IE 400: Principles of Engineering Management

Set 6: Solving Integer Programming Models with Branch-and-Bound

Geometry of IP

max x_2

s.t.

$-x_1 + x_2 \le 1/2$

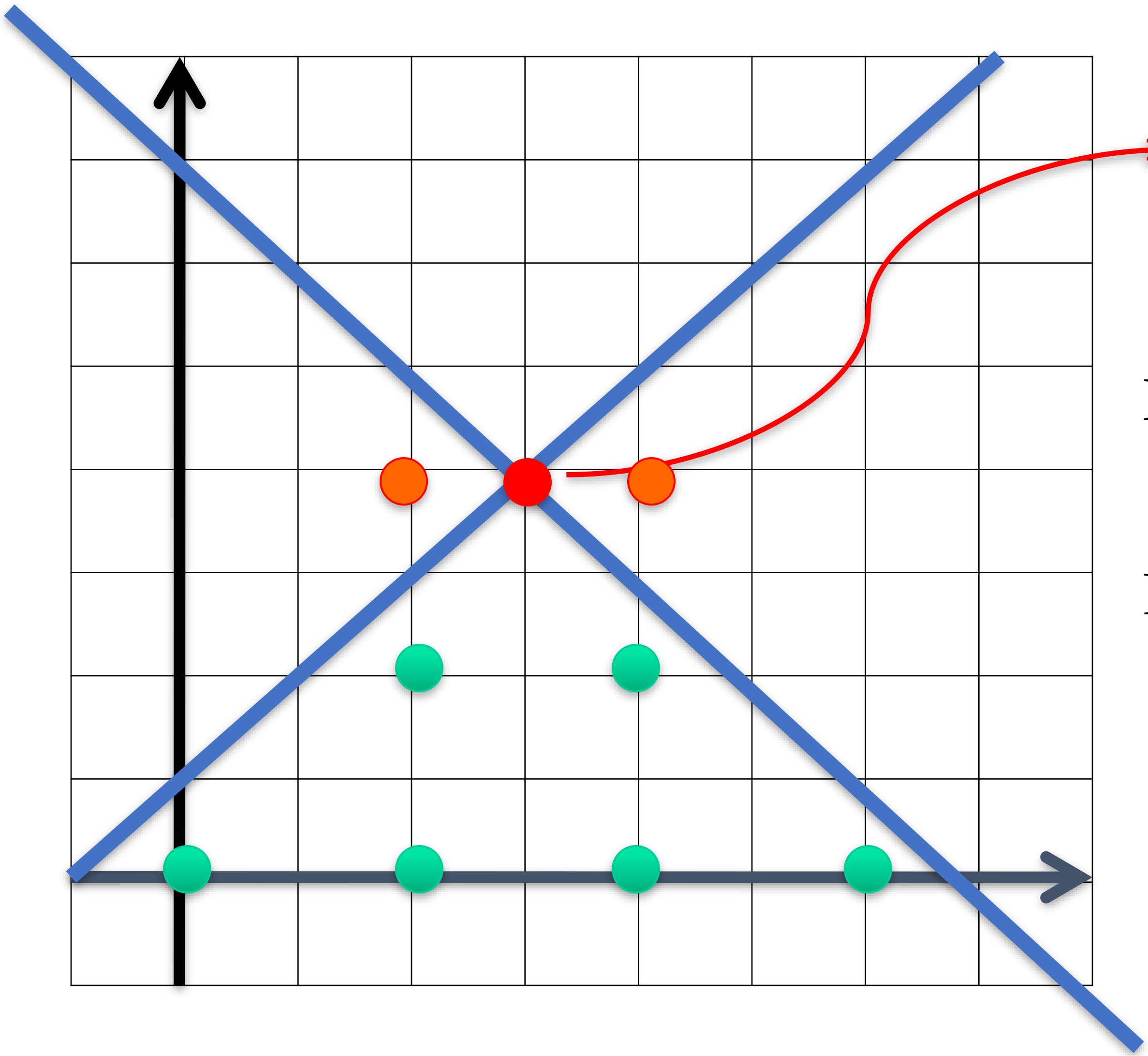
$x_1 + x_2 \le 7/2$

$x_1, x_2 \ge 0$, integer

LP Relaxation

- By removing the integrality conditions, we get an LP problem.
- This is called the LP-relaxation and is often a good way starting IP problems.
- If we solve the LP of the example:

Rounding an LP solution



$x_1 = 3/2$
$x_2 = 2$
$z = 2$

NOT INTEGER

ROUNDING

$x_1 = 1$, or 2
$x_2 = 2$
$z = 2$

Infeasible

Integer Programming

Proposition 1: For an integer programming problem, the optimal value of the LP-relaxation is at least as good as the optimal value of the IP

Proof: The LP relaxation has a larger feasible region and so more alternatives

Proposition 2: If the optimal solution of the LP relaxation is integer valued, then that solution is also optimal for the IP.

(proof of optimality)

Knapsack Problem

$$\max\ 16x_1 + 22x_2 + 12x_3 + 8x_4 + 11x_5 + 19x_6$$

s.t. $5x_1 + 7x_2 + 4x_3 + 3x_4 + 4x_5 + 6x_6 \le 14.$ (IP)

$$x_j \in \{0,1\}\ for\ j = 1, \dots, 6$$

$$\max\ 16x_1 + 22x_2 + 12x_3 + 8x_4 + 11x_5 + 19x_6$$

s.t. $5x_1 + 7x_2 + 4x_3 + 3x_4 + 4x_5 + 6x_6 \le 14.$ (LP Relaxation)

$$0 \le x_j \le 1. \quad for\ j = 1, \dots, 6$$

Solving Knapsack Problem

$$max\ 16x_1+22x_2+12x_3+8x_4+11x_5+19x_6$$

$$\text{s.t.} \quad 5x_1+7x_2+4x_3+3x_4+4x_5+6x_6 \leq 14. \qquad \text{(LP Relaxation)}$$

$$0 \leq x_j \leq 1. \quad for\ j = 1, \ldots, 6$$

Order the variables in non-increasing of per unit size utility values and fill the knapsack with respect to this order!

$$x_1 > x_6 > x_2 > x_3 > x_5 > x_4$$

$$\frac{16}{5} \geq \frac{19}{6} \geq \frac{22}{7} \geq \frac{12}{4} \geq \frac{11}{4} \geq \frac{8}{3}$$

LP Optimal Solution?

Solving IPs – Complete Enumeration

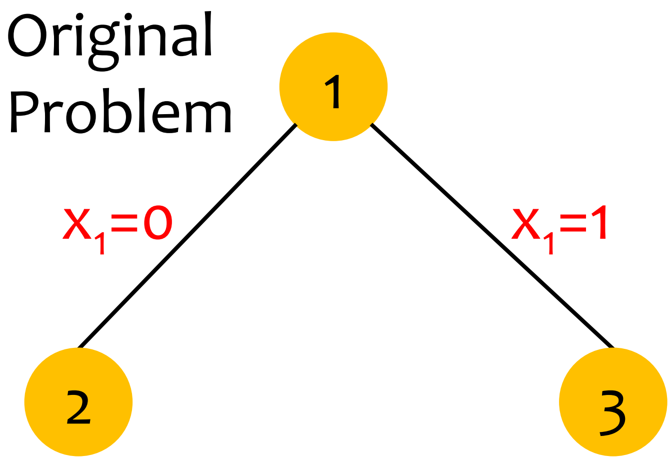
- Systematically consider all possible values of decision variables
  - If there are n binary variables, then there are 2^n different ways

- **Usual Idea:** Iteratively break the problem in two. At the first iteration, we consider separately the case that $x_1=0$ and $x_1=1$.

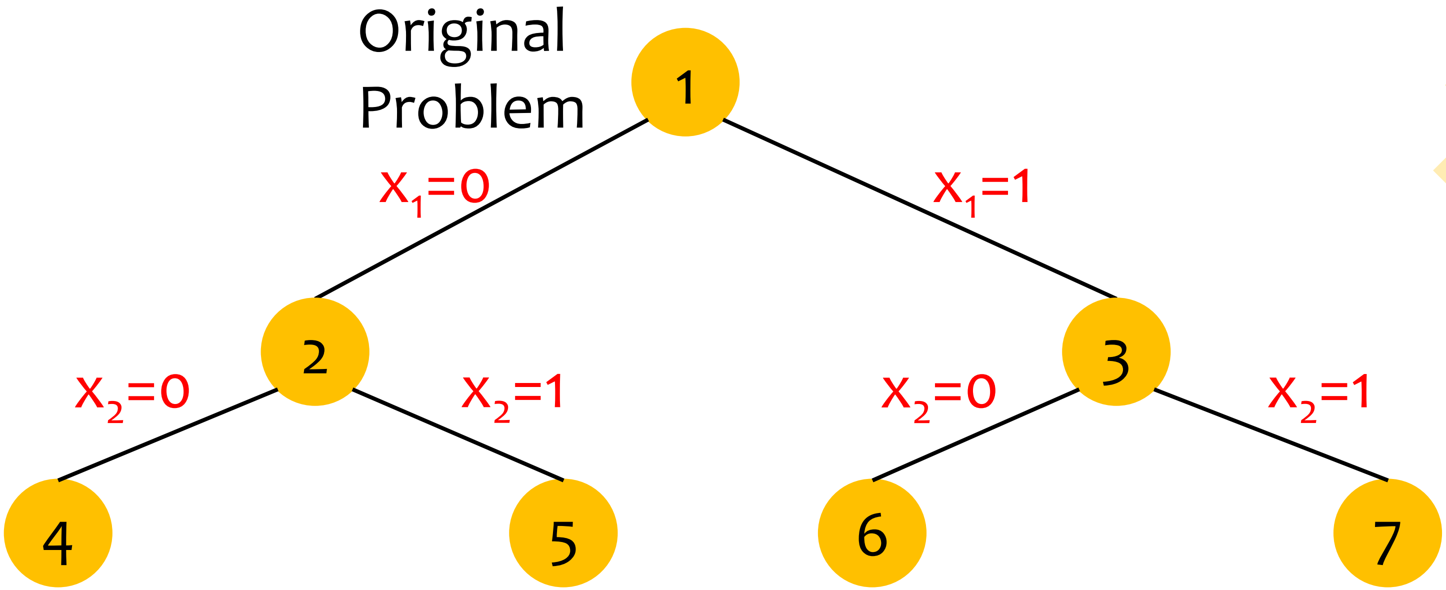
An Enumeration Tree

Original Problem

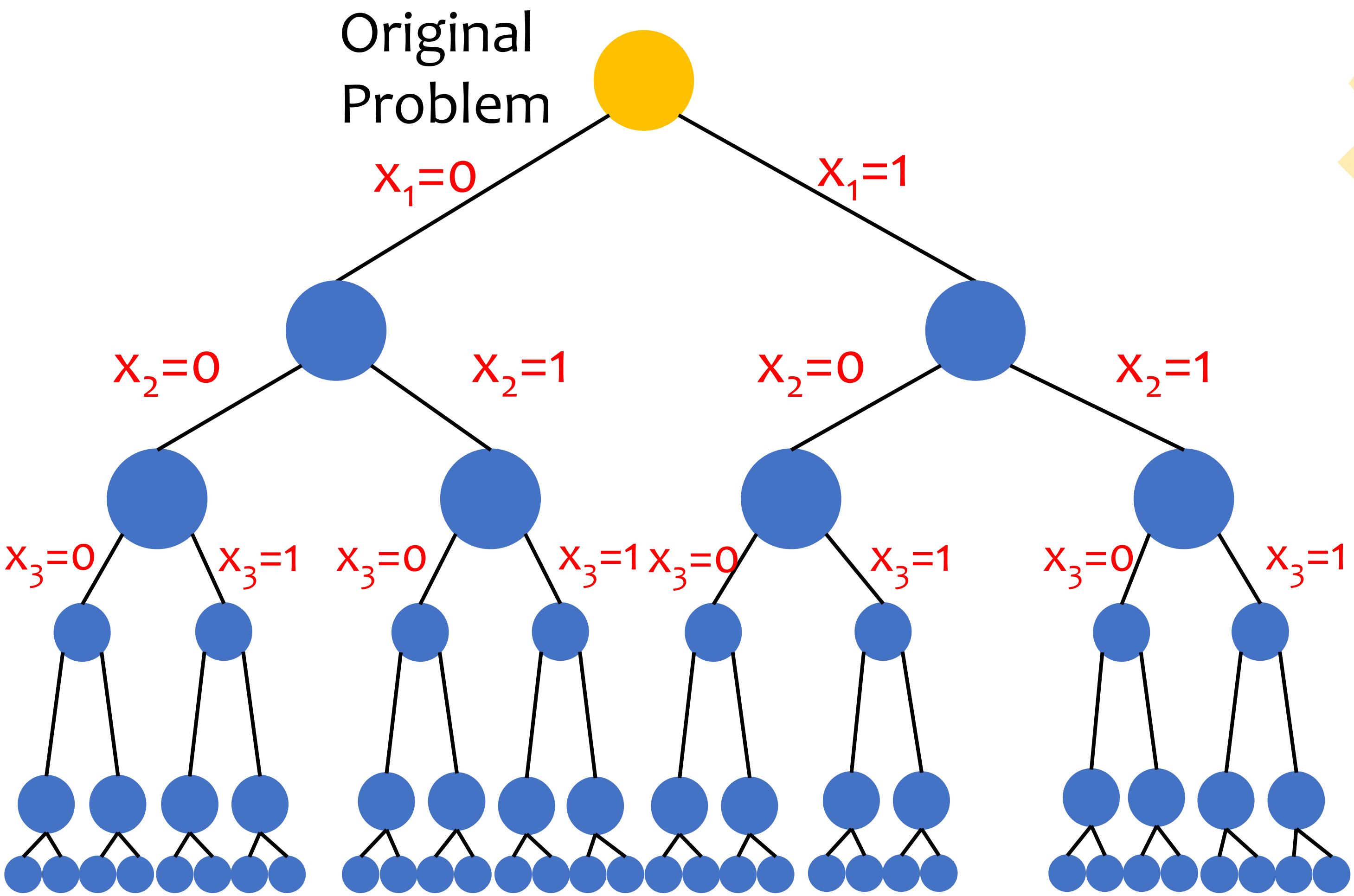
An Enumeration Tree



An Enumeration Tree



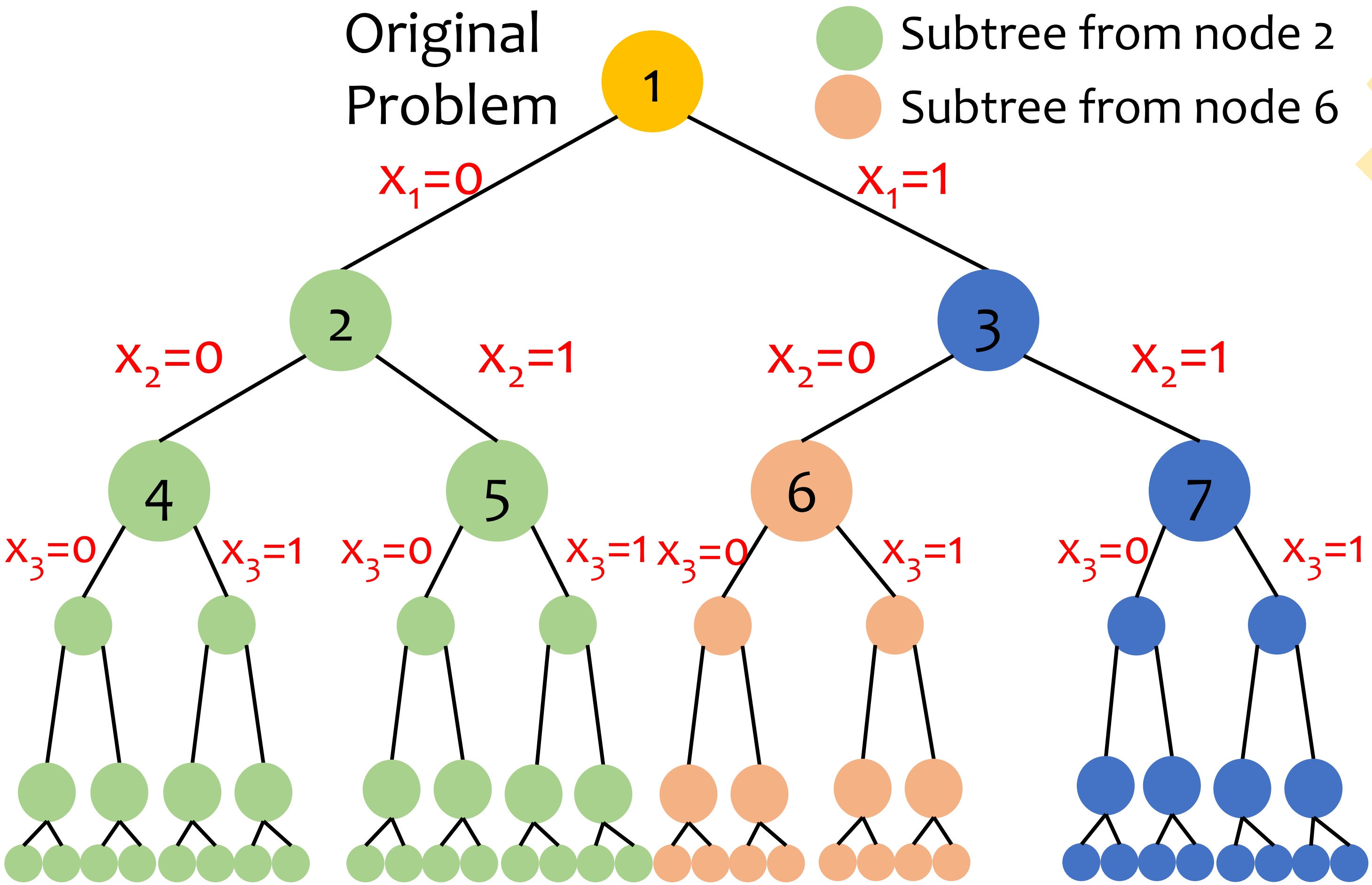
An Enumeration Tree



On Complete Enumeration

- Suppose that we could evaluate 1 trillion solutions per second and instantaneously eliminate 99.999999% of all solutions not worth considering
- Let n=number of binary variables
- Solution times
  - n=70 → 1 second
  - n=80 → 17 minutes
  - n=90 → 11.6 days
  - n=100 → 31 years
  - n=110 → 31,000 years

Solving IPs (Branch-and-Bound)



The bottom nodes are called leaves of the tree

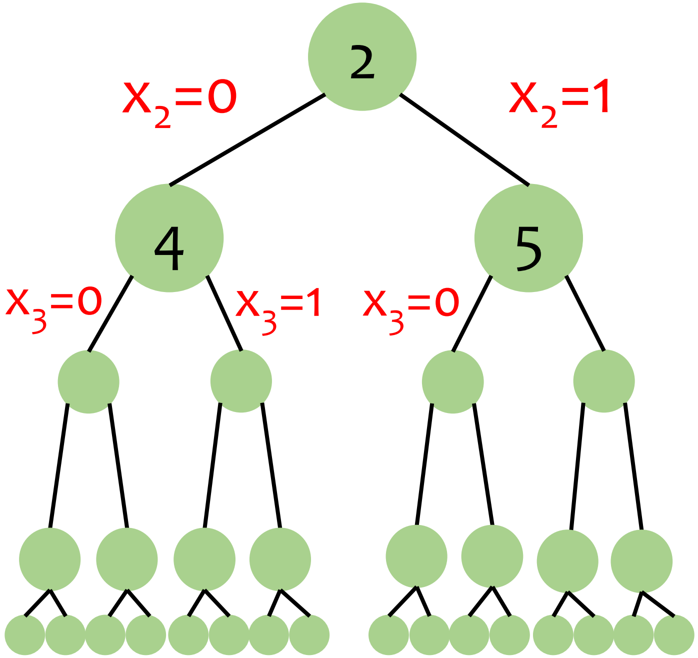
Something needed for Branch-and-Bound: **The Incumbent**

- We need a feasible solution to the integer program. We call this the incumbent.
- Suppose that $\mathbf{x}^{(I)}$ is the incumbent. Let z^I be its objective value.
- **Important question:** How does one find an incumbent?
  - We will deal with that later. Let us just assume that we have one.
- Starting incumbent (which I found by inspection)

  $x_1=1,\ x_2=1,\ x_3 = x_4 = x_5 = x_6 = 0,\ z^I=38$

The Essence of Branch-and-Bound

- Select nodes of the "enumeration tree" one at a time but do not branch from a node if none of its descendents can be a better solution than that of the incumbent



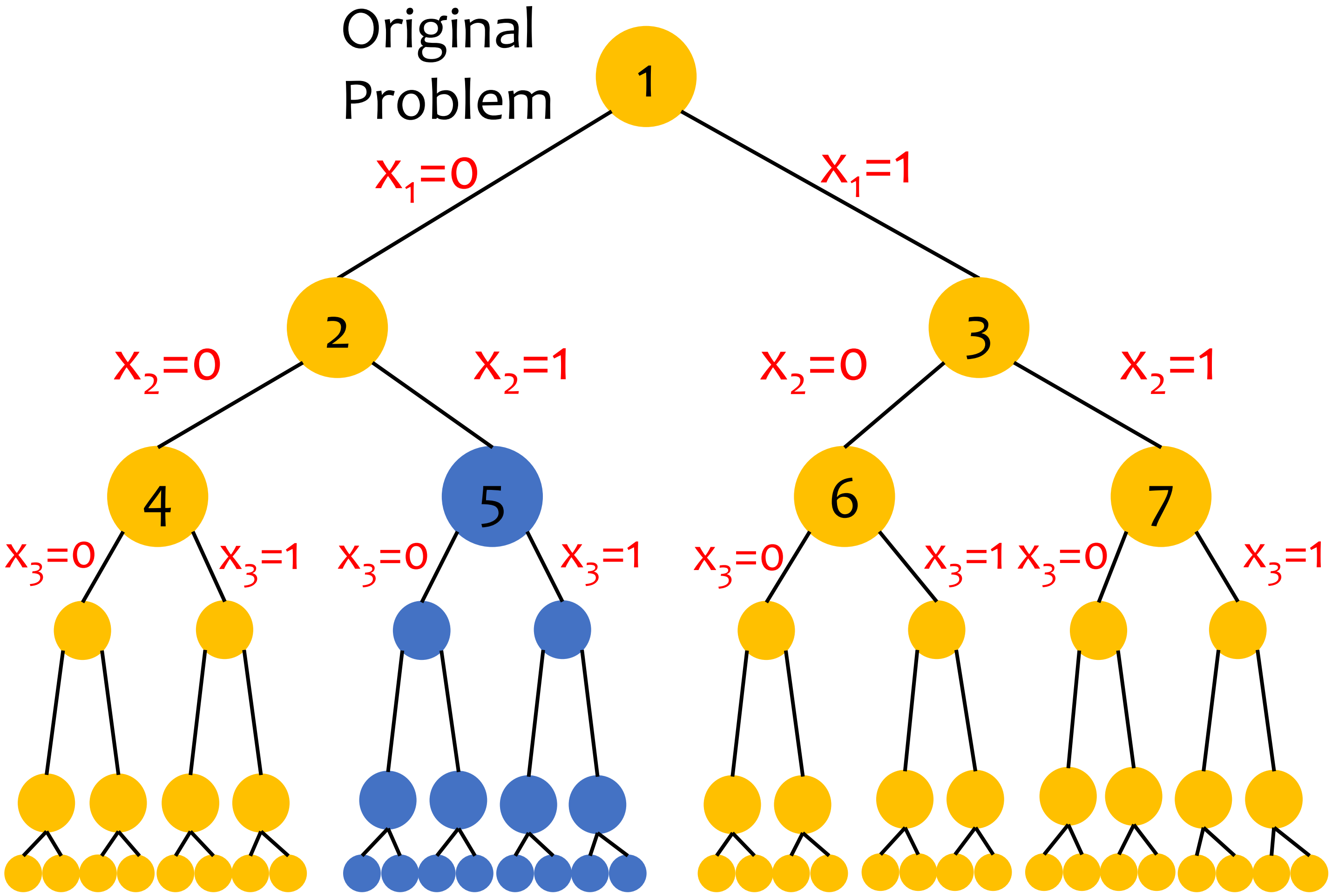
Eliminate subtree 2 if (for maximization)

$$\mathbf{z^{LP}(2) \leq z^{I}}$$

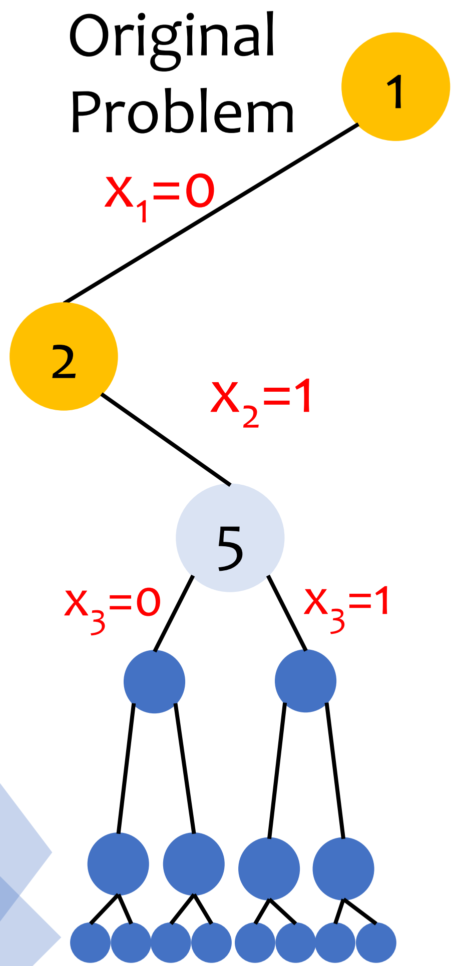
where,

- $z^{LP}(2)$: Upper Bound
- z^{I} : Incumbent

How do we find an upper bound on all descendent solutions from node 5?



Finding an upper bound for descendents of node 5



- To find the optimum descendent of node 5, we can solve the following IP called **Subproblem 5:**

Subroblem (5)

$$\begin{aligned}
&\max \quad 16x_1 + 22x_2 + 12x_3 + 8x_4 + 11x_5 + 19x_6 \\
&\text{s.t.} \quad 5x_1 + 7x_2 + 4x_3 + 3x_4 + 4x_5 + 6x_6 \le 14 \\
&\qquad x_1 = 0,\ x_2 = 1 \qquad x_j \text{ binary for } j = 3 \text{ to } 6
\end{aligned}$$

The LP relaxation:

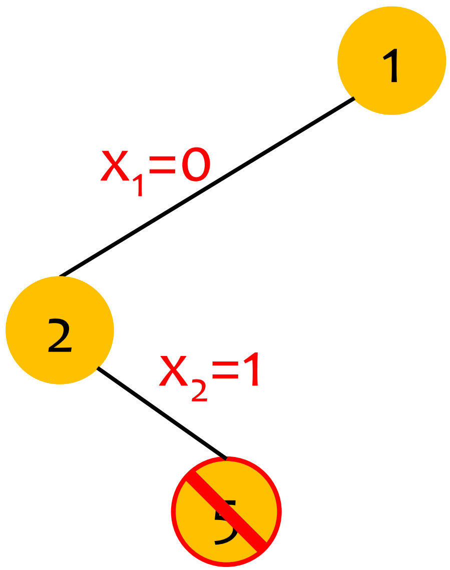
$$\begin{aligned}
&\max \quad z_{LP}(5) = 16x_1 + 22x_2 + 12x_3 + 8x_4 + 11x_5 + 19x_6 \\
&\text{s.t.} \quad 5x_1 + 7x_2 + 4x_3 + 3x_4 + 4x_5 + 6x_6 \le 14 \\
&\qquad x_1 = 0,\ x_2 = 1 \qquad 0 \le x_j \le 1 \text{ for } j = 3 \text{ to } 6.
\end{aligned}$$

- $z^{LP}(5)=44$. Found by solving LP relaxation

The solution for the LP relaxation for Node 5

x_1	x_2	x_3	x_4	x_5	x_6	
0	1	0.25	0	0	1	
Objective Value (z^{LP})						44

Can we eliminate Node 5?

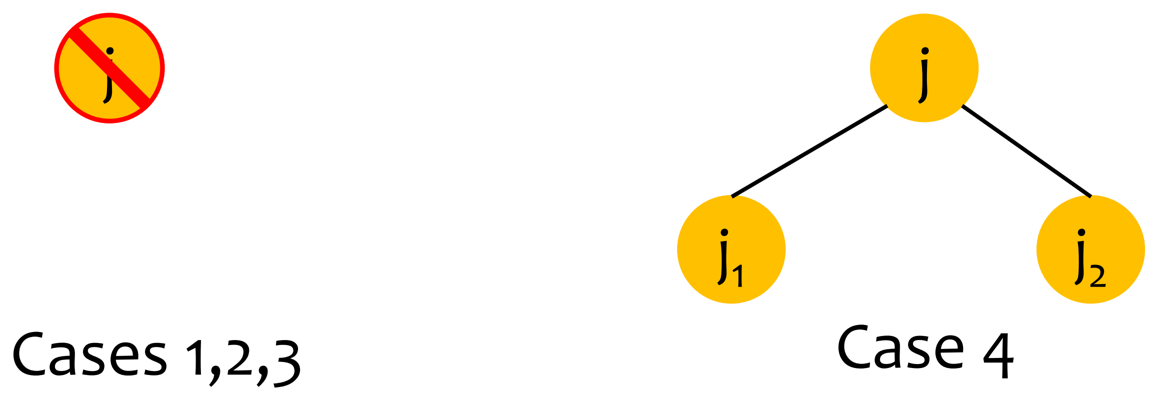


There would be no further branching from Node 5 if $z^I=45$

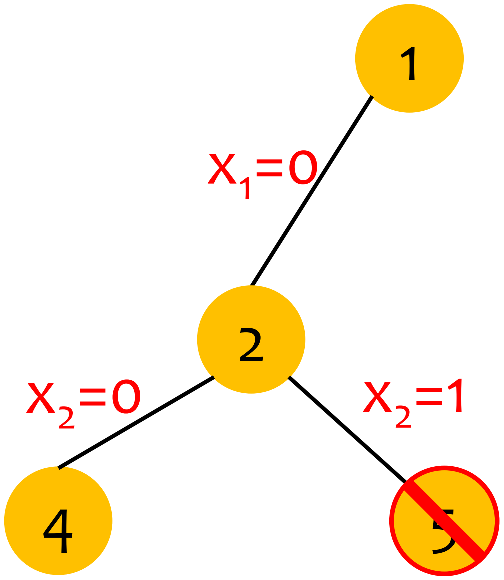
- The incumbent solution has $z^I=38$
- $z^{LP}(5)=44$
- Possibly, some descendents of Noded 5 has a better solution than z=38.
  - Conclusion: We cannot stop enumerating solutions from Node 5. We need to branch from Node 5.
- But suppose that we had an incumbent with $z^I=45$. Then, no descendent of Node 5 could be better than $\mathbf{x}^{(I)}$ and we could fathom Node 5.
- Even an incumbent with $z^I=44$ would work

Branch-and-Bound Overview

- Branch-and-Bound creates the enumeration tree one node at a time and once branch at a time.
- Before branching on a Node j, it solved LP(j). Depending on the solution of LP(j), Branch-and-Bound either fathoms Node j or branches on Node j and creates two children.

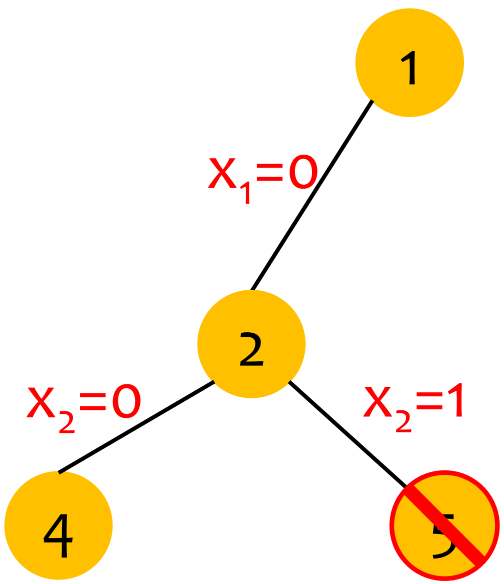


Case 1: Fathom by Infeasibility



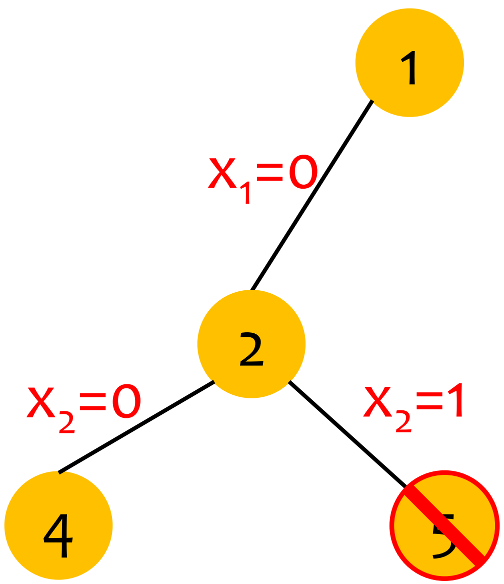
- The incumbent solution has value $z^I=38$
- **Case 1:** $z^{LP}(j) = -\infty$. That is, the LP for Subproblem j is infeasible.
  - ✓FATHOM Node j.
  - ✓Do not search any of the subtree hanging from Node j because none of these subproblems has a feasible solution.

Case 2: Fathom by Bound



- The incumbent solution has value $z^I=38$
- *Assume all feasible integer solutions have integer objective values.*
- **Case 2:** $-\infty < [z^{LP}(j)] \leq z^I$. Then, $z^{IP}(j) \leq [z^{LP}(j)] \leq z^I$.
  - ✓FATHOM Node j.
  - ✓No subproblem in the subtree hanging from Node j has a better solution than the incumbent.
- e.g., suppose $z^{LP}(5)=38.7$ → Bound(5) = $[z^{LP}(5)] = 38$

Case 3: Fathom by Integrality

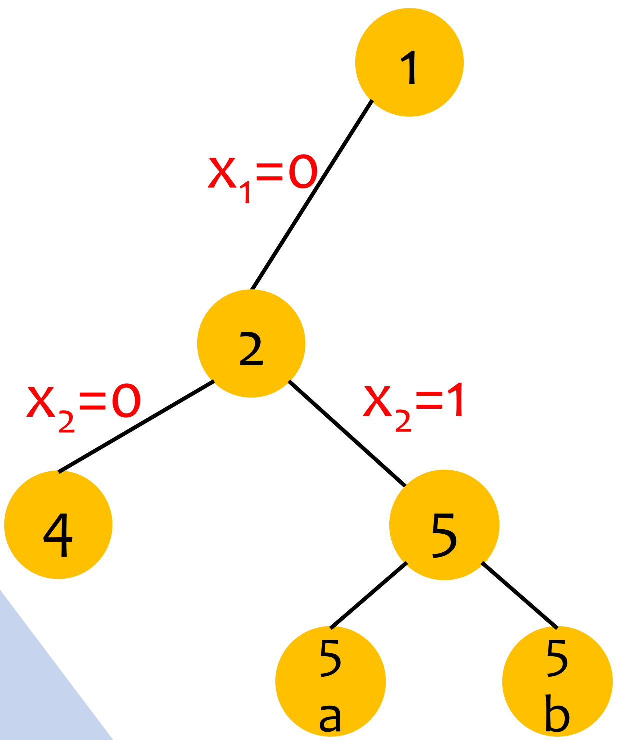


- The incumbent solution has value $z^I=38$
- **<u>Case 3</u>:** $z^I \leq z^{LP}(j)$ and the optimal solution for LP(j) is feasible for the IP.
  - ✓ Replace the current incumbent with the integral solution found by LP(j)
  - ✓ No descendant of Node j can have a better IP solution.
  - ✓ FATHOM Node j.
- e.g., suppose the optimum solution for LP(5) was:

  $x_1=0,\ x_2=1,\ x_3=0,\ x_4=1,\ x_5=1,\ x_6=0$

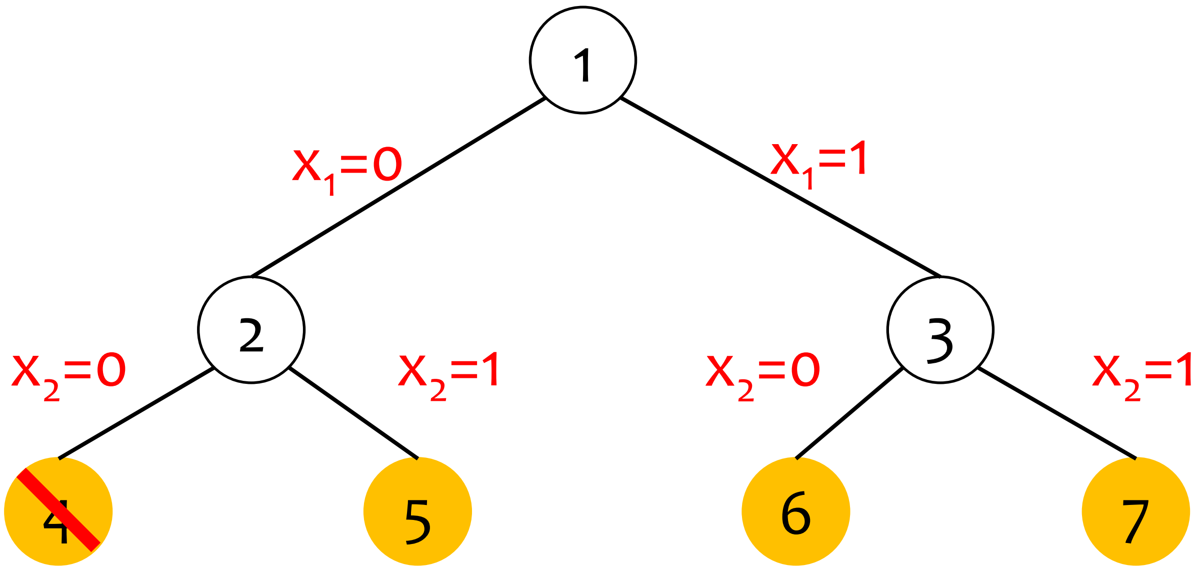
  $z^{LP}(5)=41$
- New Incumbent → $z^I=41$

Case 4: Branch Further



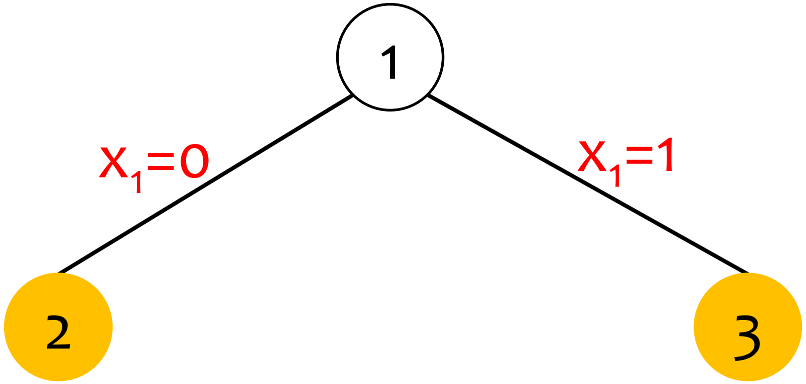
- The incumbent solution has value $z^I=38$
- **Case 4:** $z^I \leq [z^{LP}(j)]$ and the optimal solution for LP(j) is NOT feasible for the IP.
  - ✓ In this case, we cannot fathom Node j.
  - ✓ Instead, we add its two "children" to our (growing) tree and continue the branch-and-bound algorithm.
- e.g., suppose $z^{LP}(5)=41$, but the solution to LP relaxation is not feasible for the IP.
- → Bound(5)=41

Active Nodes



- A node is called active if it has no children and it has not yet been fathomed. The active nodes are 5,6,7.
- Initially, the only active node is 1. The algorithm ends when there are no active nodes.

Branch-and-Bound for 0-1 Integer Programs



LP(1)

maximize $16x_1 + 22x_2 + 12x_3 + 8x_4 + 11x_5 + 19x_6$

subject to $5x_1 + 7x_2 + 4x_3 + 3x_4 + 4x_5 + 6x_6 \leq 14$

$0 \leq x_j \leq 1$ for $j = 1$ to 6.

- $z^I=38$
- $z^{LP}(1)=44\ 3/7 \rightarrow$ Bound(1)=44
- This is Case 4. We add the two children of Node 1.

The Solution for the LP Relaxation of Node 1

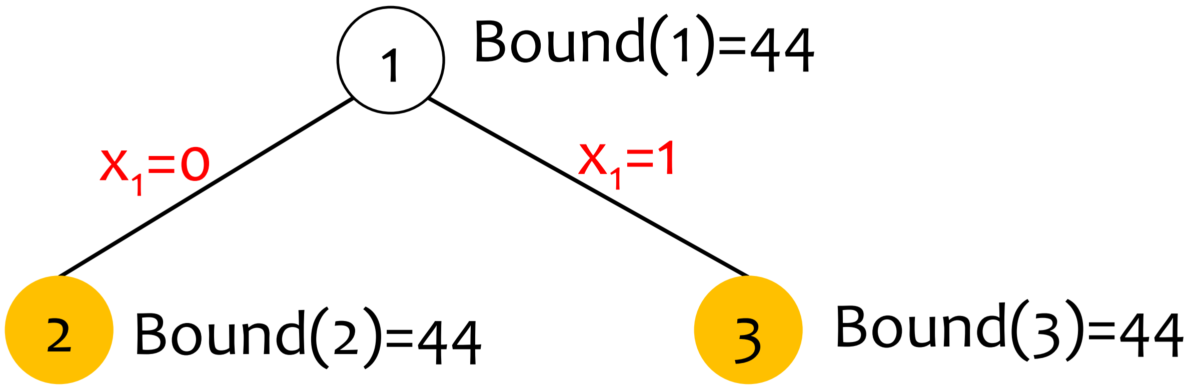
maximize $16x_1 + 22x_2 + 12x_3 + 8x_4 + 11x_5 + 19x_6$

subject to $5x_1 + 7x_2 + 4x_3 + 3x_4 + 4x_5 + 6x_6 \leq 14$

$0 \leq x_j \leq 1$ for $j = 1$ to 6

x_1	x_2	x_3	x_4	x_5	x_6	
1	0.429	0	0	0	1	
Objective Value (z^{LP})						44.43

Subproblem 2



- $z^I=38$
- LP(2):

$$\begin{array}{ll} \text{maximize} & 16x_1 + 22x_2 + 12x_3 + 8x_4 + 11x_5 + 19x_6 \\ \text{subject to} & 5x_1 + 7x_2 + 4x_3 + 3x_4 + 4x_5 + 6x_6 \leq 14 \\ & x_1 = 0, \quad 0 \leq x_j \leq 1 \quad \text{for } j = 2 \text{ to } 6. \end{array}$$

- $z^{LP}(2)=44$
- **Optimum Solution:** $x_1=0$, $x_2=1$, $x_3=0.25$, $x_4=0$, $x_5=0$, $x_6=1$
- Bound(2) = 44. This is Case 4. We add the two children of Node 2.

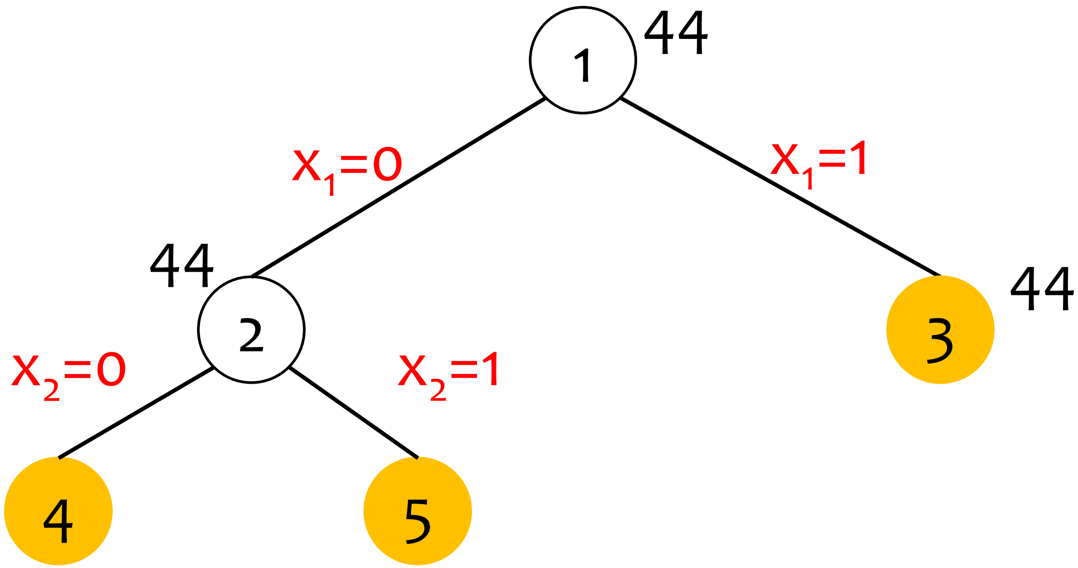
The Solution for the LP Relaxation of Node 2

maximize	$16x_1 + 22x_2 + 12x_3 + 8x_4 + 11x_5 + 19x_6$
subject to	$5x_1 + 7x_2 + 4x_3 + 3x_4 + 4x_5 + 6x_6 \le 14$
	$0 \le x_j \le 1$ for $j = 1$ to 6, and $x_1 = 0$

x_1	x_2	x_3	x_4	x_5	x_6	
0	1	0.25	0	0	1	
Objective Value (z^{LP})						44

- Case 4: Cannot Fathom → Add two children of Node 2.

Subproblem 3



- $z^I=38$
- $z^{LP}(3)=44\ 3/7 \rightarrow$ Bound(3) = 44
- This is Case 4. So we add the two children of Node 3.
- Active nodes are colored yellow. Other nodes are white. Fathomed nodes have a "no nodes permitted" label

The Solution for the LP Relaxation of Node 3

maximize $16x_1 + 22x_2 + 12x_3 + 8x_4 + 11x_5 + 19x_6$

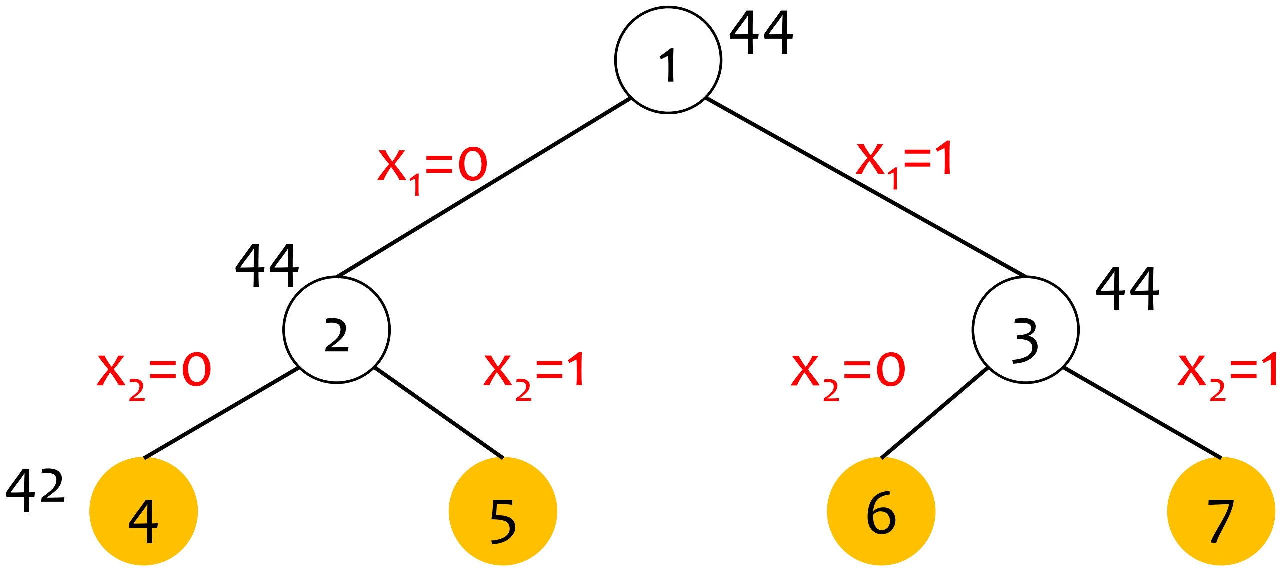
subject to $5x_1 + 7x_2 + 4x_3 + 3x_4 + 4x_5 + 6x_6 \leq 14$

$0 \leq x_j \leq 1$ for $j = 1$ to 6, and $x_1 = 1$

x_1	x_2	x_3	x_4	x_5	x_6	
1	0.429	0	0	0	1	
Objective Value (z^{LP})						44.43

- Case 4: Cannot Fathom → Add two children of Node 2.

Subproblem 4



- z^I=38
- $z^{LP}(4)$=42.
- The solution for LP(4) is: x_1=0, x_2=0, x_3 =1, x_4 =0, x_5 =1, x_6 = 1
- It is a <u>**feasible solution**</u> for the IP that is better than the Incumbent → This is Case 3.

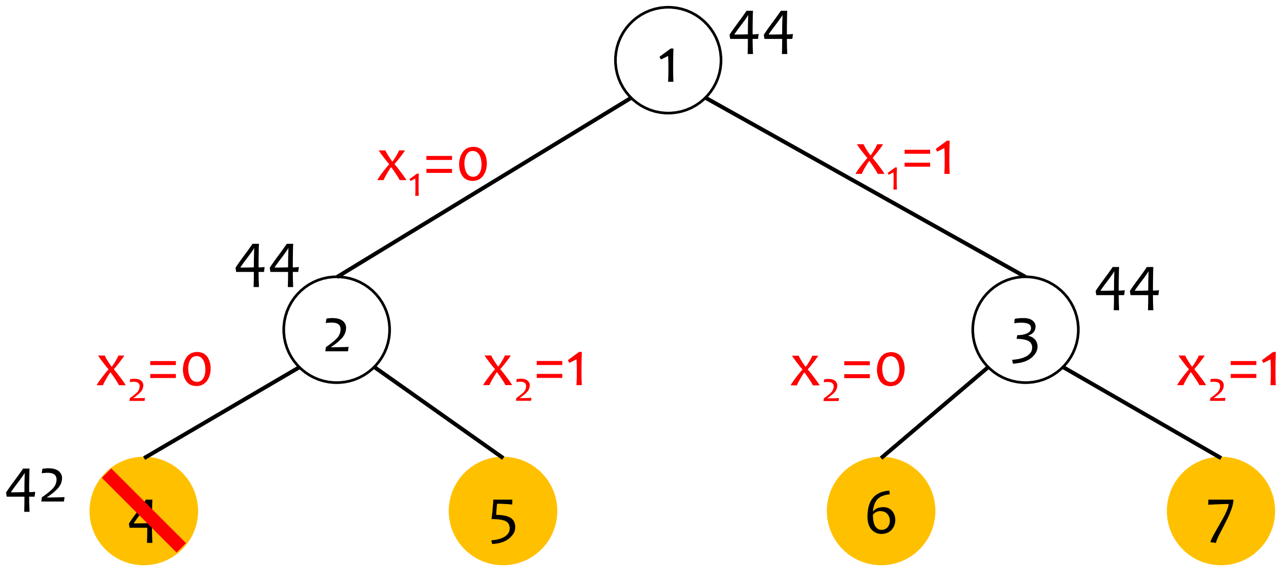
The Solution for the LP Relaxation of Node 4

$$\text{maximize} \quad 16x_1 + 22x_2 + 12x_3 + 8x_4 + 11x_5 + 19x_6$$
$$\text{subject to} \quad 5x_1 + 7x_2 + 4x_3 + 3x_4 + 4x_5 + 6x_6 \leq 14$$
$$0 \leq x_j \leq 1 \ \text{ for } j = 1 \text{ to } 6, \ \text{and } x_1 = x_2 = 0.$$

x_1	x_2	x_3	x_4	x_5	x_6	
0	0	1	0	1	1	
Objective Value (z^{LP})						42

- Case 3: Found a feasible IP solution better than the Incumbent.

Result of Subproblem 4



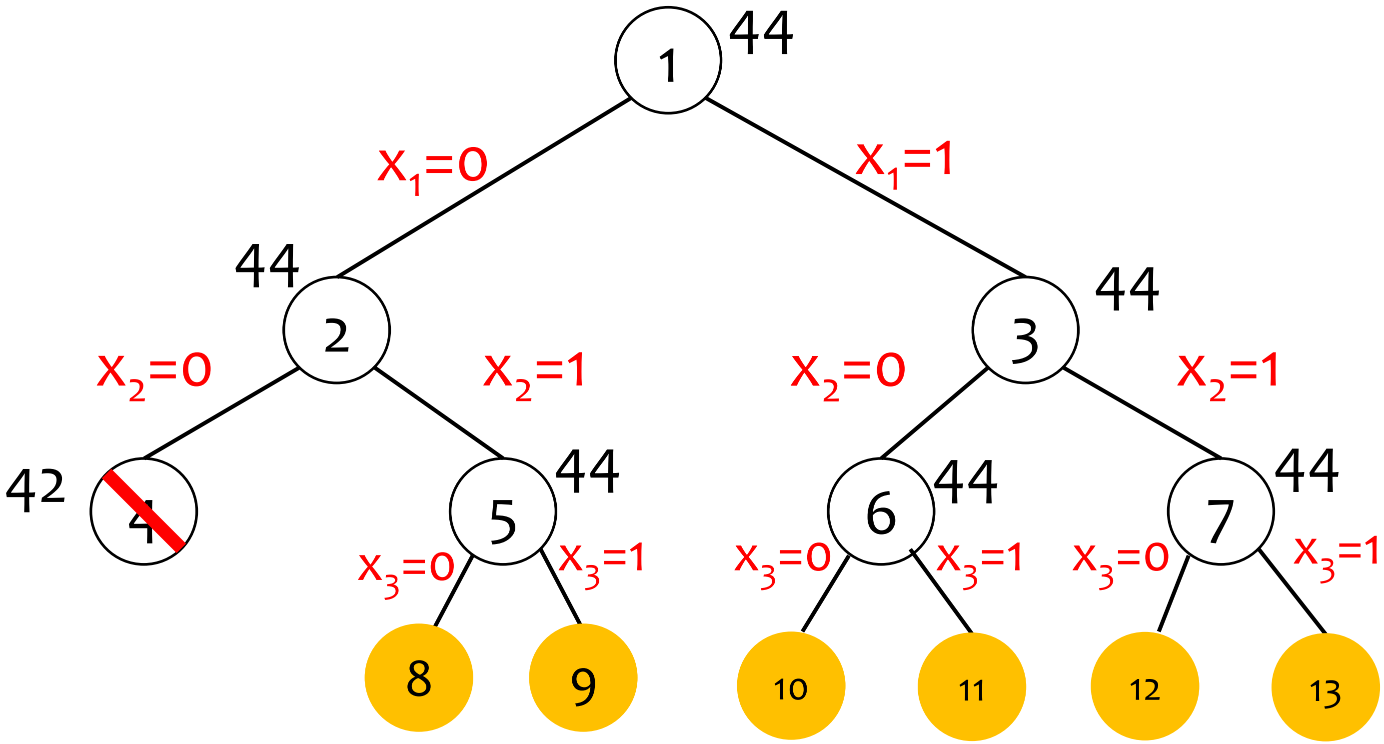
- Replace the Incumbent by

  $x_1=0,\ x_2=0,\ x_3=1,\ x_4=0,\ x_5=1,\ x_6=1$

  $z^I=42$

- Fathom Node 4.

Result of Subproblems 5,6,7



- Nodes 5,6, and. 7. each led to Case 4. In each case, we created two new children and made the nodes active.

The Solution for the LP Relaxation of Node 5

x_1	x_2	x_3	x_4	x_5	x_6	
0	1	0.25	0	0	1	
Objective Value (z^{LP})						44

The Solution for the LP Relaxation of Node 6

x_1	x_2	x_3	x_4	x_5	x_6	
1	0	0.75	0	0	1	
Objective Value (z^{LP})						44

The Solution for the LP Relaxation of Node 7

x_1	x_2	x_3	x_4	x_5	x_6	
1	1	0	0	0	0.333	
Objective Value (z^{LP})						44.33

The Solution for the LP Relaxation of Node 8

x_1	x_2	x_3	x_4	x_5	x_6	
0	1	0	0	0.25	1	
Objective Value (z^{LP})						43.75

The Solution for the LP Relaxation of Node 9

x_1	x_2	x_3	x_4	x_5	x_6	
0	1	1	0	0	0.5	
Objective Value (z^{LP})						43.5

The Solution for the LP Relaxation of Node 10

x_1	x_2	x_3	x_4	x_5	x_6	
1	0	0	0	0.75	1	
Objective Value (z^{LP})						43.25

The Solution for the LP Relaxation of Node 11

x_1	x_2	x_3	x_4	x_5	x_6	
1	0	1	0	0	0.833	
Objective Value (z^{LP})						43.83

The Solution for the LP Relaxation of Node 12

x_1	x_2	x_3	x_4	x_5	x_6	
1	1	0	0	0	0.333	
Objective Value (z^{LP})						44.33

The Solution for the LP Relaxation of Node 13

This subproblem has no feasible solution!

Getting to the end a little quicker

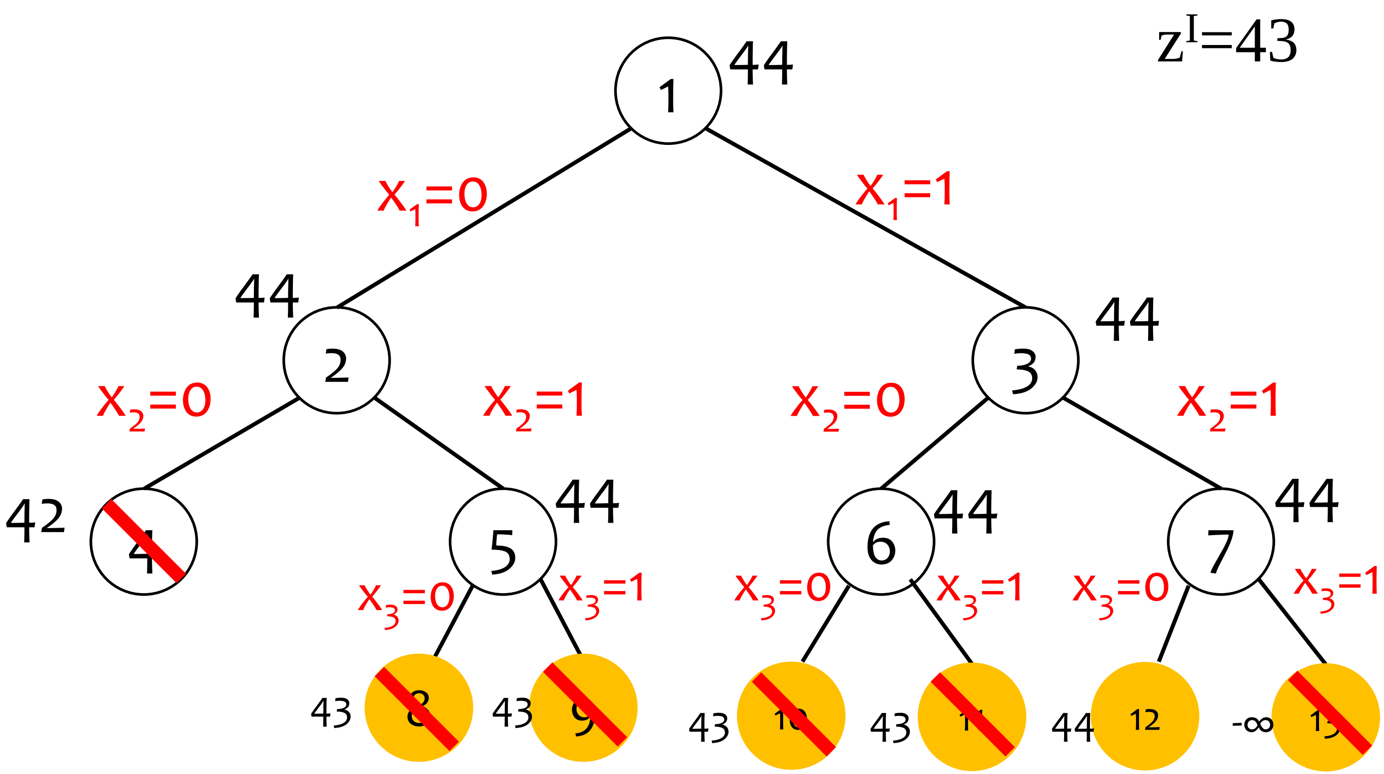
- This algorithms, if continues in its current way, would explore almost all of the nodes.
- Say, we somehow had:
- $\mathbf{x}^l$=(1,0,0,1,0,1)
- z^l=43

$$max\ 16x_1+22x_2+12x_3+8x_4+11x_5+19x_6$$

$$\text{s.t.} \qquad 5x_1+7x_2+4x_3+3x_4+4x_5+6x_6 \leq 14. \qquad \text{(IP)}$$

$$x_j \in \{0,1\}\ for\ j = 1, \dots, 6$$

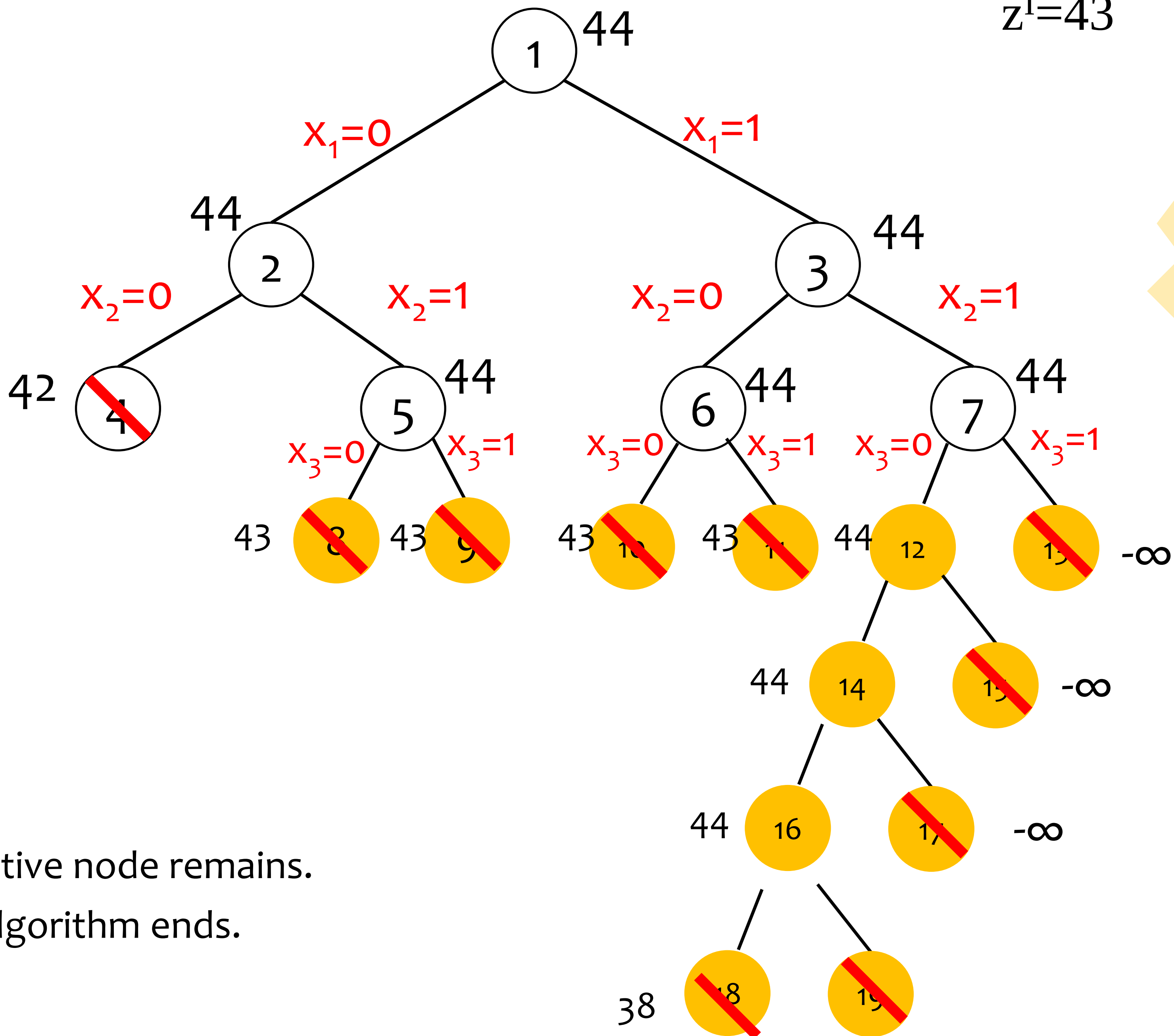
Nodes 8 to 13



- We next solved the LPs associated with Nodes 8-13.
- Nodes 8-11 were fathomed by Case 2 (Their bounds were 43). Node 13 was fathomed because the LP was infeasible.

Finishing Up

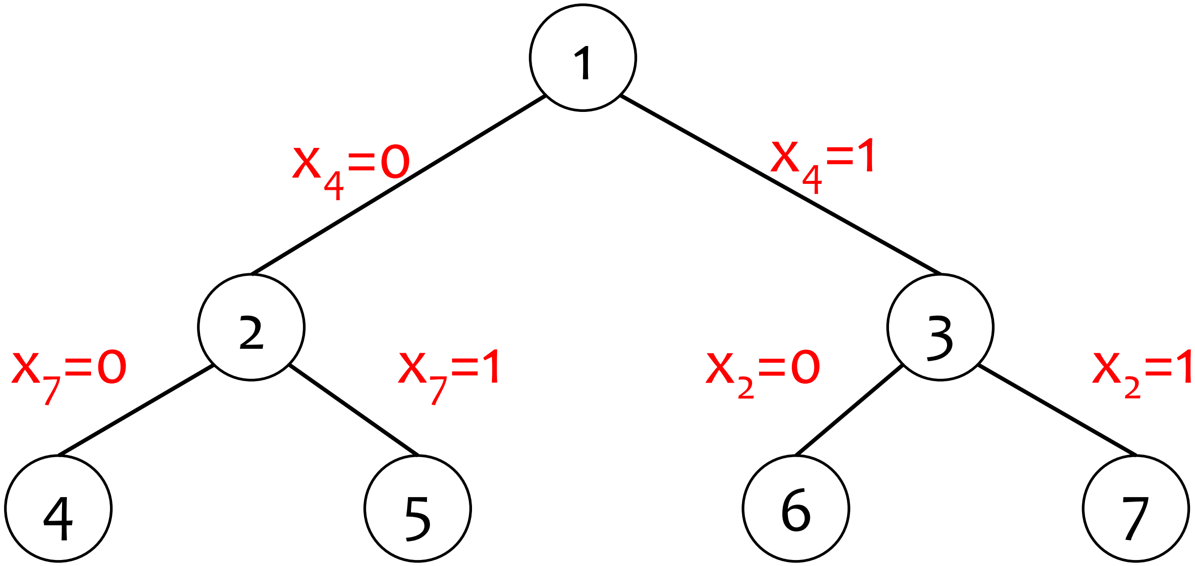
$z^I=43$



No active node remains.

The algorithm ends.

Other Branching Rules



- We don’t need to branch on any specific variables in any order.
- Each branch should divide the “population of solutions” into two parts.
- Commercial algorithms use good branching rules that will lead to faster run times.

Branch and Bound Algorithm

- **<u>Initialize</u>**
  - Active={1} (node 1 is the original problem)
  - Incumbent= Ø (or some heuristic finds an incumbent)
- **<u>Select</u>**
  - If Active= Ø, then
    - The incumbent is optimal if it exists.
    - The problem is infeasible if no incumbent exists.
  - Else, let j be a node from Active. Remove j from active

CASE 1. $z_{LP}(j) = -\infty$. Then fathom node j.
CASE 2. $-\infty < \lfloor z_{LP}(j) \rfloor \le z_i$. Then fathom node j.
CASE 3. $z_{LP}(j) > z_i$ and the optimal solution for LP(j) is feasible for the IP. Then fathom node j, and replace the incumbent with this new solution. CASE 4. $\lfloor z_{LP}(j) \rfloor > z_i$ and the optimal solution for LP(j) is not feasible for the IP. Then create two children for node j.

B&B for Pure Integer Programs

maximize $8x_1 + 5x_2$

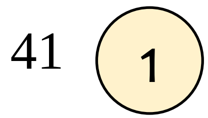
s.t.

$$x_1 + x_2 \le 6$$

$$9x_1 + 5x_2 \le 45$$

$$x_1, x_2 \ge 0, \; x_1, x_2 \text{ integer}$$

The first node of the B&B tree



There is no initial incumbent.
$z^I = -\infty$

maximize $8x_1 + 5x_2$

s.t.

$$x_1 + x_2 \le 6 \qquad\qquad \text{LP(1)}$$

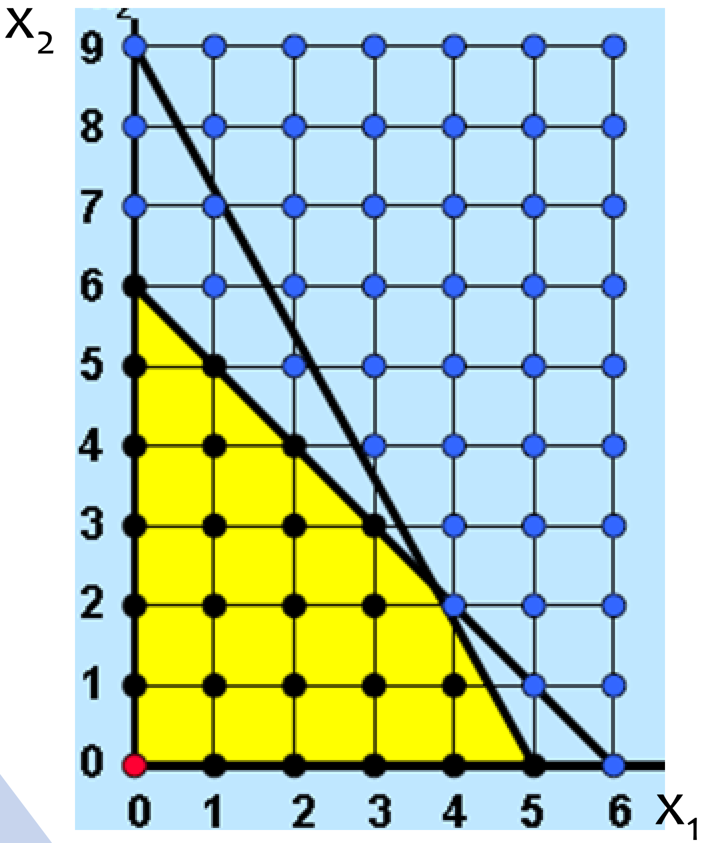
$$9x_1 + 5x_2 \le 45$$

$$x_1, x_2 \ge 0$$

- Optimal solution to LP(1) is:

  $x_1 = 3.75$, $x_2 = 2.25$, $z_{LP}(1) = 41.25$
- This is Case 4 of B&B. Create two children.

The Graphical Representation



maximize $8x_1 + 5x_2$

s.t.

$x_1 + x_2 \leq 6$

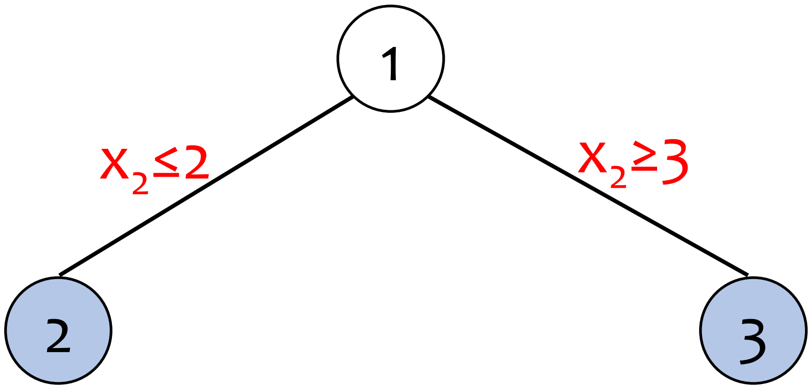
$9x_1 + 5x_2 \leq 45$

$x_1, x_2 \geq 0$, x_1, x_2 integer

Optimal solution to LP relaxation is:

$x_1=3.75$ $x_2=2.25$ $z=41.25$

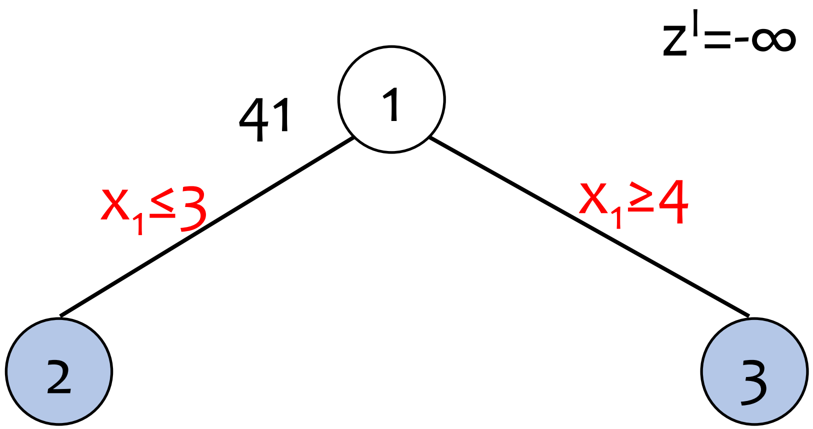
Note: We can create subproblems any way that we want so long as eventually every solution would be enumerated if we did not fathom.



That is, no feasible solution to the integer program ever gets eliminated by branching.

It will be feasible for one of the branches.

Node 2 of the B&B Tree



Standard branching choice:

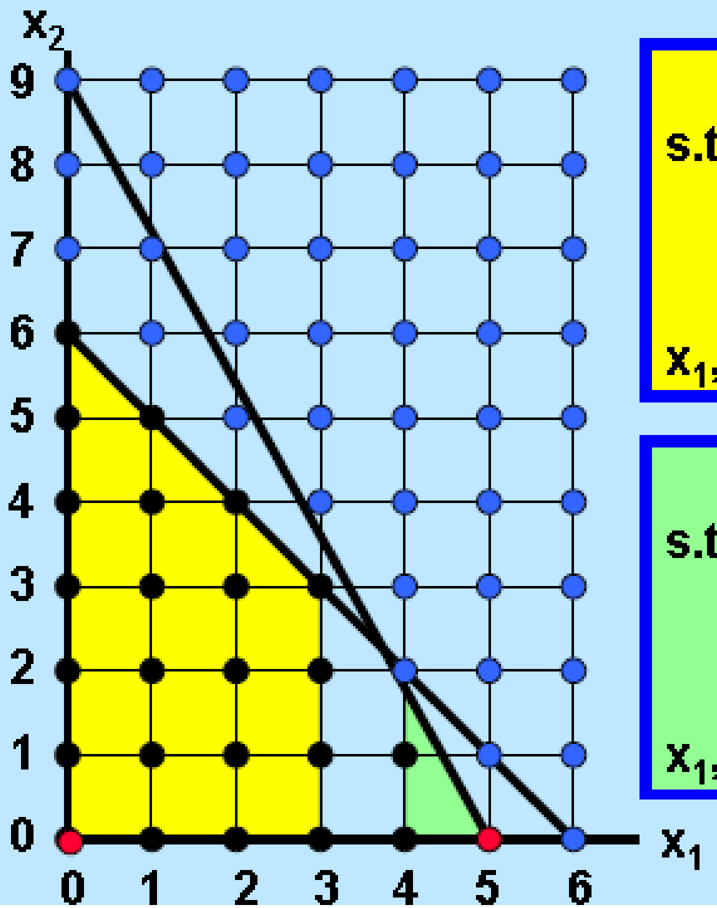
Create children so that the solution for LP(1) is not feasible for either subproblem.

- Optimal solution to LP(2) is:
  $x_1=3$, $x_2=3$, $z_{LP}(2)=39$.

- This gives us our first integer solution. It is Case3 of B&B.

- We replace the incumbent and fathom the node.

Subproblems 1 and 2

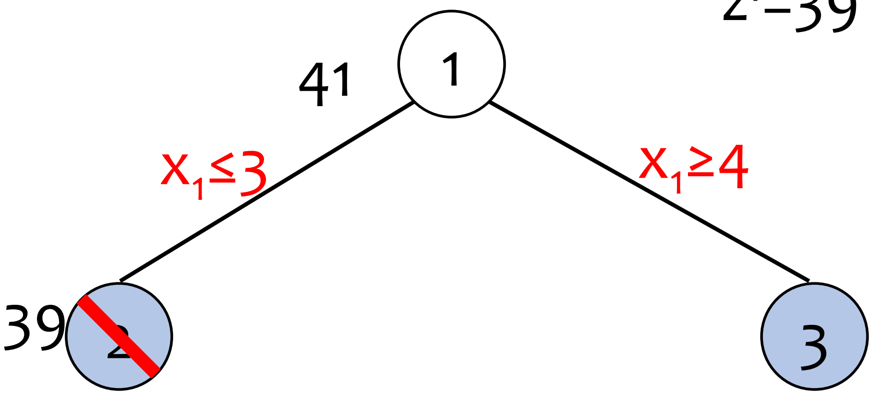


$$
\begin{aligned}
&\text{Max} \quad 8x_1 + 5x_2 \\
\text{s.t} \quad & x_1 + x_2 \le 6 \\
& 9x_1 + 5x_2 \le 45 \\
& x_1 \le 3 \\
& x_1, x_2 \ge 0, \quad x_1, x_2 \text{ integer}
\end{aligned}
$$

$$
\begin{aligned}
&\text{Max} \quad 8x_1 + 5x_2 \\
\text{s.t} \quad & x_1 + x_2 \le 6 \\
& 9x_1 + 5x_2 \le 45 \\
& x_1 \ge 4 \\
& x_1, x_2 \ge 0, \quad x_1, x_2 \text{ integer}
\end{aligned}
$$

Node 3 of the B&B Tree

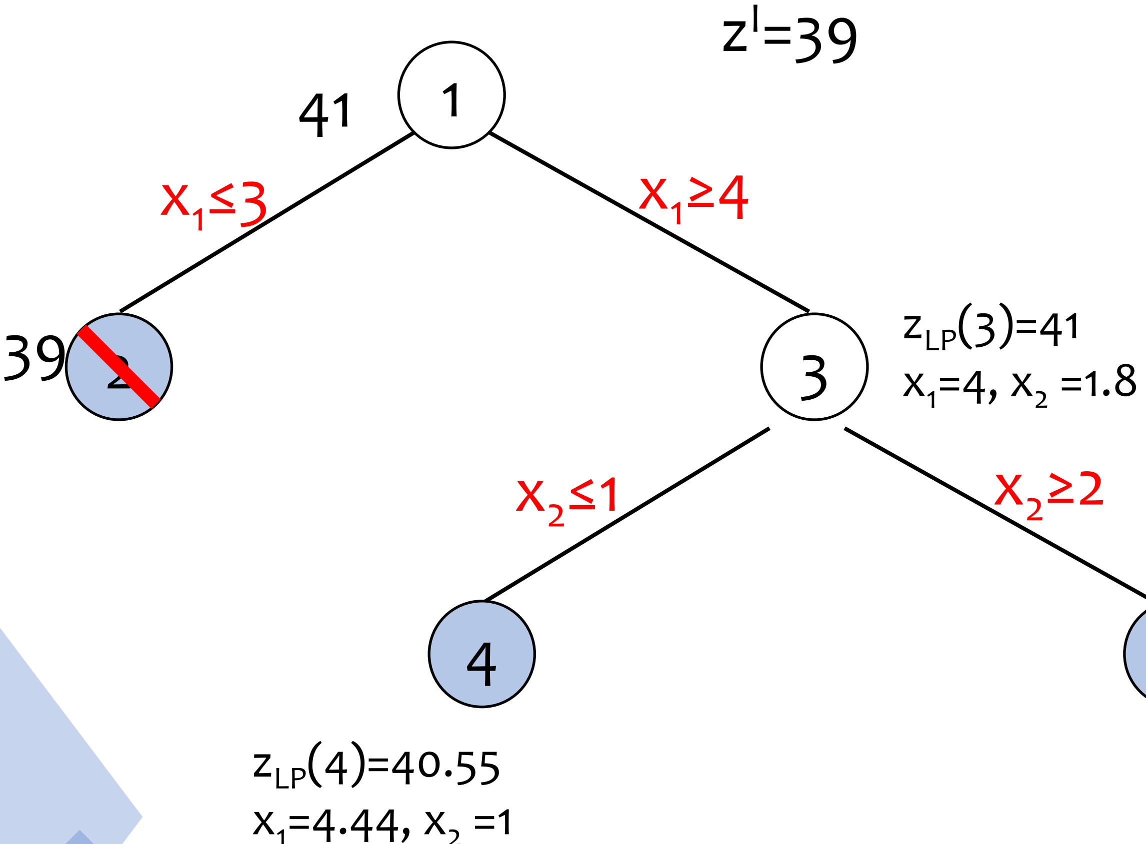
$z^l = 39$



$z_{LP}(3) = 41.\ x_1 = 4,\ x_2 = 1.8$

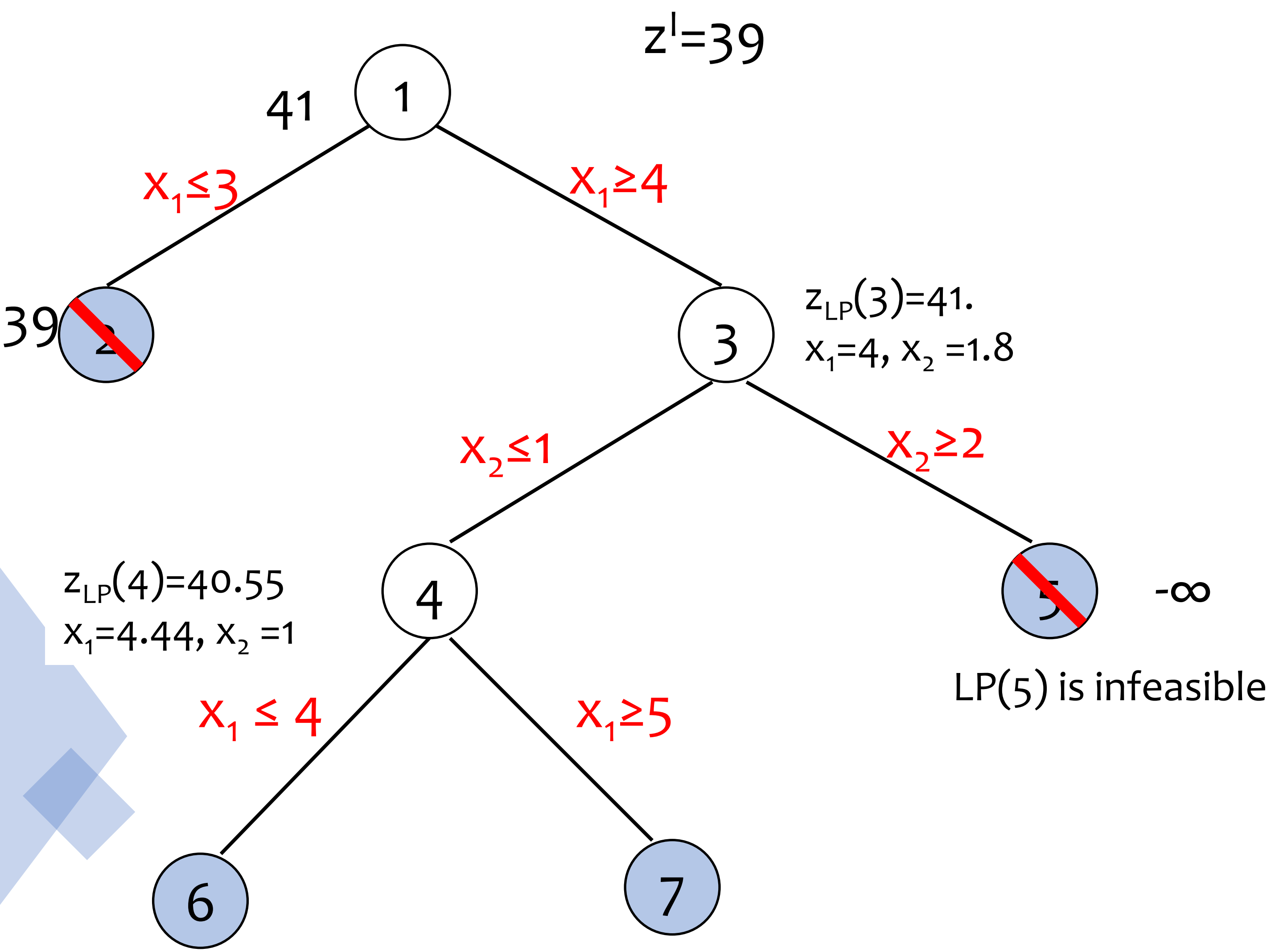
This is Case 4 of B&B.
We create two children.

Node 4 of the B&B Tree

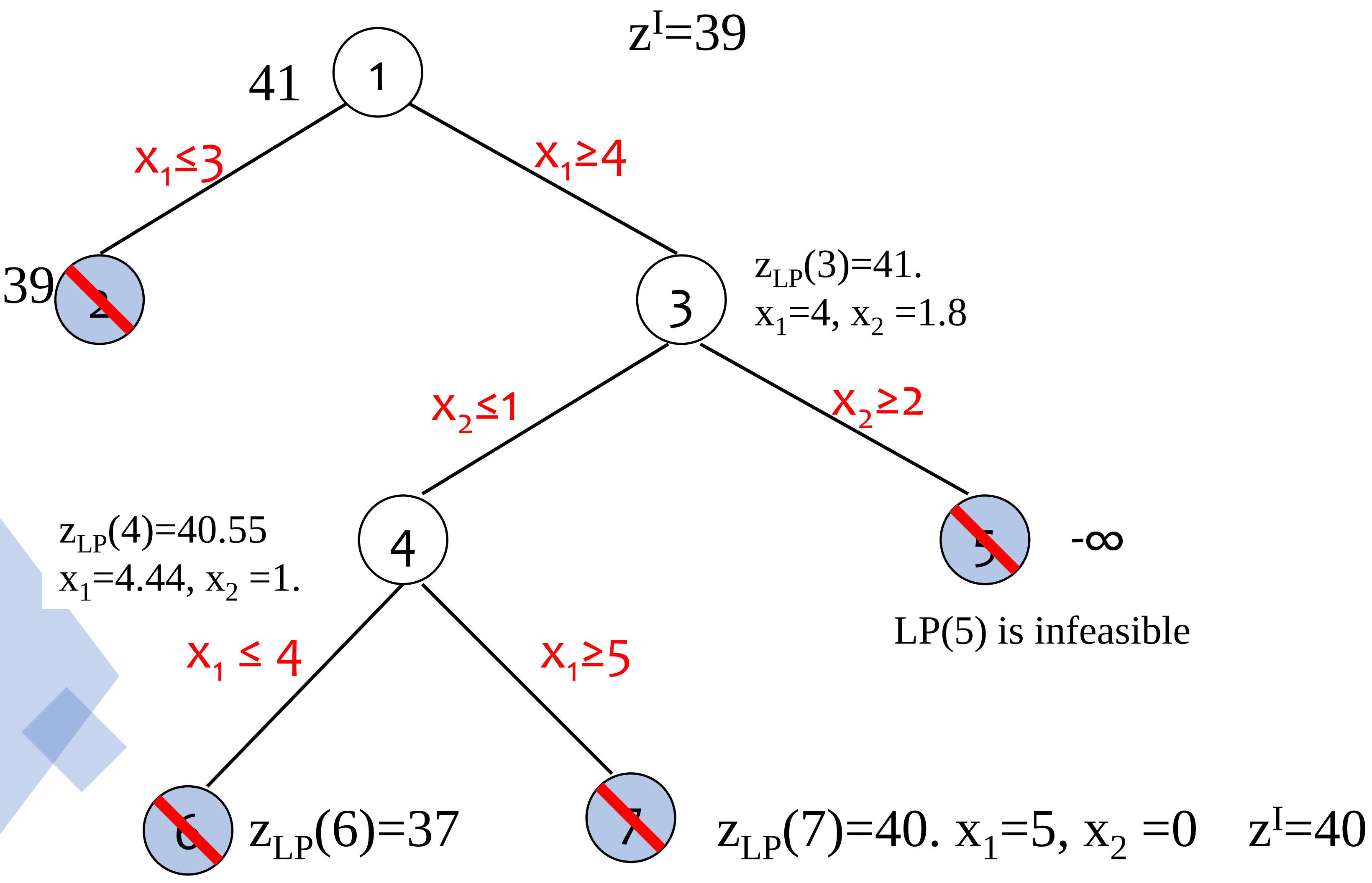


Two children are created.

Node 5 of the B&B Tree



Node 6 and 7 of the B&B Tree



Branch and Bound in General

Notation:

- Incumbent: Best solution on hand (Its value is z^I).
- $z_{IP}(j)$: the optimum value of subproblem j
- $z_{LP}(j)$: value of the LP relaxation of node j
- Bound (j). In cases where we are maximizing and where integer solutions have integer objective values, we let Bound (j)= $\lfloor z_{LP}(j) \rfloor$
- Children of a node: the two problems created for a node such that every solution that is a descendent of the original node is also a descendent of one of its children.
- Active: the collection of active (not fathomed) subproblems.

On Branch and Bound

- Branch and Bound is a stardard way of solving integer programs.
- Lots of art and engineering can go into making it work efficiently.
- Next: A couple of issues that arise.

Different Branching and Bounding Rules are Possible

- The more accurate the bound, the quicker the fathoming.
  - The better the bounding technique, the more accurate the bound and the quicker the fathoming.
  - The quicker the important decisions are resolved, the quicker the fathoming.

- The better the incumbent, the quicker the fathoming.