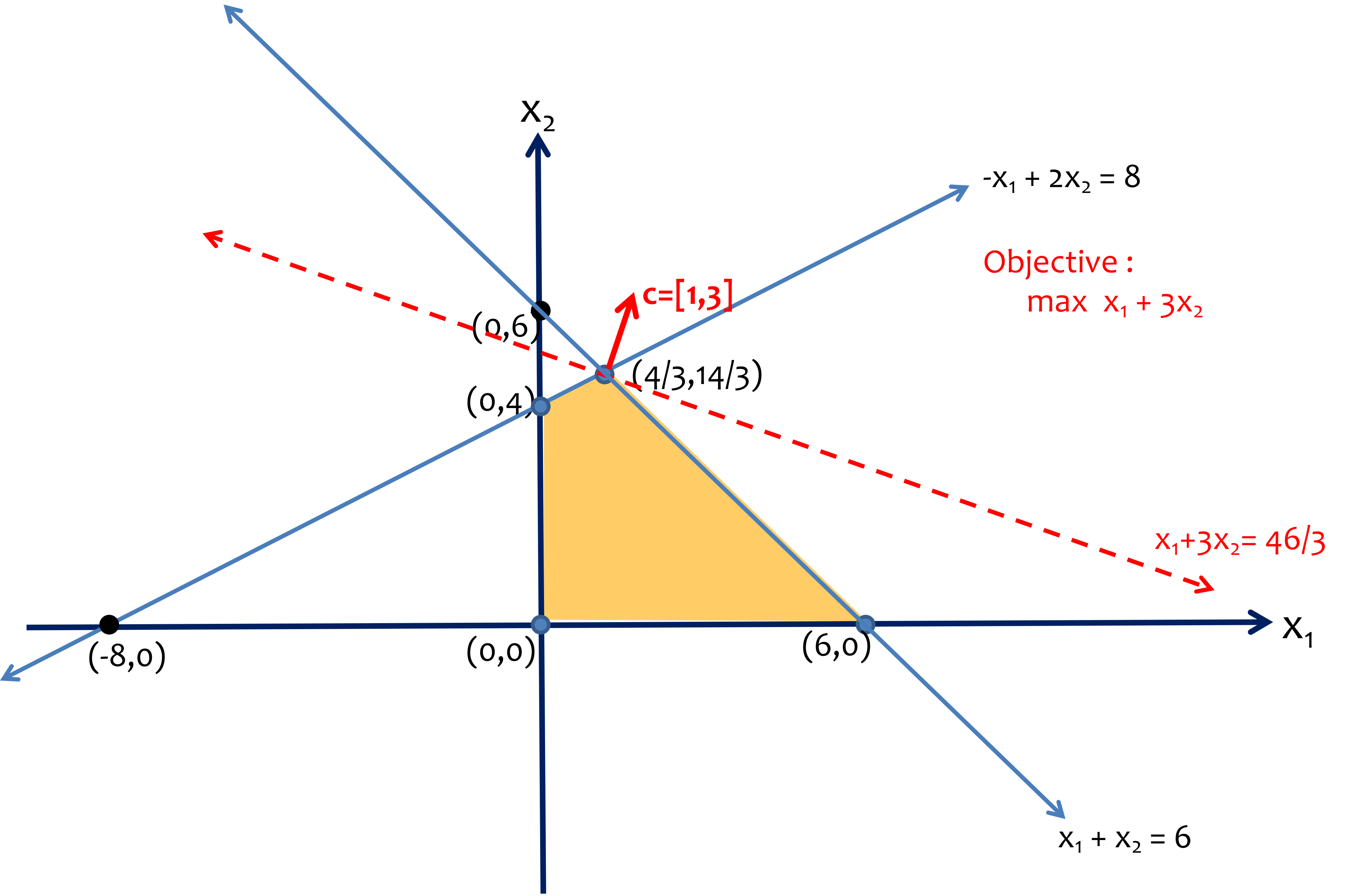
x_2
x_1
$-x_1 + 2x_2 = 8$
Objective :
max $x_1 + 3x_2$
(0,6)
(4/3,14/3)
(0,4)
(0,3)
(0,2)
$c=[1\ \ 3]$
(1,0)
(-8,0)
(0,0)
(3,0)
(6,0)
(9,0)
$x_1 + 3x_2 = 9$
$x_1 + 3x_2 = 6$
$x_1 + 3x_2 = 3$
$x_1 + x_2 = 6$
$x_1 + 3x_2 = 0$



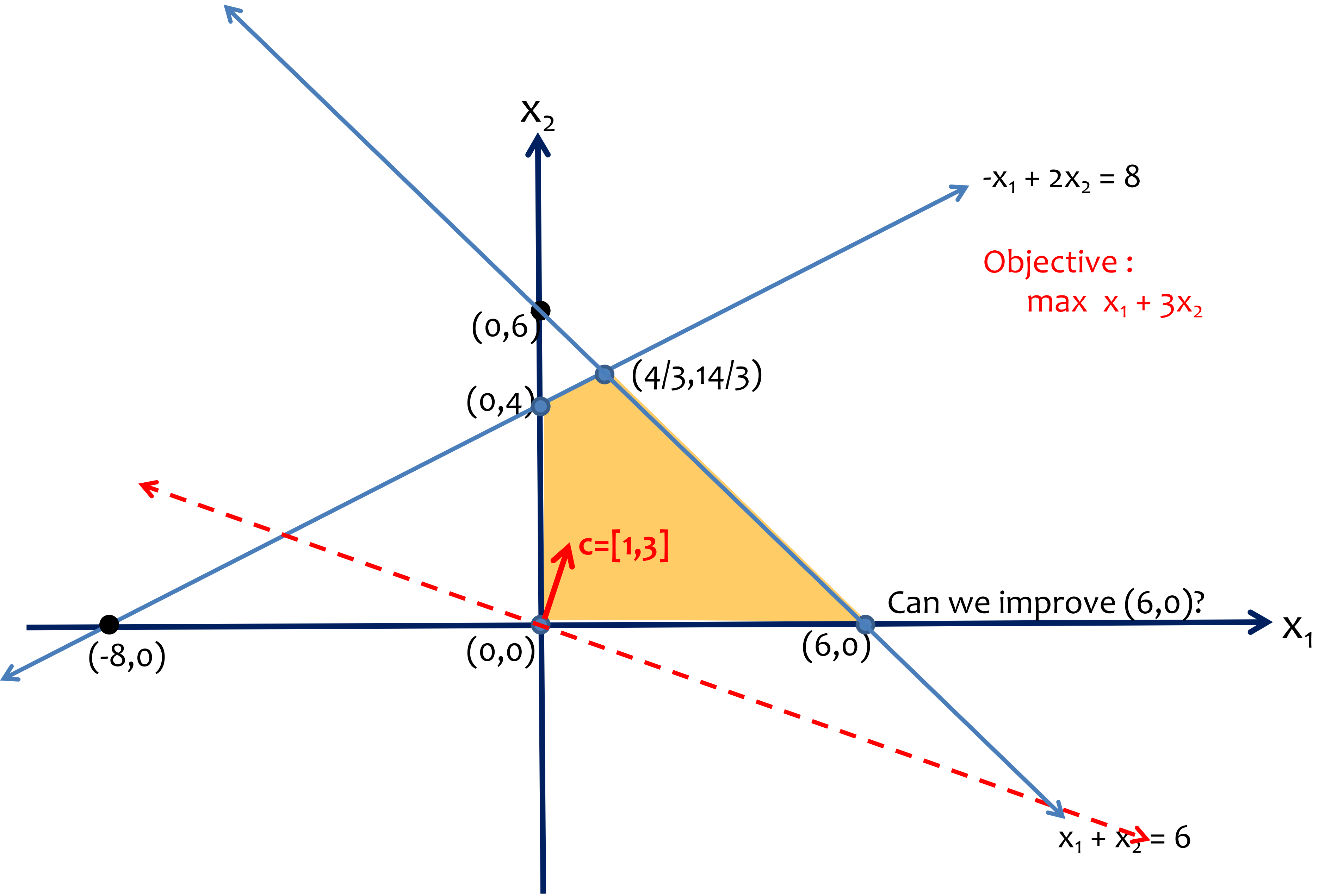
x_2
x_1
$-x_1 + 2x_2 = 8$
Objective :
max $x_1 + 3x_2$
(0,6)
(4/3,14/3)
(0,4)
c=[1 3]
(0,3)
c=[1 3]
(-8,0)
(0,0)
(6,0)
(9,0)
$x_1 + 3x_2 = 9$
$x_1 + x_2 = 6$



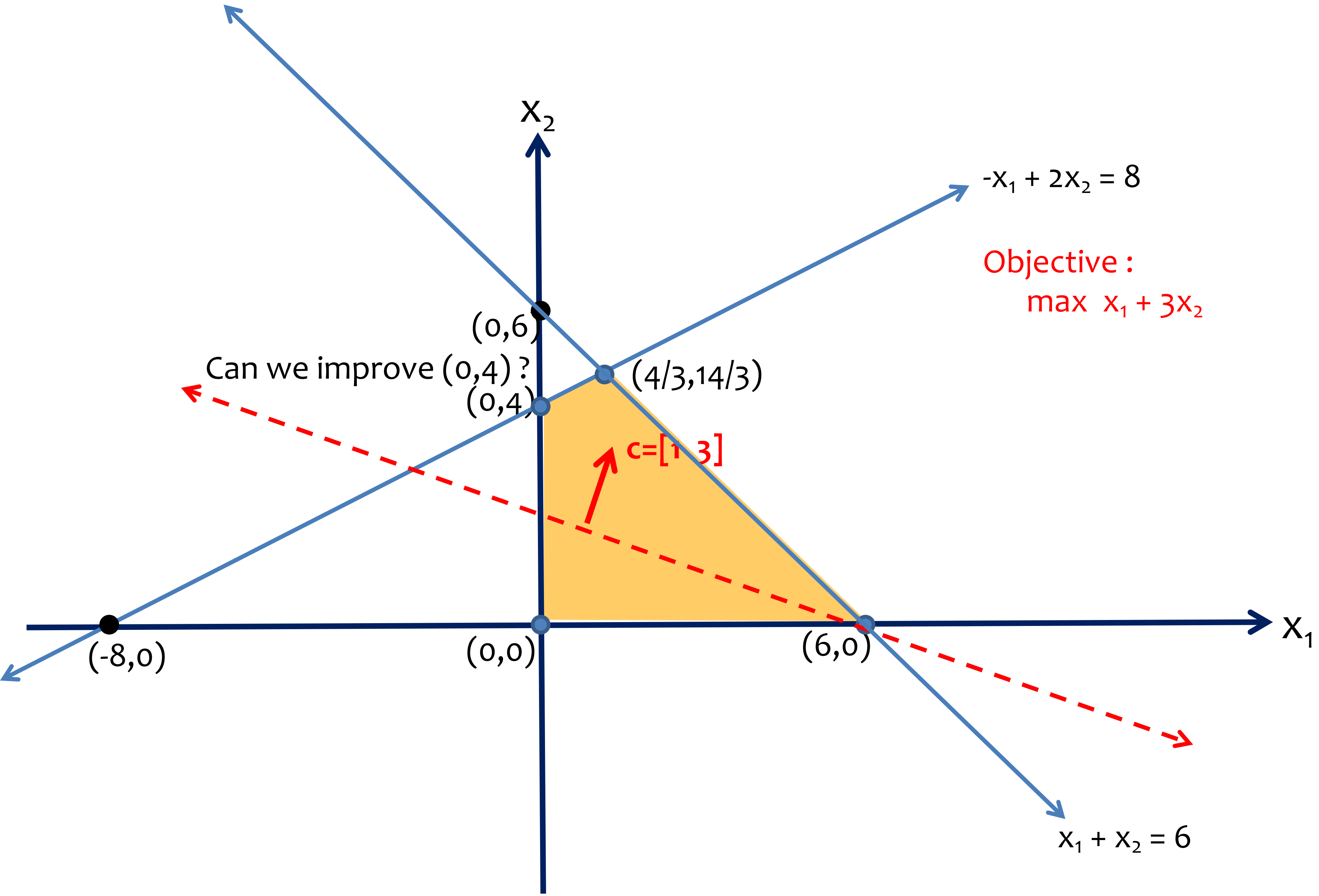
x_2
x_1
$-x_1 + 2x_2 = 8$
Objective :
max $x_1 + 3x_2$
(0,6)
(4/3,14/3)
(0,4)
c=[1 3]
c=[1 3]
(0,3)
(-8,0)
(0,0)
(6,0)
(9,0)
(12,0)
$x_1 + 3x_2 = 12$
$x_1 + 3x_2 = 9$
$x_1 + x_2 = 6$



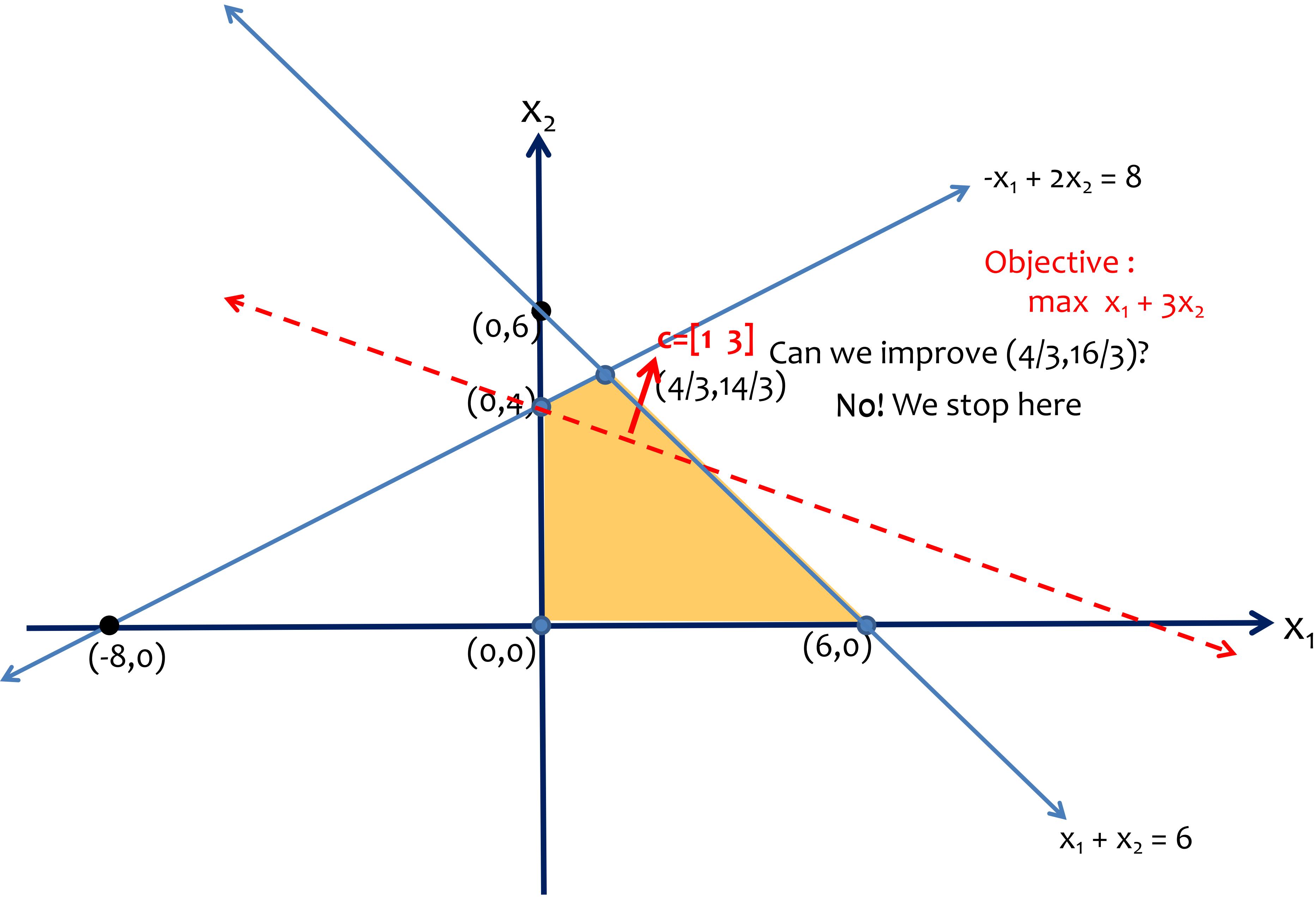
x2
-x1 + 2x2 = 8
Objective :
max x1 + 3x2
c=[1,3]
(0,6)
(4/3,14/3)
(0,4)
x1+3x2= 46/3
(-8,0)
(0,0)
(6,0)
x1
x1 + x2 = 6



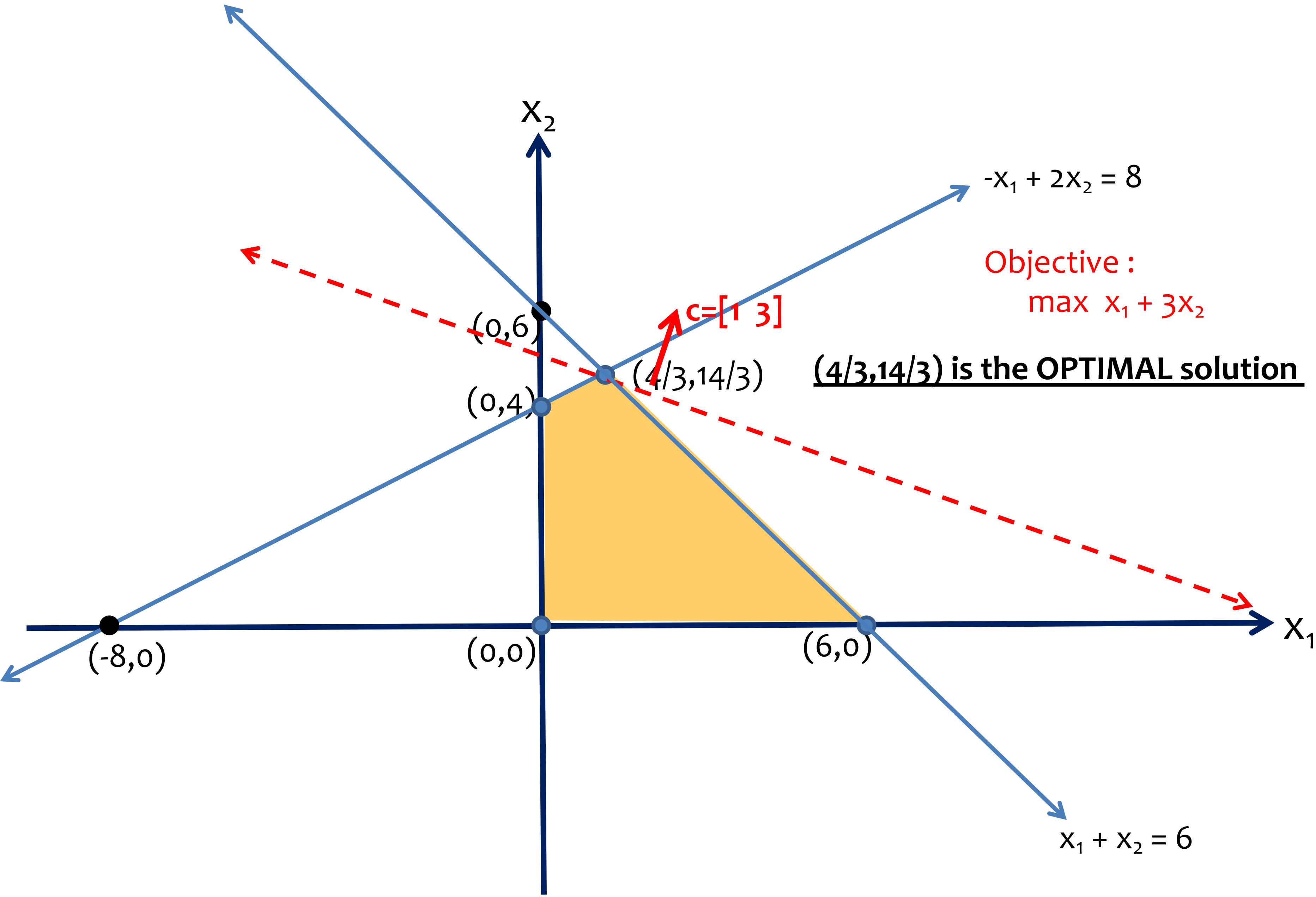
x_2
$-x_1 + 2x_2 = 8$
Objective :
max $x_1 + 3x_2$
(0,6)
(4/3,14/3)
(0,4)
c=[1,3]
Can we improve (6,0)?
x_1
(-8,0)
(0,0)
(6,0)
$x_1 + x_2 = 6$



x_2
$-x_1 + 2x_2 = 8$
Objective :
max $x_1 + 3x_2$
(0,6)
Can we improve (0,4) ?
(0,4)
(4/3,14/3)
c=[1 3]
(-8,0)
(0,0)
(6,0)
x_1
$x_1 + x_2 = 6$



x_2
$-x_1 + 2x_2 = 8$
Objective :
max $x_1 + 3x_2$
(0,6)
c=[1 3]
Can we improve (4/3,16/3)?
(4/3,14/3)
No! We stop here
(0,4)
(-8,0)
(0,0)
(6,0)
x_1
$x_1 + x_2 = 6$



x_2
$-x_1 + 2x_2 = 8$
Objective :
max $x_1 + 3x_2$
c=[1 3]
(0,6)
(4/3,14/3)
(4/3,14/3) is the OPTIMAL solution
(0,4)
(-8,0)
(0,0)
(6,0)
x_1
$x_1 + x_2 = 6$

Graphical Solution of 2-variable LP Problems

- Point (4/3, 14/3) is the last point in the feasible region when we move in the improving direction. So, if we move in the same direction any further, there is no feasible point.

- Therefore, point $x^*= (4/3, 14/3)$ is the maximizing point. The **optimal value** of the LP problem is $z^*=46/3$.

Graphical Solution of 2-variable LP Problems

- Let us modify the objective function while keeping the constraints unchanged:

Example 1.a.)

max $x_1 + 3x_2$

s.t.

$x_1 + x_2 \leq 6$ (1)

$-x_1 + 2x_2 \leq 8$ (2)

$x_1, x_2 \geq 0$ (3), (4)

Example 1.b.)

max $x_1 - 2x_2$

s.t.

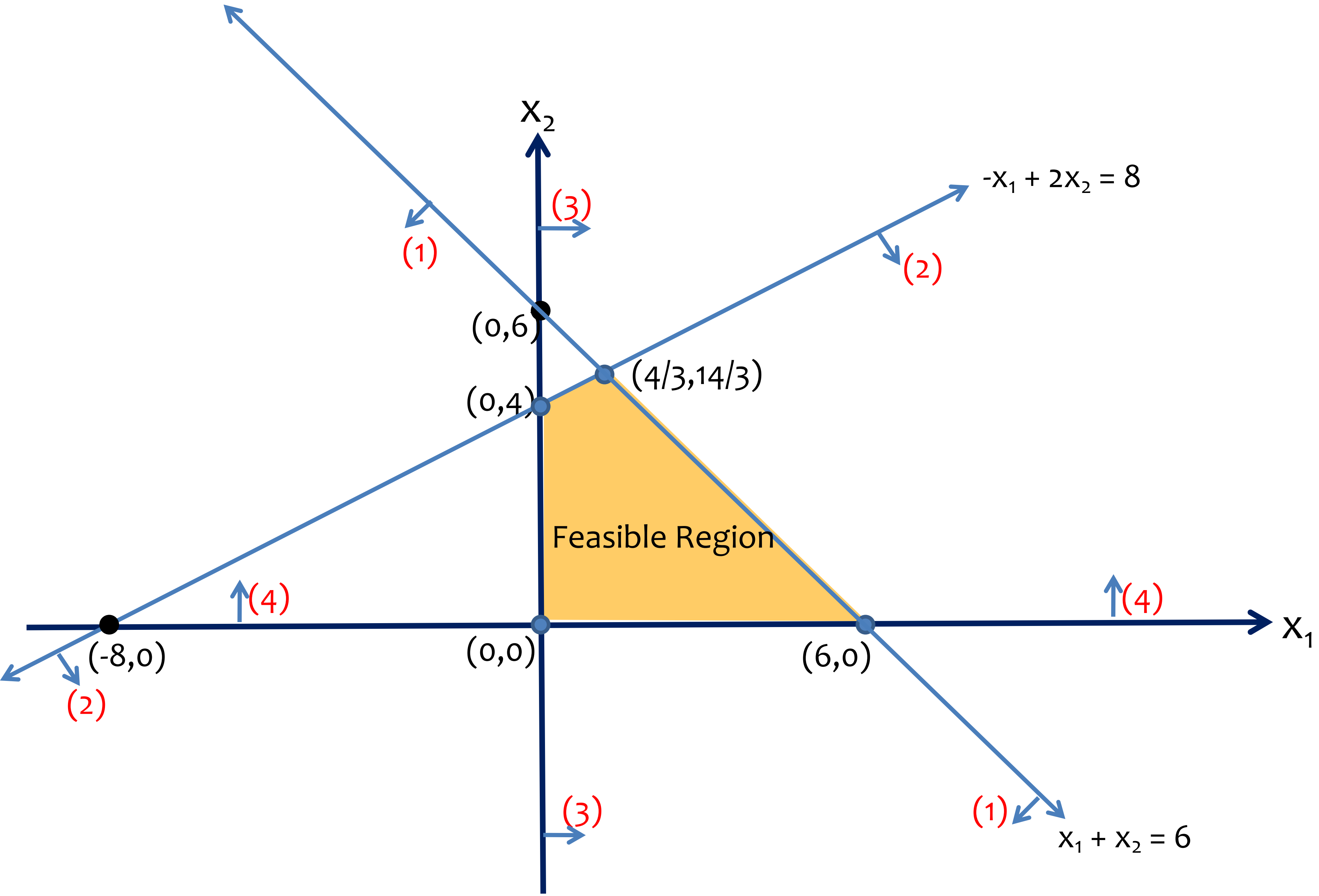
$x_1 + x_2 \leq 6$ (1)

$-x_1 + 2x_2 \leq 8$ (2)

$x_1, x_2 \geq 0$ (3), (4)

Graphical Solution of 2-variable LP Problems

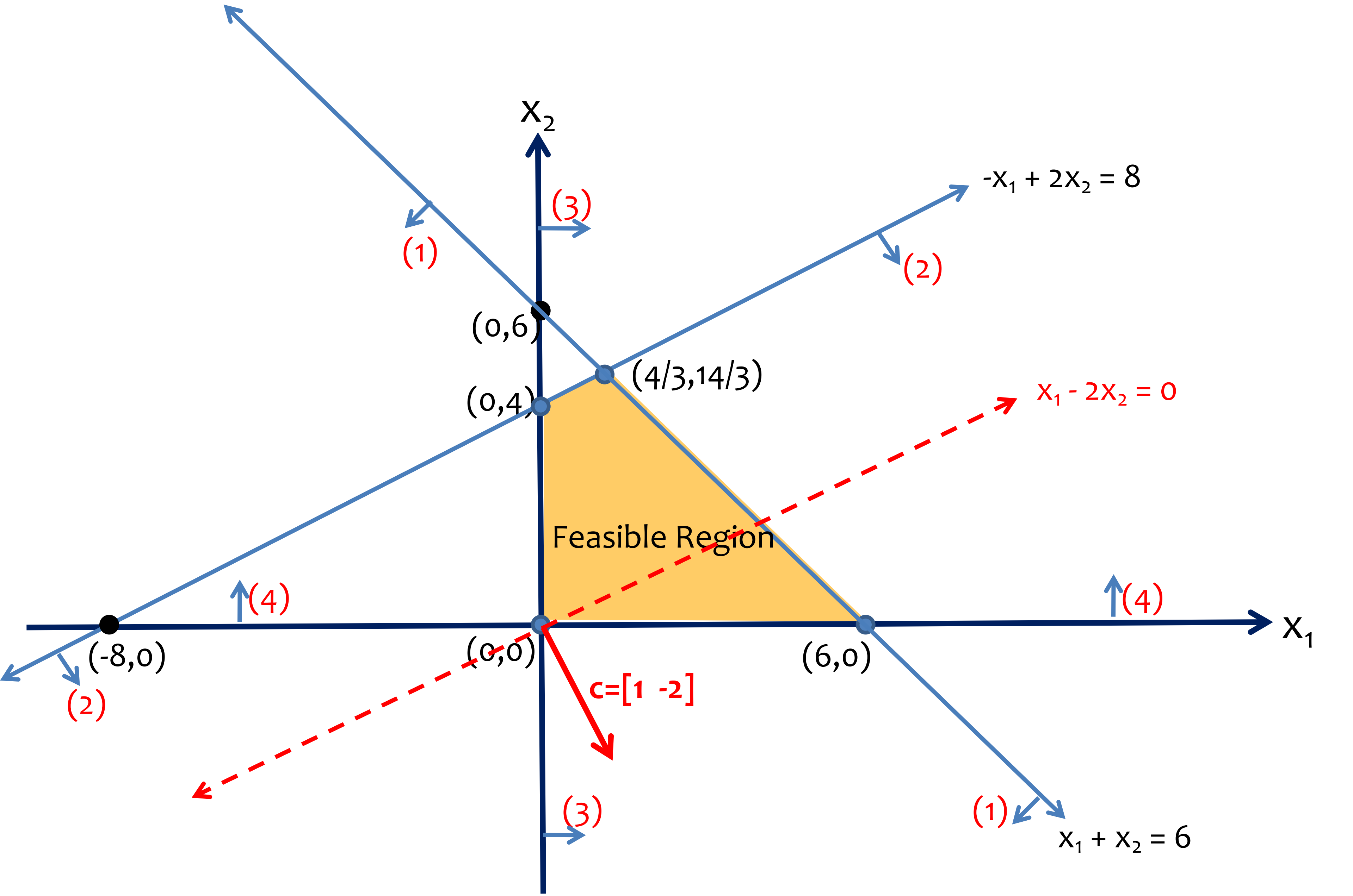
- Does feasible region change?



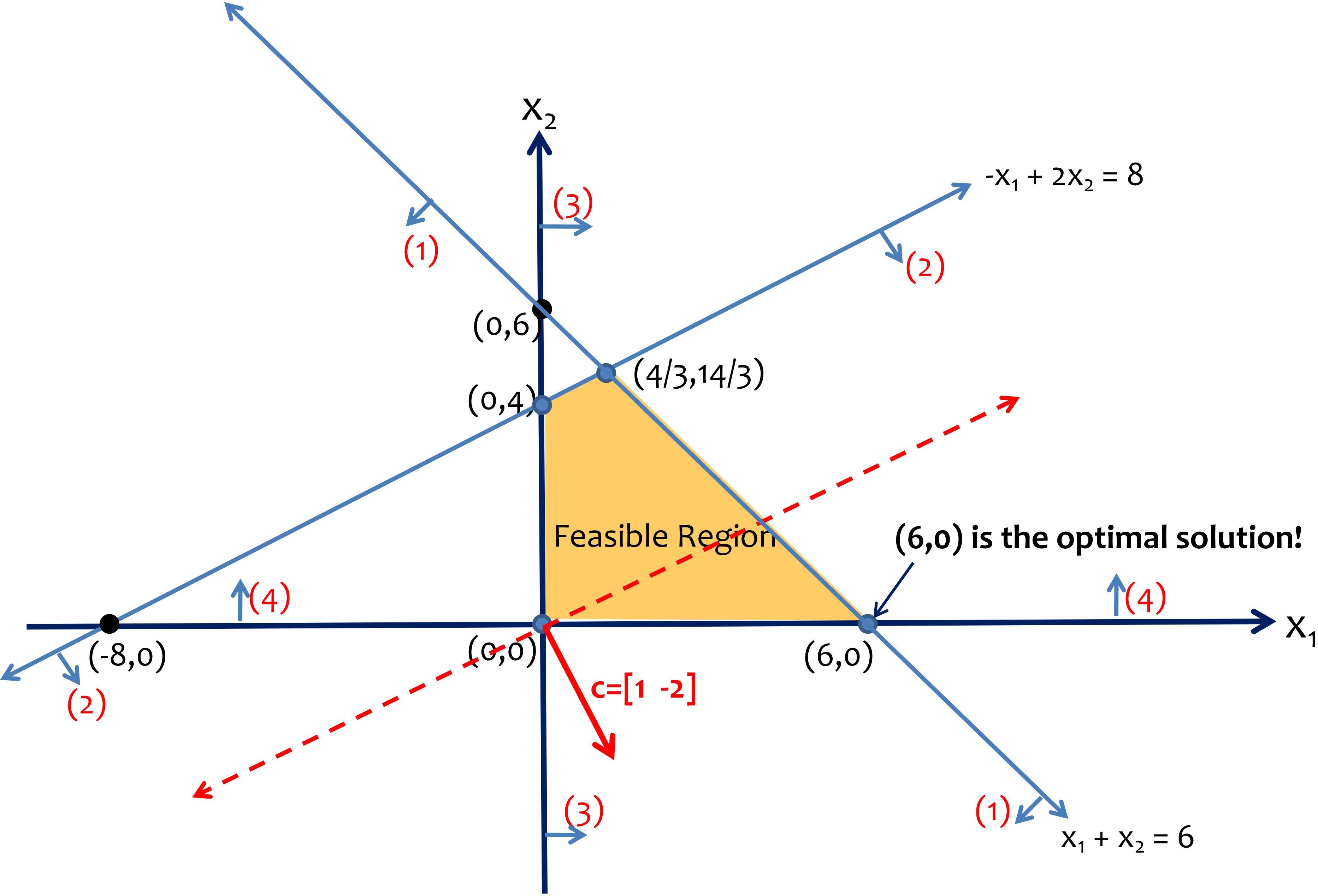
x_2
x_1
$-x_1 + 2x_2 = 8$
$x_1 + x_2 = 6$
(1)
(2)
(3)
(4)
(0,6)
(0,4)
(4/3,14/3)
(-8,0)
(0,0)
(6,0)
Feasible Region

Graphical Solution of 2-variable LP Problems

- What about the improving direction?



x_2
x_1
$-x_1 + 2x_2 = 8$
$x_1 + x_2 = 6$
$x_1 - 2x_2 = 0$
(1)
(2)
(3)
(4)
(4)
(2)
(3)
(1)
(0,6)
(4/3,14/3)
(0,4)
Feasible Region
(-8,0)
(0,0)
(6,0)
c=[1 -2]



x_2
$-x_1 + 2x_2 = 8$
(3)
(1)
(2)
(0,6)
(4/3,14/3)
(0,4)
Feasible Region
(6,0) is the optimal solution!
(4)
(4)
x_1
(-8,0)
(0,0)
(6,0)
(2)
c=[1 -2]
(3)
(1)
$x_1 + x_2 = 6$

Graphical Solution of 2-variable LP Problems

- Now, let us consider:

Example 1.c.)

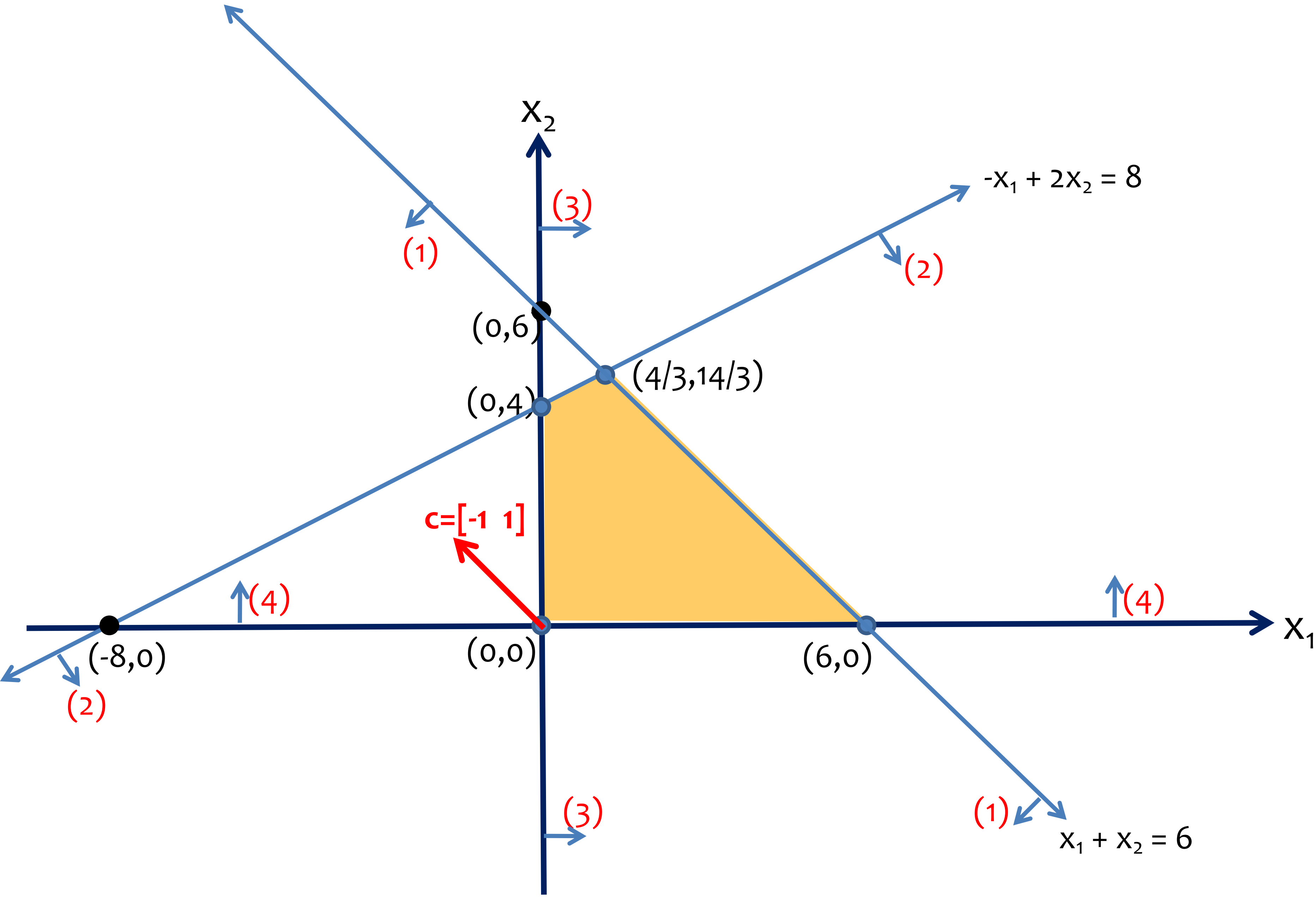
$$\max \quad -x_1 + x_2$$

s.t.

$$x_1 + x_2 \leq 6 \quad (1)$$

$$-x_1 + 2x_2 \leq 8 \quad (2)$$

$$x_1, x_2 \geq 0 \quad (3), (4)$$



x_2
$-x_1 + 2x_2 = 8$
(3)
(1)
(2)
(0,6)
(4/3,14/3)
(0,4)
c=[-1 1]
(4)
(4)
x_1
(-8,0)
(0,0)
(6,0)
(2)
(3)
(1)
$x_1 + x_2 = 6$

Graphical Solution of 2-variable LP Problems

- Now, let us consider:

Example 1.c.)

$$\max \quad -x_1 + x_2$$

s.t.

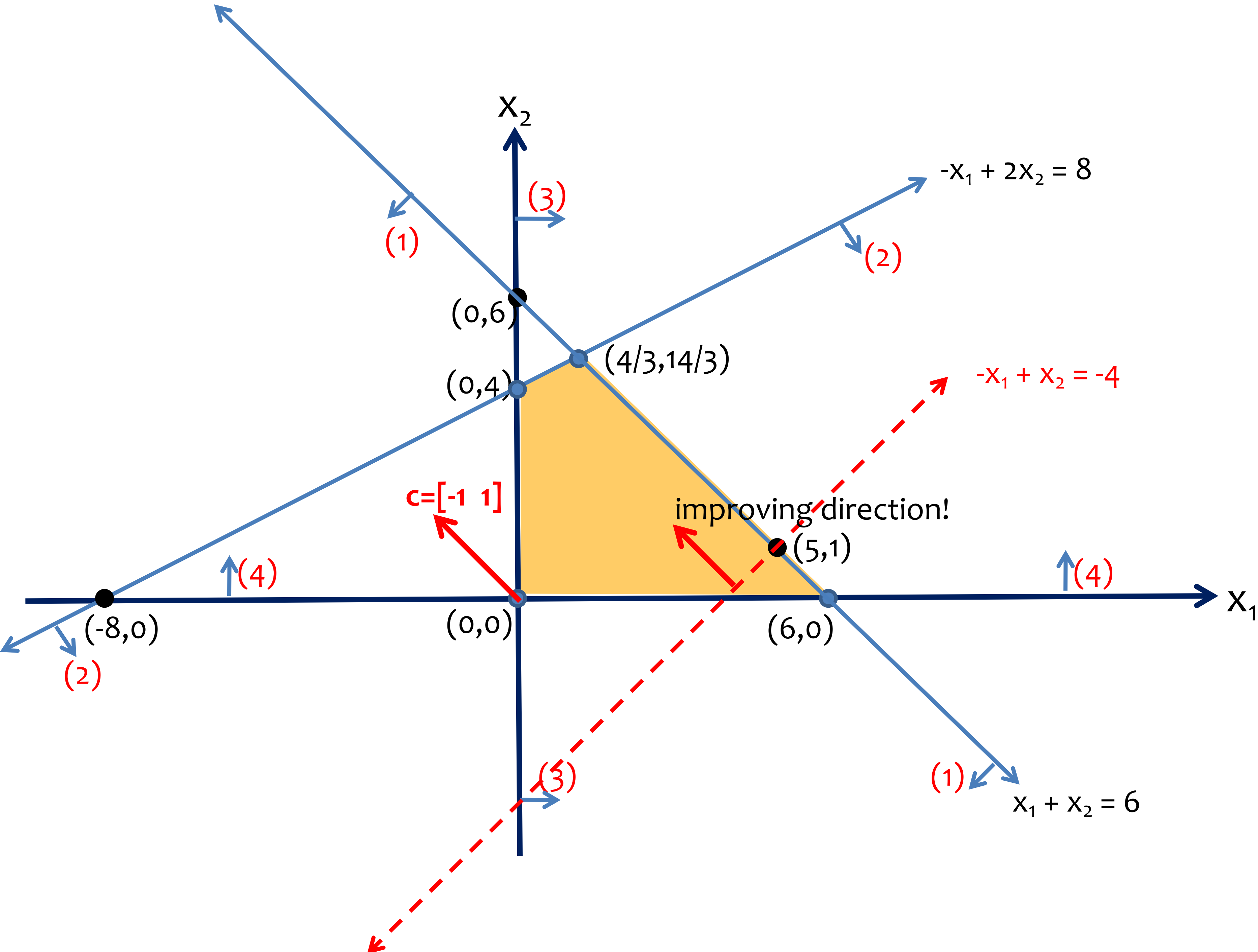
$$x_1 + x_2 \leq 6 \quad (1)$$

$$-x_1 + 2x_2 \leq 8 \quad (2)$$

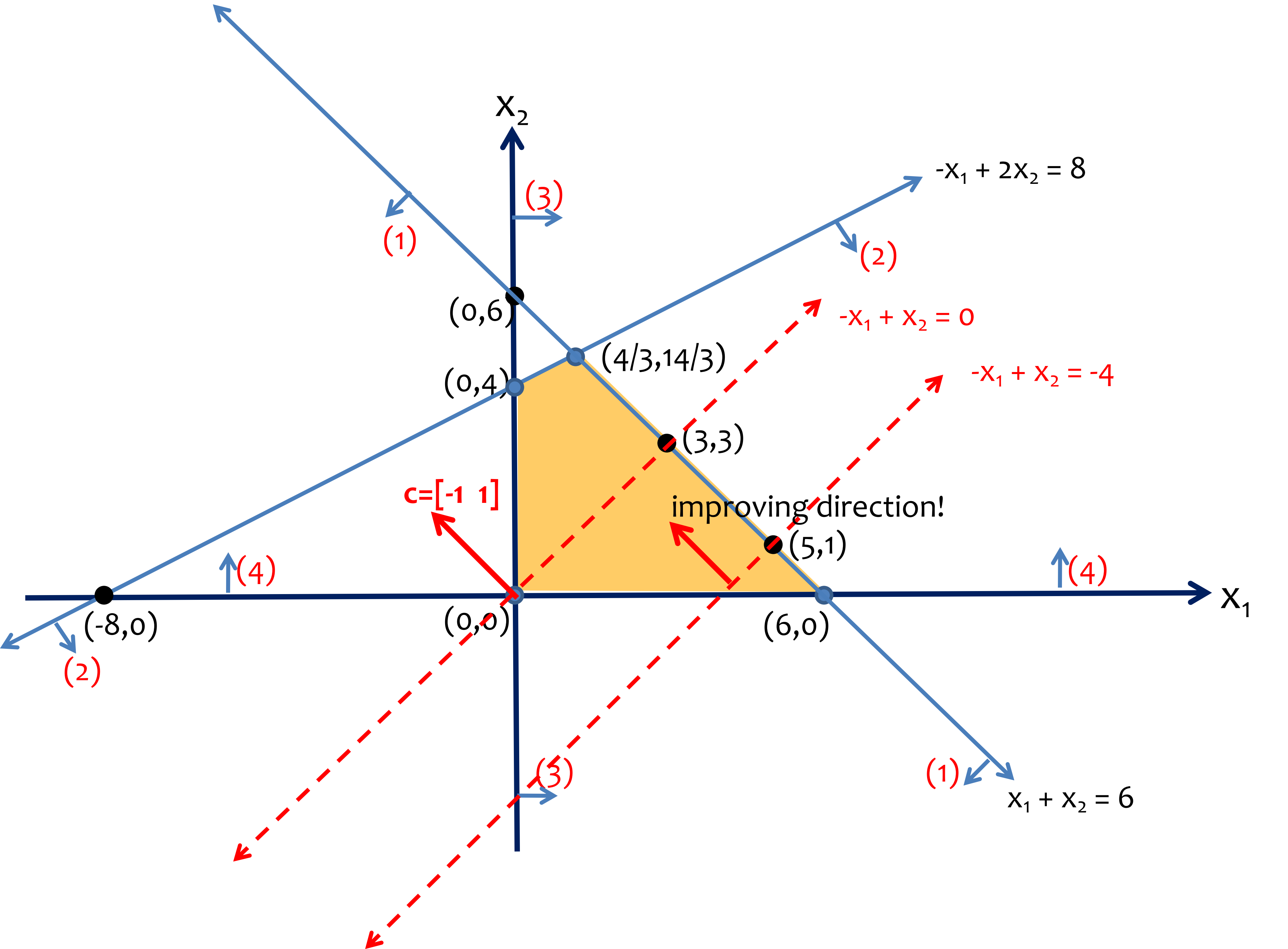
$$x_1, x_2 \geq 0 \quad (3), (4)$$

Question:

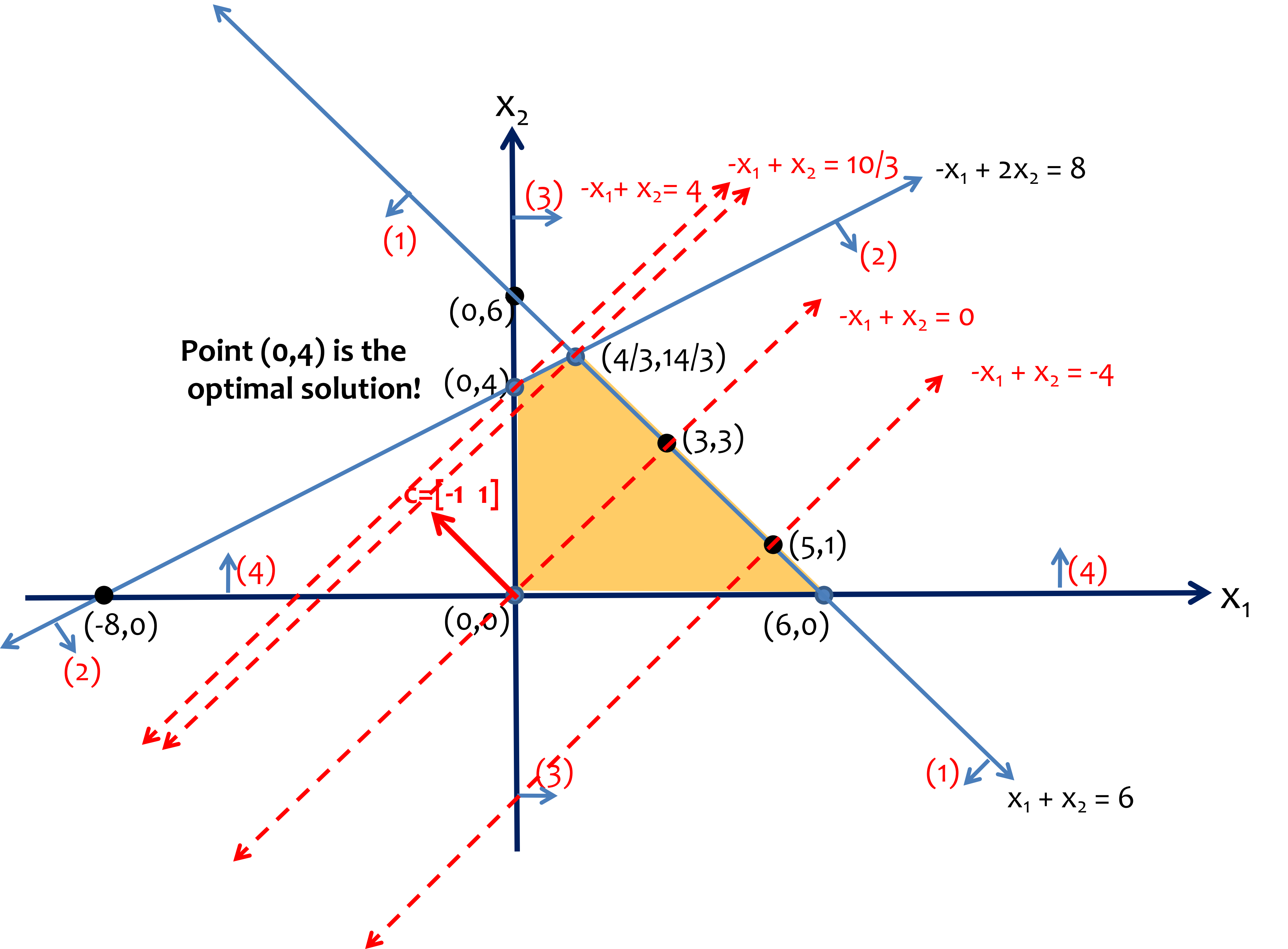
Is it possible for (5,1) be the optimal?



x_2
$-x_1 + 2x_2 = 8$
(3)
(1)
(2)
(0,6)
(4/3,14/3)
(0,4)
$-x_1 + x_2 = -4$
c=[-1 1]
improving direction!
(5,1)
(4)
(4)
x_1
(-8,0)
(0,0)
(6,0)
(2)
(3)
(1)
$x_1 + x_2 = 6$



x_2
x_1
$-x_1 + 2x_2 = 8$
$-x_1 + x_2 = 0$
$-x_1 + x_2 = -4$
$x_1 + x_2 = 6$
(1)
(2)
(3)
(4)
(0,6)
(0,4)
(4/3,14/3)
(3,3)
(5,1)
(-8,0)
(0,0)
(6,0)
c=[-1 1]
improving direction!



x2
x1
-x1 + x2 = 10/3
-x1 + 2x2 = 8
-x1+ x2= 4
-x1 + x2 = 0
-x1 + x2 = -4
x1 + x2 = 6
(1)
(2)
(3)
(4)
(0,6)
(0,4)
(4/3,14/3)
(3,3)
(5,1)
(-8,0)
(0,0)
(6,0)
c=[-1 1]
Point (0,4) is the optimal solution!

Graphical Solution of 2-variable LP Problems

- Now, let us consider:

Example 1.d.)

$$\max \quad -x_1 - x_2$$

s.t.

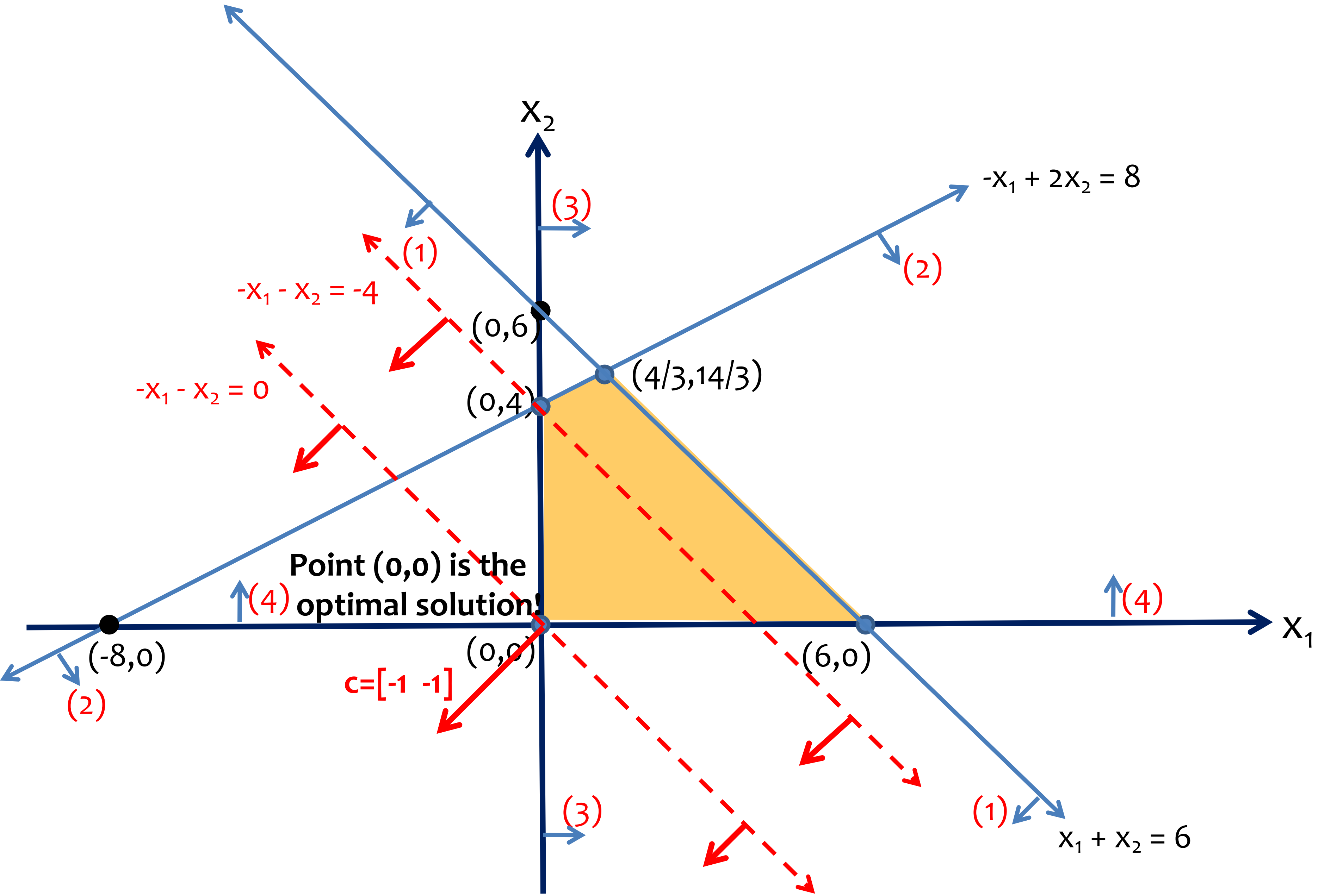
$$x_1 + x_2 \leq 6 \quad (1)$$

$$-x_1 + 2x_2 \leq 8 \quad (2)$$

$$x_1, x_2 \geq 0 \quad (3), (4)$$

Question:

Is it possible for (0,4) be the optimal?



x_2
x_1
$-x_1 + 2x_2 = 8$
$x_1 + x_2 = 6$
$-x_1 - x_2 = -4$
$-x_1 - x_2 = 0$
(1)
(2)
(3)
(4)
(0,6)
(4/3,14/3)
(0,4)
(-8,0)
(0,0)
(6,0)
c=[-1 -1]
Point (0,0) is the optimal solution!

Graphical Solution of 2-variable LP Problems

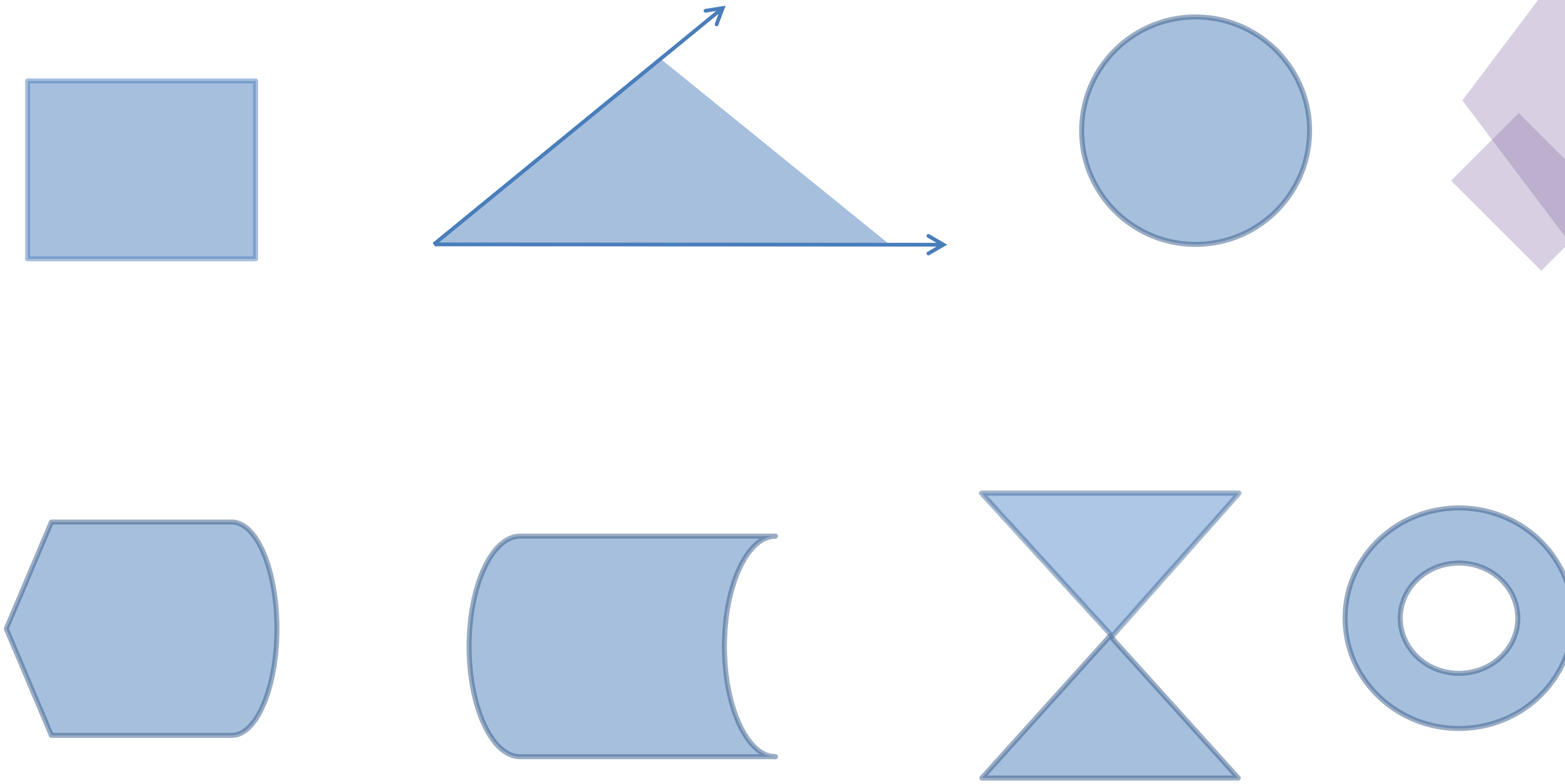
- Have you noticed anything about the optimal points?

Convex Set

- A set of points S is a convex set if for any two points $x \in S$ and $y \in S$, their convex combination $\lambda x+(1-\lambda)y$ is also in S for all $\lambda \in [0,1]$.

- In other words, a set is a convex set if the line segment joining any pair of points in S is wholly contained in S.

Are these convex set?



Are these convex set?

- Feasible set of an LP $x=\{x \in R^n: Ax \leq b,\ x \geq 0\}$ is a convex set.

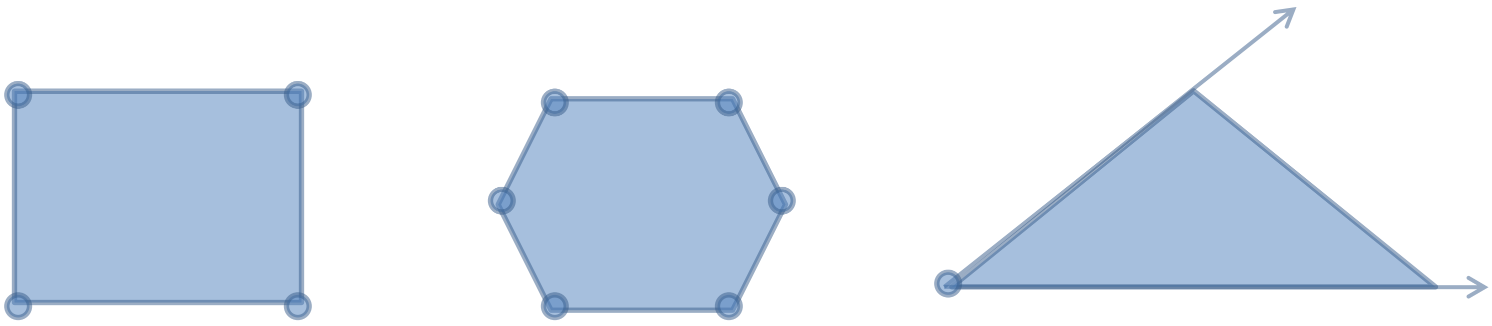
Extreme Points (Corner Points)

- A point P of a convex set S is an extreme point (corner point) if it cannot be represented as a ***strict convex combination*** of two distinct points of S.

- ***Strict convex combination:*** x is a strict convex combination of x_1 and x_2 if $x=\lambda x_1+(1-\lambda)x_2$ for some $\lambda \in (0,1)$.

Extreme Points (Corner Points)

- In other words, for any convex set S, a point P in S is an extreme point (corner point) if each line segment that lies completely in S and contains the point P, has P as an endpoint of the line segment.



Graphical Solution of 2-variable LP Problems

Example 2:

A company wishes to increase the demand for its product through advertising.

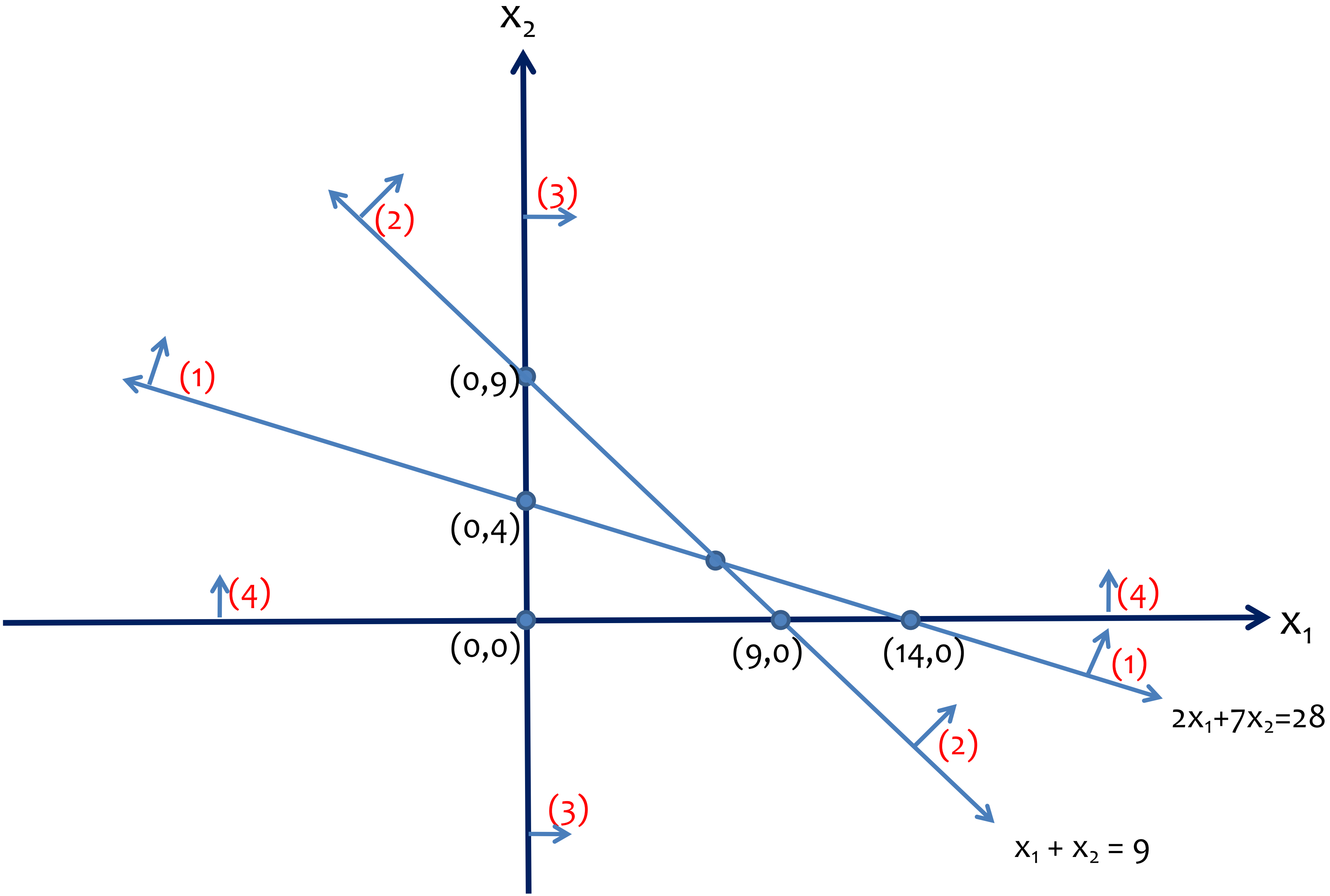
- Each minute of radio ad costs \$1 and each minute of TV ad costs \$2.
- Each minute of radio ad increases the daily demand by 2 units and each minute of TV ad by 7 units.
- The company would wish to place at least 9 minutes of daily ad in total. It wishes to increase daily demand by at least 28 units.

How can the company meet its advertising requirements at minimum total cost?

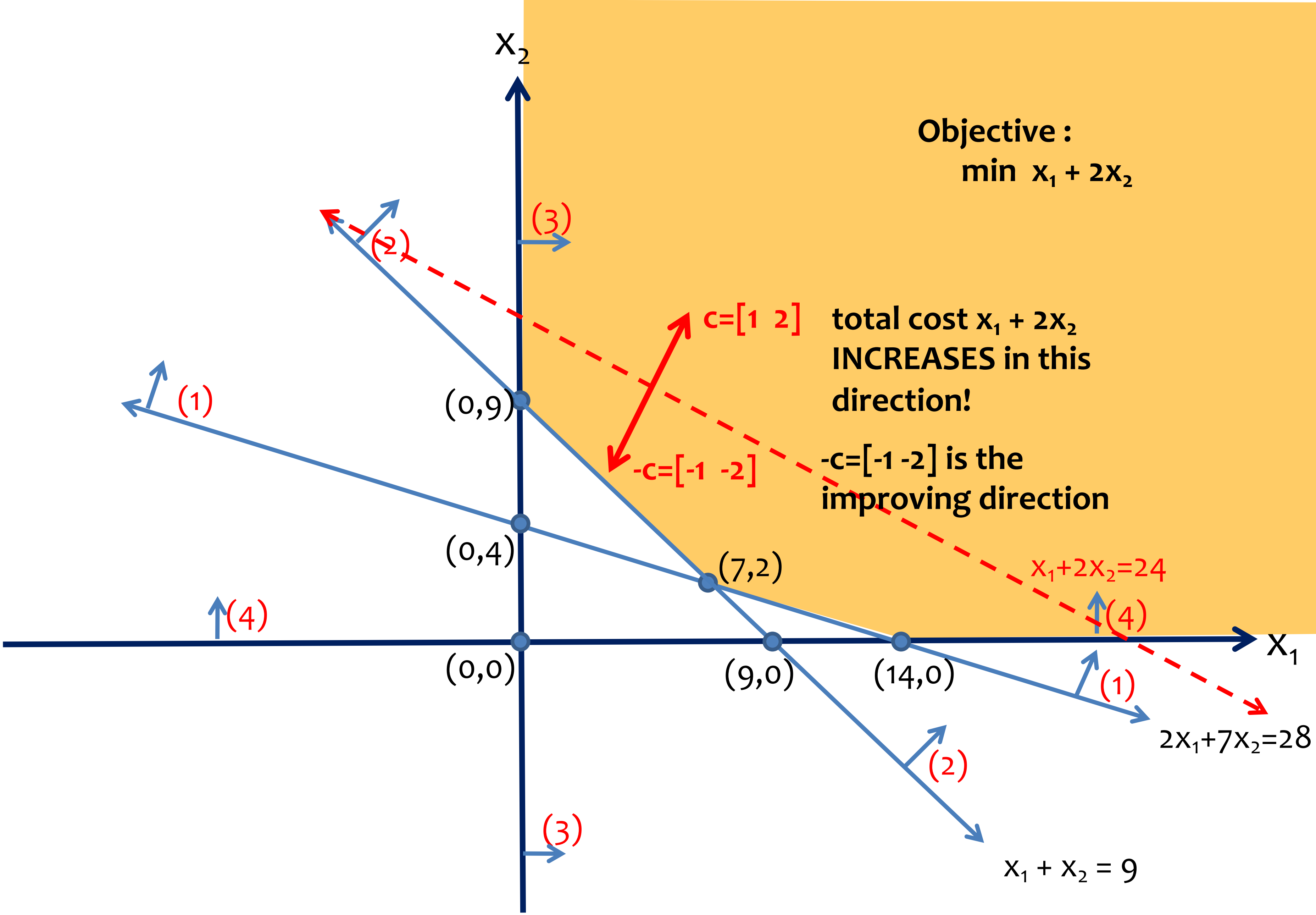
Example 2

- x_1: # of minutes of radio ads purchased
- x_2: # of minutes of TV ads purchased

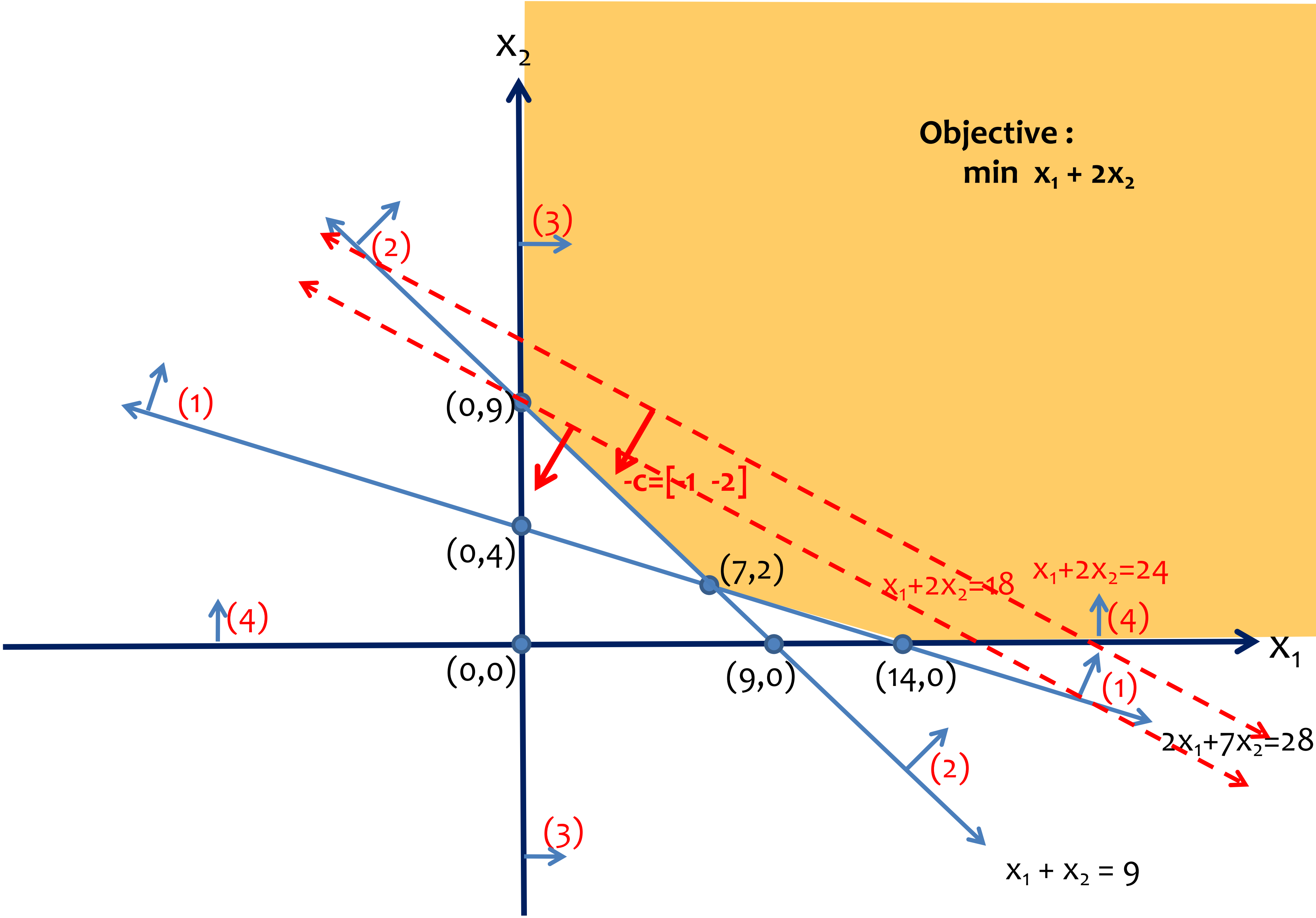
$$
\begin{array}{lrl}
\min & x_1 + 2x_2 & \\
\text{s.t.} & & \\
& 2x_1 + 7x_2 \geq 28 & (1) \\
& x_1 + x_2 \geq 9 & (2) \\
& x_1, x_2 \geq 0 & (3), (4)
\end{array}
$$



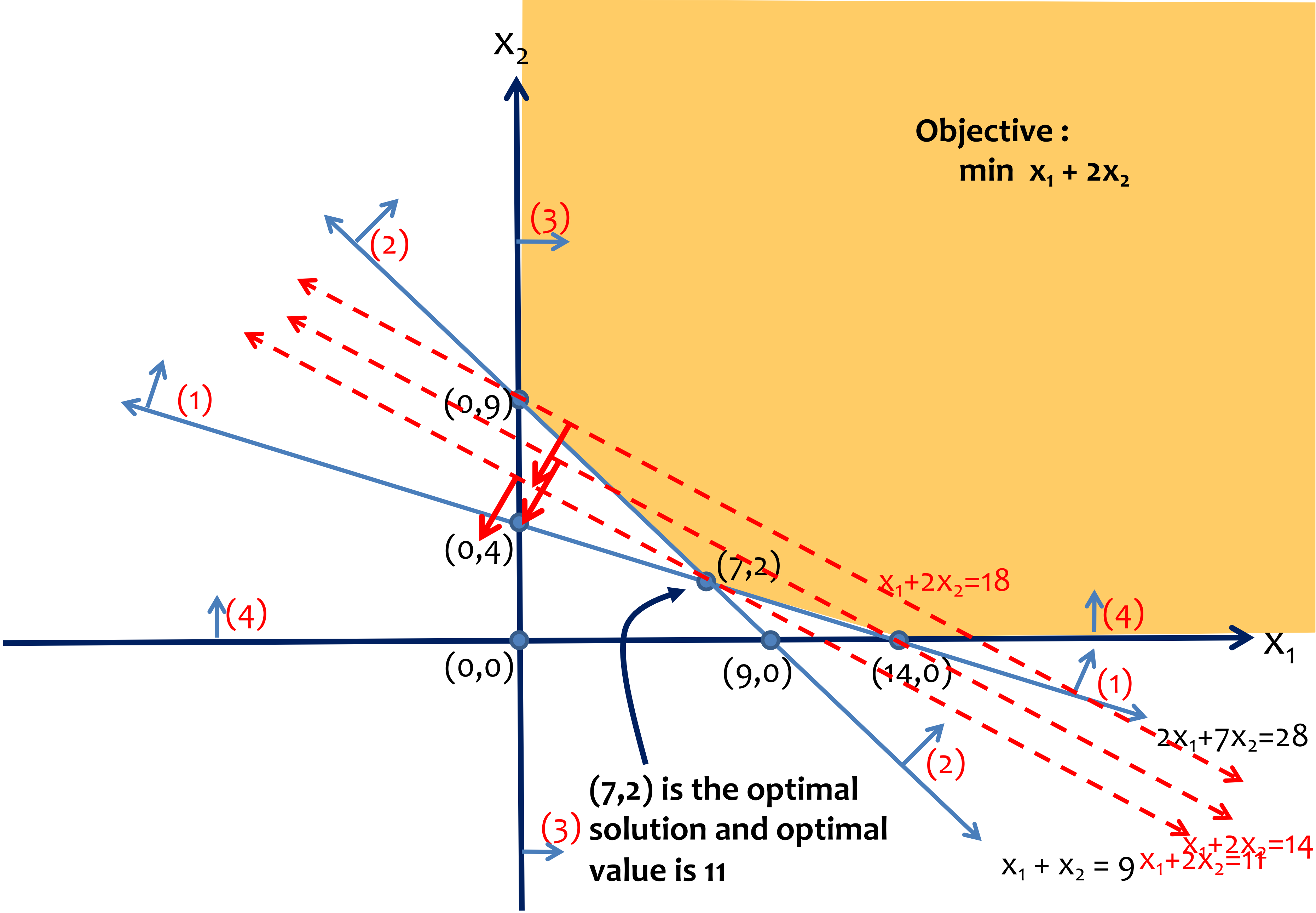
x2
x1
(0,9)
(0,4)
(0,0)
(9,0)
(14,0)
(1)
(2)
(3)
(4)
2x1+7x2=28
x1 + x2 = 9



x_2
x_1
Objective :
min $x_1 + 2x_2$
(3)
(2)
(1)
(4)
c=[1 2]
-c=[-1 -2]
total cost $x_1 + 2x_2$
INCREASES in this
direction!
-c=[-1 -2] is the
improving direction
(0,9)
(0,4)
(7,2)
(0,0)
(9,0)
(14,0)
$x_1+2x_2=24$
(4)
(1)
$2x_1+7x_2=28$
(2)
(3)
$x_1 + x_2 = 9$



x2
Objective :
min x1 + 2x2
(3)
(2)
(1)
(0,9)
-c=[-1 -2]
(0,4)
(7,2)
x1+2x2=18
x1+2x2=24
(4)
(4)
(0,0)
(9,0)
(14,0)
x1
(1)
2x1+7x2=28
(2)
(3)
x1 + x2 = 9



Objective :
min $x_1 + 2x_2$
x_2
x_1
(1)
(2)
(3)
(4)
(0,9)
(0,4)
(0,0)
(7,2)
(9,0)
(14,0)
$x_1+2x_2=18$
$2x_1+7x_2=28$
$x_1 + x_2 = 9$
$x_1+2x_2=14$
$x_1+2x_2=11$
(7,2) is the optimal solution and optimal value is 11

Graphical Solution of 2-variable LP Problems

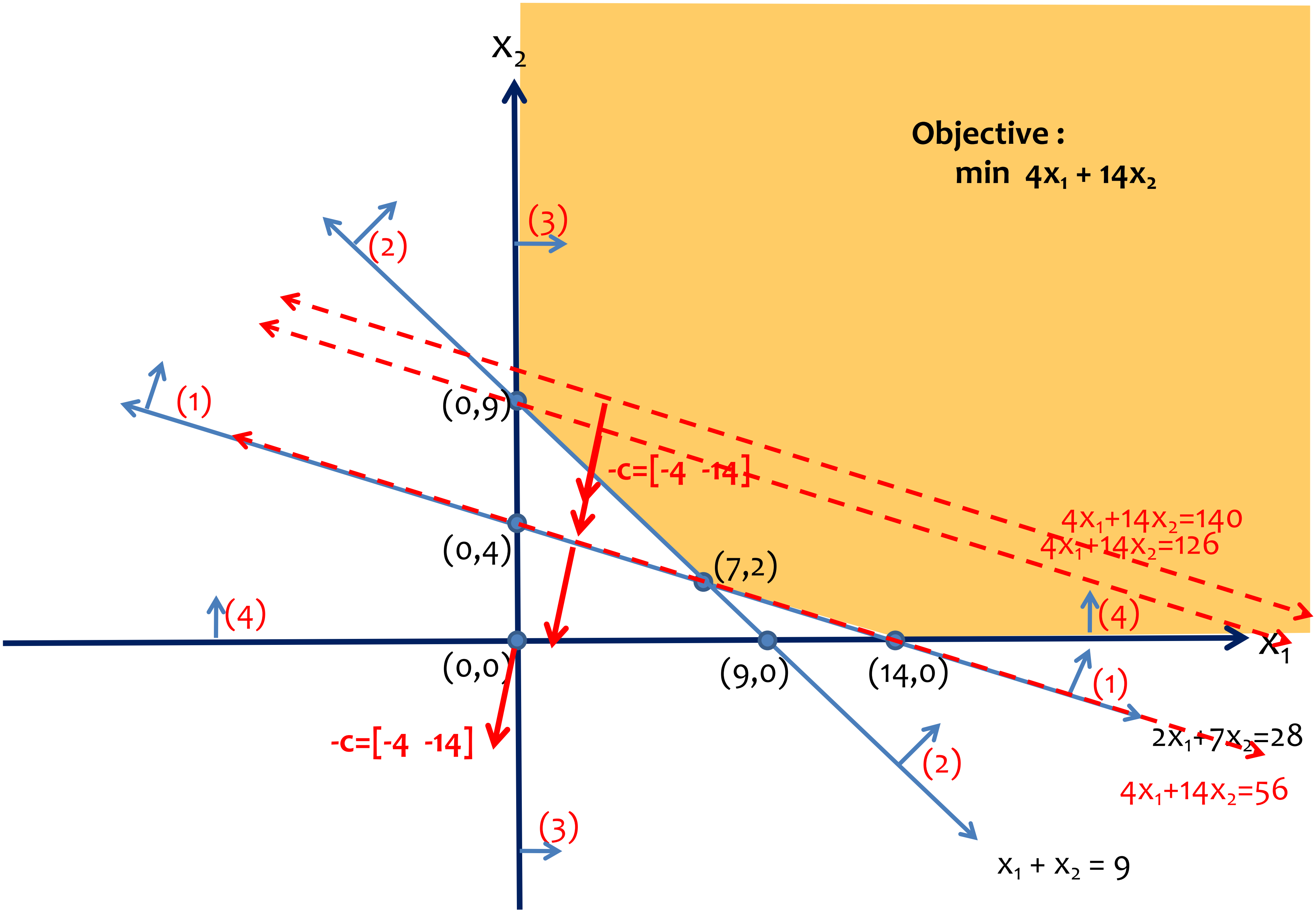
Example 3:

Suppose that in the previous example, it costed \$4 to place a minute of radio ad and \$14 to place a minute of TV ad

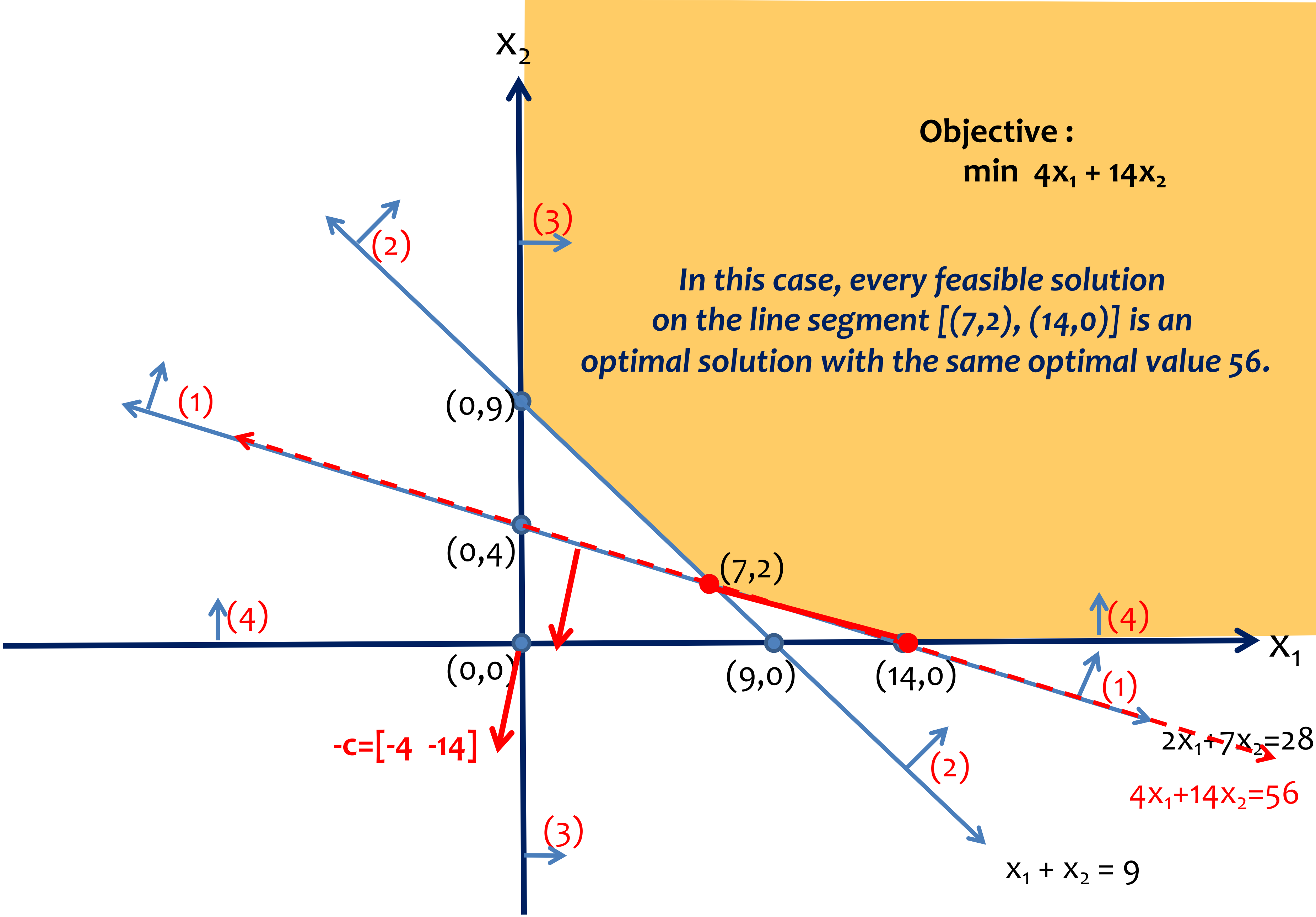
How can the company meet its advertising requirements at minimum total cost?

Example 3

$$
\begin{aligned}
\min \quad & 4x_1 + 14x_2 \\
\text{s.t.} \quad & \\
& 2x_1 + 7x_2 \geq 28 \quad (1) \\
& x_1 + x_2 \geq 9 \quad (2) \\
& x_1, x_2 \geq 0 \quad (3), (4)
\end{aligned}
$$



Objective :
min $4x_1 + 14x_2$
x_2
x_1
(1)
(2)
(3)
(4)
(0,9)
(0,4)
(0,0)
(7,2)
(9,0)
(14,0)
-c=[-4 -14]
-c=[-4 -14]
$4x_1+14x_2=140$
$4x_1+14x_2=126$
$2x_1+7x_2=28$
$4x_1+14x_2=56$
$x_1 + x_2 = 9$



Objective :
min 4x₁ + 14x₂
In this case, every feasible solution
on the line segment [(7,2), (14,0)] is an
optimal solution with the same optimal value 56.
x2
x1
(0,9)
(0,4)
(0,0)
(7,2)
(9,0)
(14,0)
(1)
(2)
(3)
(4)
-c=[-4 -14]
2x1+7x2=28
4x1+14x2=56
x1 + x2 = 9

Graphical Solution of 2-variable LP Problems

At point A= $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$, the objective function value is: [4 14] $\begin{bmatrix} 7 \\ 2 \end{bmatrix} = 56$,

At point B= $\begin{bmatrix} 14 \\ 0 \end{bmatrix}$, the objective function value is: [4 14] $\begin{bmatrix} 14 \\ 0 \end{bmatrix} = 56$.

This is also true for any feasible point on the line segment [A,B]. We say that LP has **multiple** or **alternate optimal** solutions.

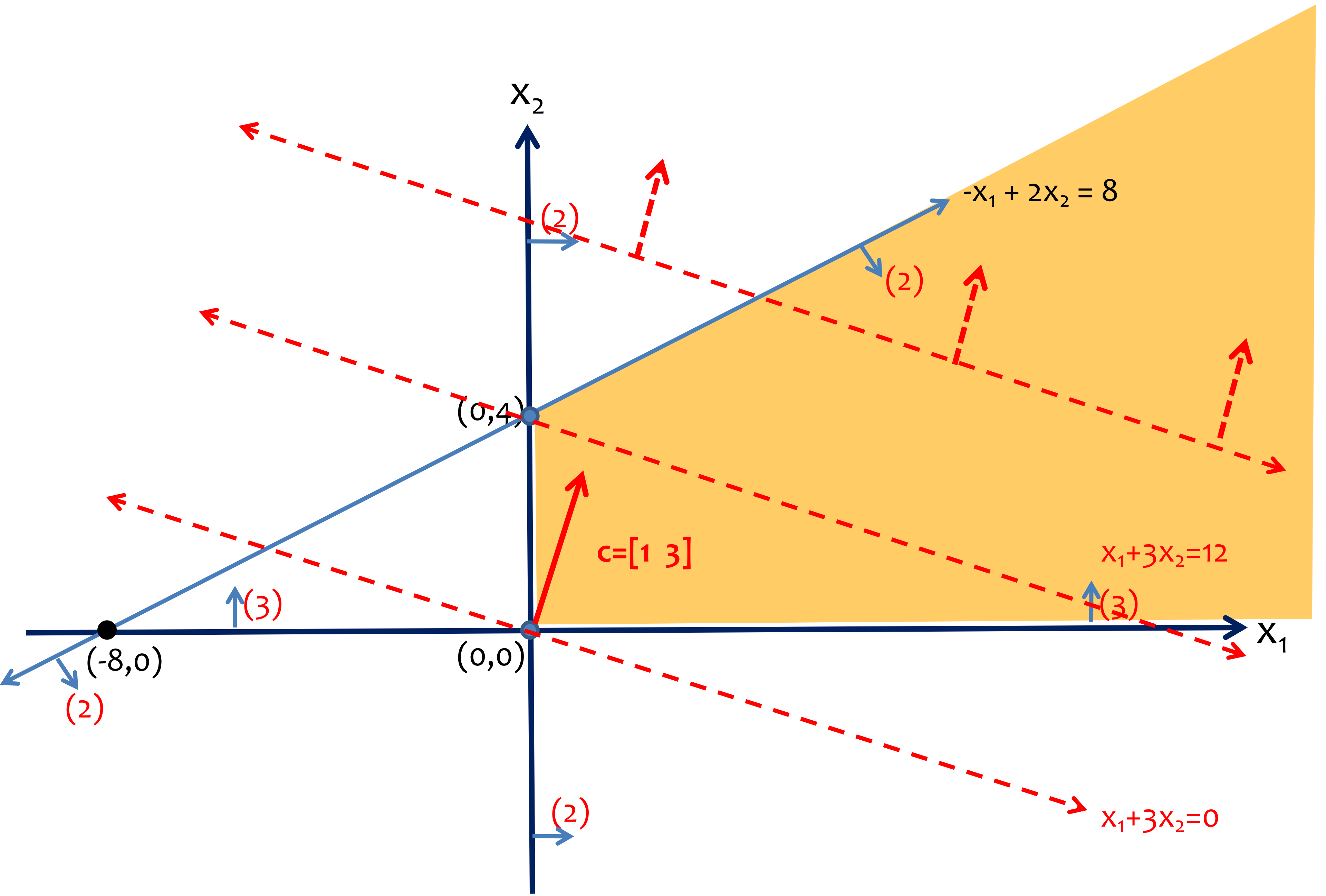
$$\text{Line Segment } [A,B] = \left\{ \lambda \begin{bmatrix} 7 \\ 2 \end{bmatrix} + (1-\lambda) \begin{bmatrix} 14 \\ 0 \end{bmatrix} : \lambda \in [0,1] \right\}$$

Graphical Solution of 2-variable LP Problems

- Consider the following LP problem:

Example 4.)

$$
\begin{aligned}
\max \quad & x_1 + 3x_2 \\
\text{s.t.} \quad & \\
& -x_1 + 2x_2 \le 8 \quad (1) \\
& x_1, x_2 \ge 0 \quad (2), (3)
\end{aligned}
$$



x_2
$-x_1 + 2x_2 = 8$
(2)
(2)
(0,4)
c=[1 3]
$x_1+3x_2=12$
(3)
(3)
x_1
(-8,0)
(0,0)
(2)
(2)
$x_1+3x_2=0$

Unbounded LP Problems:

- The objective function for this **maximization problem** can be increased by moving in the improving direction c as much as we want while still staying in the feasible region. In this case, we say that the **LP is unbounded**.

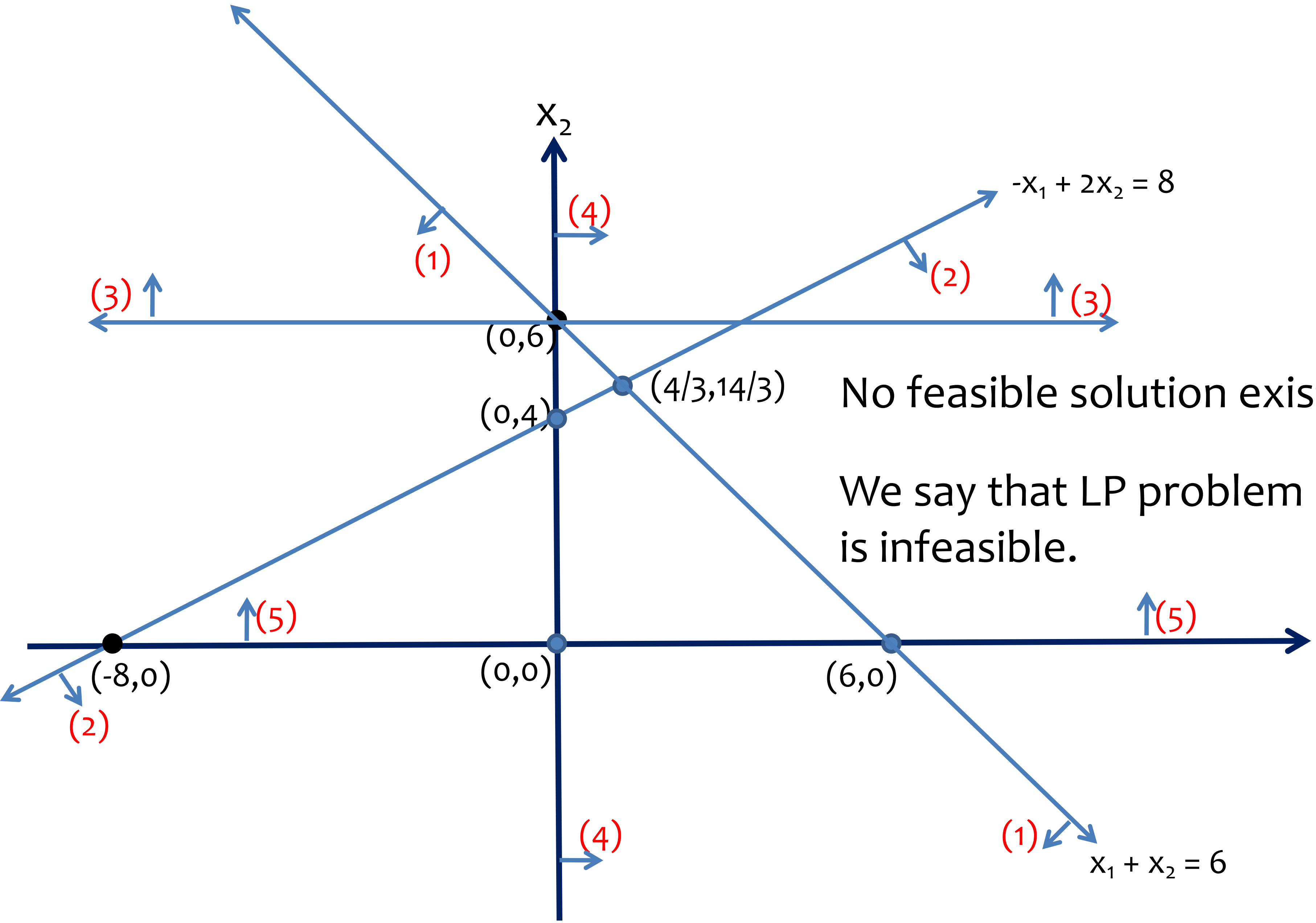
- For unbounded LP problems, there is no optimal solution and the optimal value is defined to be $+\infty$.

Graphical Solution of 2-variable LP Problems

- Consider the following LP problem:

Example 5:

$$
\begin{aligned}
\min \quad & x_1 + 3x_2 \\
\text{s.t.} \quad & \\
& x_1 + x_2 \leq 6 \quad (1) \\
& -x_1 + 2x_2 \leq 8 \quad (2) \\
& x_2 \geq 6 \quad (3) \\
& x_1, x_2 \geq 0 \quad (4), (5)
\end{aligned}
$$



No feasible solution exists!

We say that LP problem is infeasible.

Every LP Problem falls into one of the 4 cases:

- **Case 1:** The LP problem has a unique optimal solution (see Ex.1 and Ex.2)
- **Case 2:** The LP problem has alternative or multiple optimal solution (see Ex.3). In this case, there are **infinitely many optimal solutions**.
- **Case 3:** The LP problem is unbounded (see Ex.4). In this case, there is **no optimal solution**.
- **Case 4:** The LP problem is infeasible (see Ex.5). In this case, there is no feasible solution.

Extreme Points (Corner Points):

- **Theorem:** If the feasible set $\{x \in R^n: Ax \leq b,\ x \geq 0\}$ is not empty, that is if there is a feasible solution, then there is an extreme point.

- **Theorem:** If an LP has an optimal solution, then there is an extreme (or corner) point of the feasible region which is an optimal solution to the LP

- **P.S:** Not every optimal solution needs to be an extreme point! (Remember the alternate optimal solution case)

Graphical Solution of 2-variable LP Problems

- Consider the following LP problem:

Example 6:

$$
\begin{aligned}
\min \quad & 2x_1 + x_2 \\
\text{s.t.} \quad & \\
& x_1 + 2x_2 \le 10 \quad (1) \\
& x_1 + x_2 \le 6 \quad (2) \\
& x_1 - x_2 \le 2 \quad (3) \\
& x_1 - 2x_2 \le 1 \quad (4) \\
& x_1, x_2 \ge 0 \quad (5), (6)
\end{aligned}
$$

Graphical Solution of 2-variable LP Problems

Example 7:

A firm plans to purchase at least 200 quintals of scrap containing high-quality metal X and low-quality metal Y. It decides that the scrap to be purchased must contain at least 100 quintals of metal X and not more than 35 quintals of metal Y. The firm can purchase the scrap from two suppliers (A and B) in unlimited quantities. The percentage of X and Y metals in terms of weight in the scrap supplied by A and B is given below.

Metals	Supplier A	Supplier B
X	25%	75%
Y	10%	20%

The price of A's scrap is Rs 200 per quintal and that of B is Rs 400 per quintal. The firm wants to determine the quantities that it should buy from the two suppliers so that the total cost is minimized.

Graphical Solution of LP Problems:

We may graphically solve an LP with two decision variables as follows:

Step 1: Graph the feasible region.

Step 2: Draw an isoprofit line (for max problem) or an isocost line (for min problem).

Step 3: Move parallel to the isoprofit/isocost line in the improving direction. The last point in the feasible region that contacts an isoprofit/isocost line is an optimal solution to the LP.