IE 400: Principles of Engineering Management
Study Set 4 Solutions

Fall 2022

Question 1.

First calculate the ratios:

x_1 : 9/6

x_2 : 13/3

x_3 : 10/2

x_4 : 8/4

x_5 : 8/7

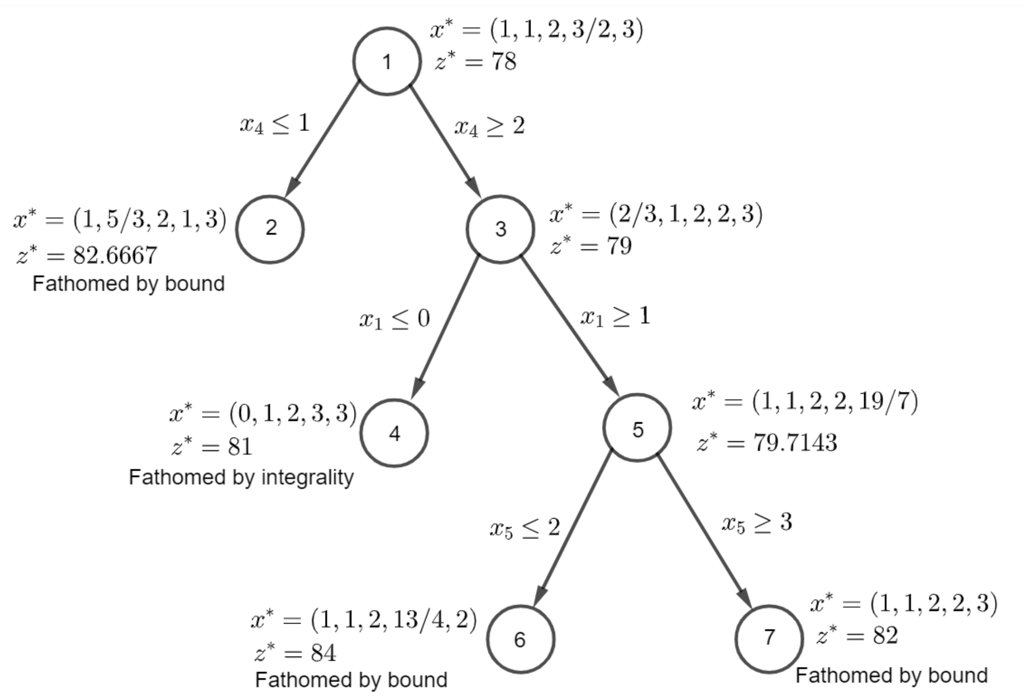
For the **minimization problem**, sort the ratios from smallest to largest:

$$x_5 < x_1 < x_4 < x_2 < x_3$$

Solve the LP relaxation of the knapsack model according to the above priority list (x_5 is the most preferable and x_3 is the least preferable one). The solution to the LP relaxation of the given model is displayed at node 1. Since this is not an integer solution, we branch from node 1, and add new constraints as given in the Branch and Bound tree below. Note that at each node we solve the LP relaxation of the IP model along with the newly added constraints.

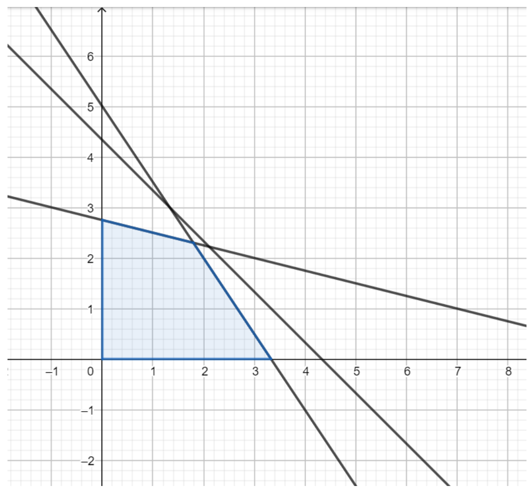
Note that node 2 is fathomed by bound, because we have found an integer solution at node 4 that has a better objective value than the one at node 2.

(If we branch node 2, then the resulting nodes would have an objective value ≥ 82.6667 which will always result in a solution worse than node 4)

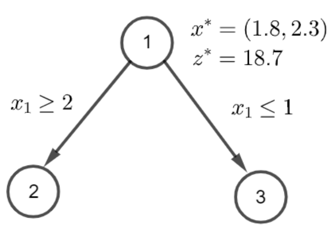


The optimal solution is $x^* = (0, 1, 2, 3, 3)$.

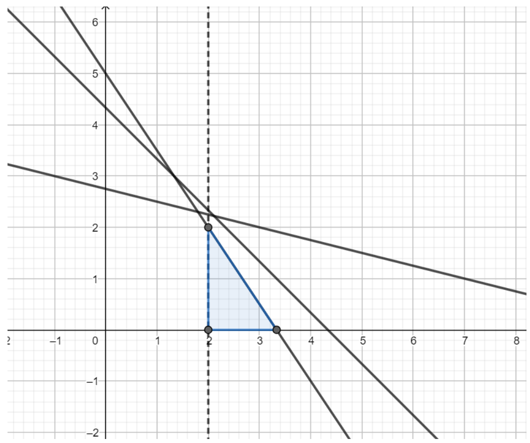
Question 2.
Solve the LP relaxation of the original model using graphical method:



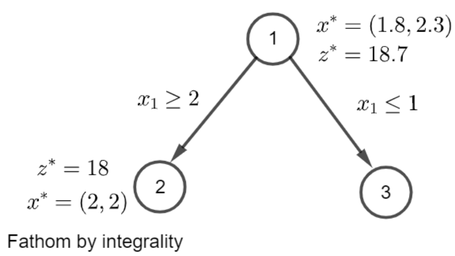
For c=(4,5), we obtain $x^* = (1.8, 2.3)$. This is not an integer solution, so start branching:



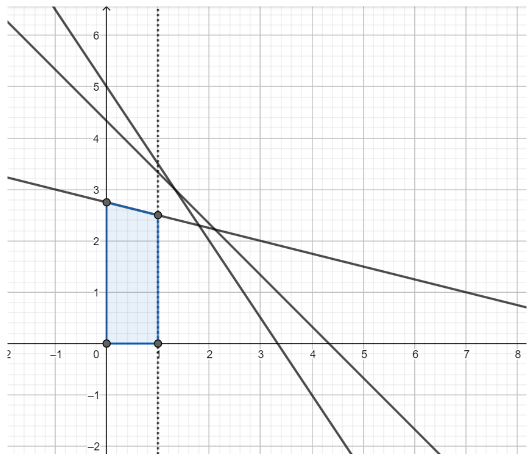
Now move to node 2. Here the constraint $x_1 \geq 2$ is added to the model:



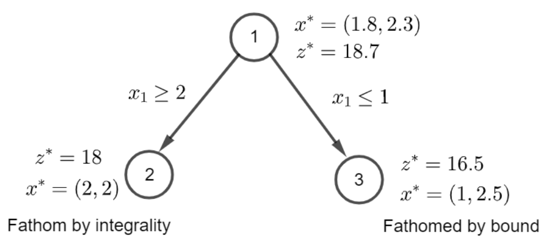
Solve the new model with graphical method and update the branch and bound tree:



We obtained an integer solution at node 2. Hence, it is fathomed by integrality. Now move to the third node. Add the constraint $x_1 \leq 1$ to the original model:



Solving via the graphical method we obtain:



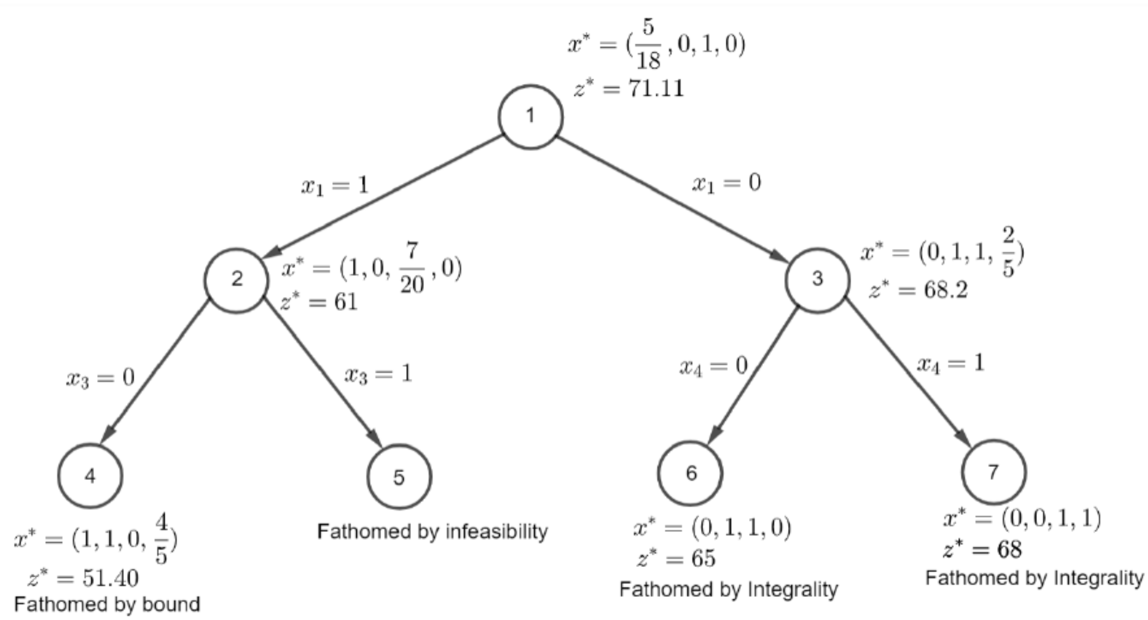
Node 3 is fathomed by bound, because we have already obtained an integer solution at node 2 that has a better objective value (for maximization) than the model at node 3 . If we branch the third node, the objective value will be ≤ 16.5 !

We cannot make further branches. Then, the optimal solution is $x^* = (2, 2)$ and $z^* = 18$.

Question 3.

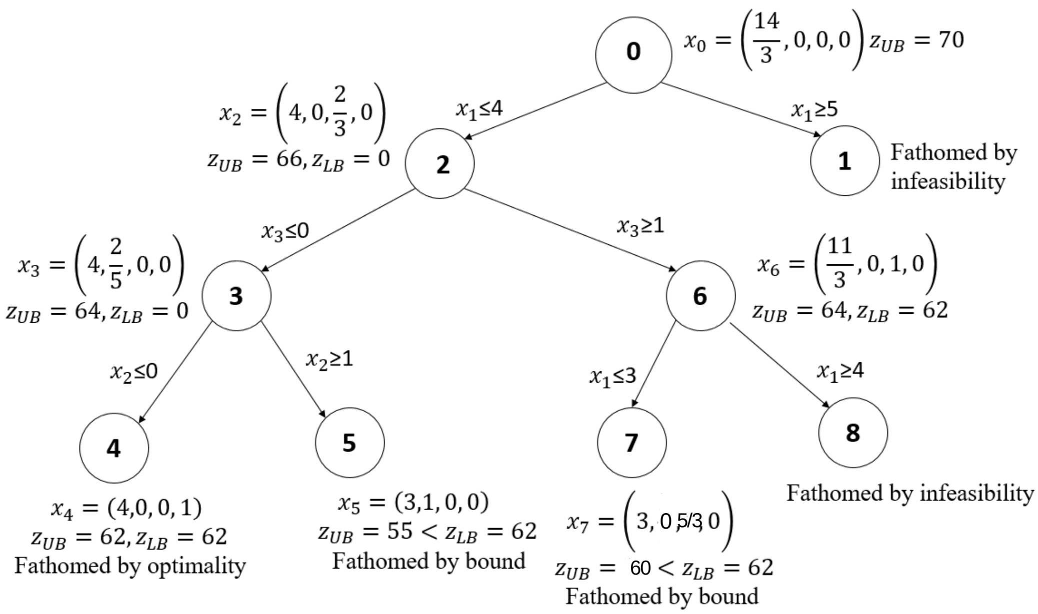
Order of the variables:

$$x_3 > x_1 > x_2 > x_4$$



Optimal Solution is: $x^* = (0, 0, 1, 1)$.

Question 4.


0
$x_0 = \left(\frac{14}{3}, 0, 0, 0\right) z_{UB} = 70$
$x_1 \leq 4$
$x_1 \geq 5$
1
Fathomed by infeasibility
2
$x_2 = \left(4, 0, \frac{2}{3}, 0\right)$
$z_{UB} = 66, z_{LB} = 0$
$x_3 \leq 0$
$x_3 \geq 1$
3
$x_3 = \left(4, \frac{2}{5}, 0, 0\right)$
$z_{UB} = 64, z_{LB} = 0$
6
$x_6 = \left(\frac{11}{3}, 0, 1, 0\right)$
$z_{UB} = 64, z_{LB} = 62$
$x_2 \leq 0$
$x_2 \geq 1$
$x_1 \leq 3$
$x_1 \geq 4$
4
5
7
8
Fathomed by infeasibility
$x_4 = (4, 0, 0, 1)$
$z_{UB} = 62, z_{LB} = 62$
Fathomed by optimality
$x_5 = (3, 1, 0, 0)$
$z_{UB} = 55 < z_{LB} = 62$
Fathomed by bound
$x_7 = \left(3, 0, 5/3, 0\right)$
$z_{UB} = 60 < z_{LB} = 62$
Fathomed by bound

Question 5.

(a) Since x_6, $x_7 \in \{0, 1\}$, the node will be fathomed by infeasibility if

$$a_1 + a_2 < 15$$

(b) To have integer solution we must have:

$$a_1 + a_2 = 15 \text{ or } a_1 = 15 \text{ or } a_2 = 15$$

(c) Since current incumbet is 34, the problem is minimization we fathom by bound if we get an objective greater than or equal to 34.

$$c_1 x_6 + c_2 x_7 \geq 34$$

This is like a knapsack problem:

$$\begin{aligned} &\min\ c_1 x_6 + c_2 x_7 \\ &\text{s.t.}\ \ a_1 x_6 + a_2 x_7 \geq 15 \end{aligned}$$

Case i: $\frac{c_1}{a_1} \leq \frac{c_2}{a_2}$

- if $a_1 \geq 15$ and $c_1 \frac{15}{a_1} \geq 34$, then the node will be fathomed by bound.
- if $a_1 < 15$ and $c_1 + c_2 \frac{15 - a_1}{a_2} \geq 34$, then the node will be fathomed by bound.

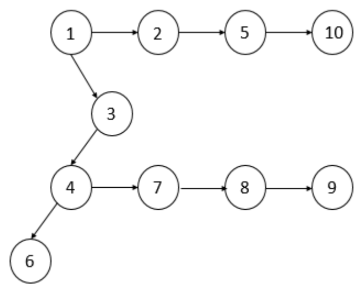
Case ii: $\frac{c_2}{a_2} \leq \frac{c_1}{a_1}$

- if $a_2 \geq 15$ and $c_2 \frac{15}{a_2} \geq 34$, then the node will be fathomed by bound.
- if $a_2 < 15$ and $c_2 + c_1 \frac{15 - a_2}{a_1} \geq 34$, then the node will be fathomed by bound.

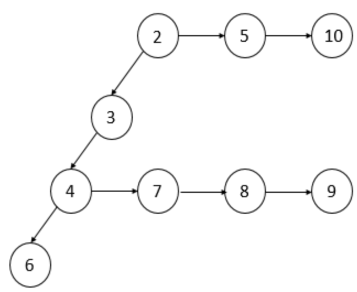
(d) If none of the conditions from a, b, c is satisfied, the node is not fathomed.

Question 6.

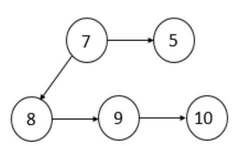
(a) From 1 to 10 (Bob): Shortest path: 1-2-5-10, length=20



From 2 to 10 (Ed): Shortest path: 2-5-10, length=17



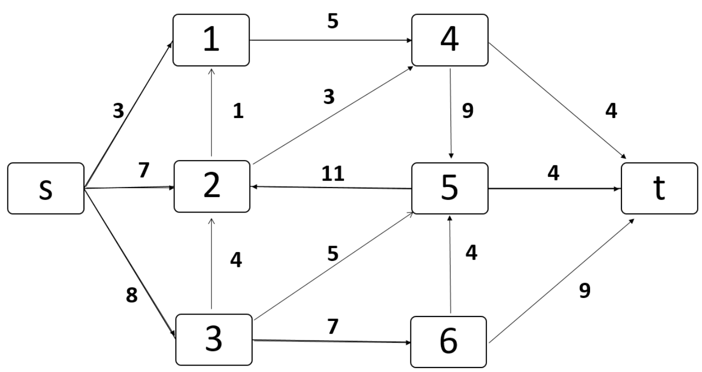
From 7 to 10 (Stu): Shortest path: 7-8-9-10, length=8



(b) Bob should leave 20 minutes earlier than the meeting time, Ed 17 minutes, and Stu 8 minutes.

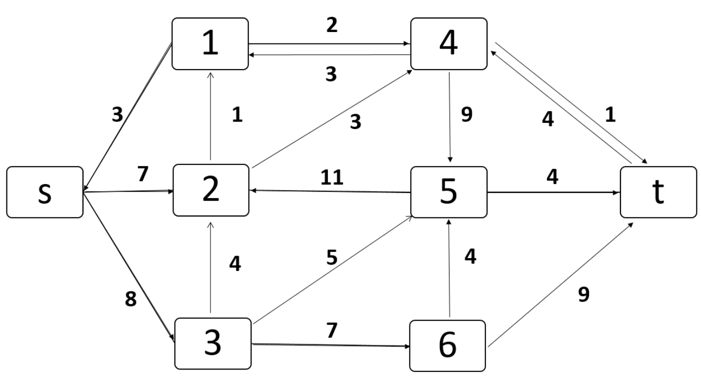
(c) Bob can give a ride to Ed since he is on his shortest path.

Question 7. Consider costs as capacities. Using Ford - Fulkerson algorithm find max-flow.



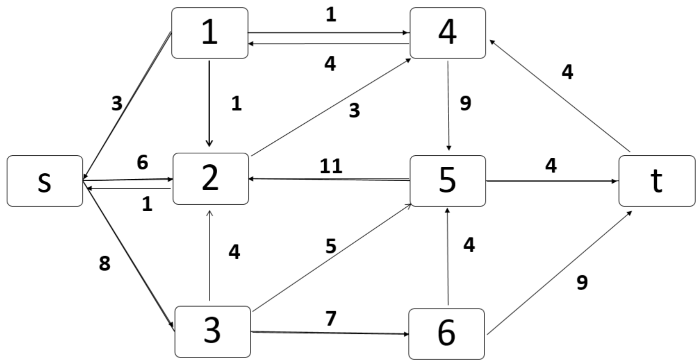
Step 1: Choose path s - 1 - 4 - t

min{3, 5, 4}= 3



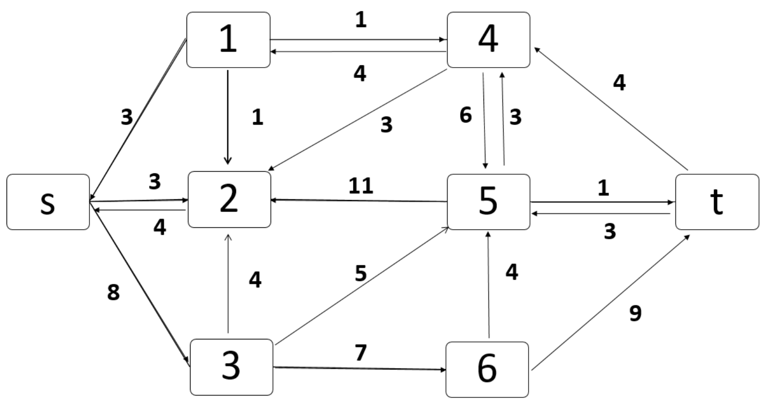
Step 2: Choose path s - 2 - 1 - 4 - t

min{7, 1, 2, 1}= 1



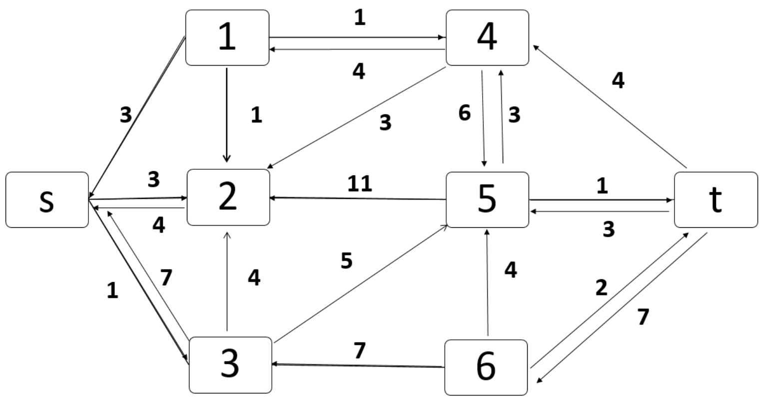
Step 3: Choose path s - 2 - 4 - 5 - t

min{6, 3, 9, 4}= 3



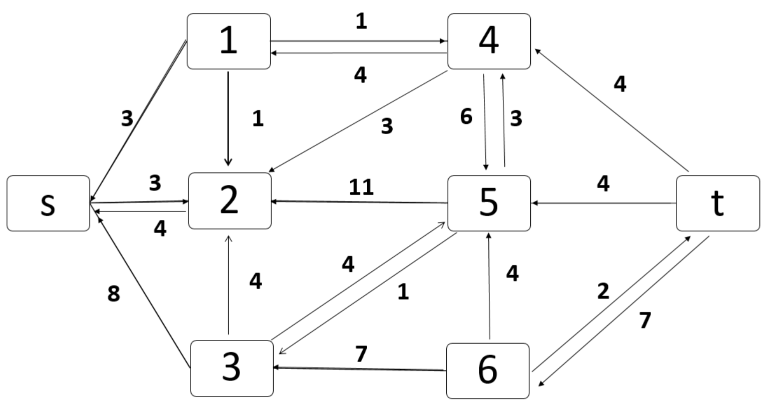
Step 4: Choose path s - 3 - 6 - t

min{8, 7, 9}= 7



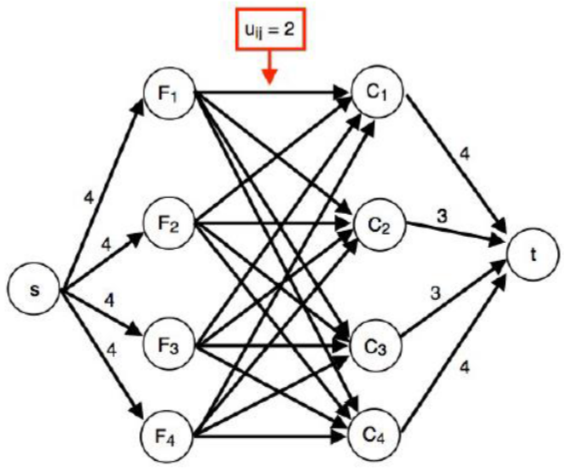
Step 5: Choose path s - 3 - 5 - t

min{1, 5, 1}= 1



There is no other path from s to t.
Since max-flow = min-cut in optimal solution. The minimum cost to block flood's way to city is 15.

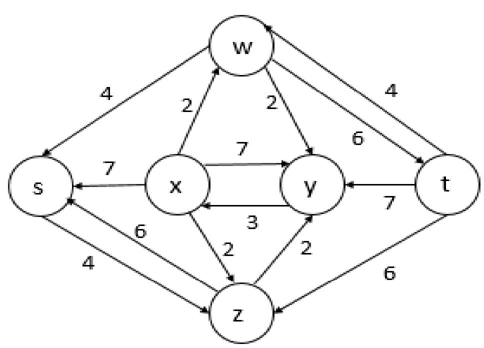
Question 8. The problem of transporting the maximum possible number of people to the picnic can be formulated as a maximum-flow problem on the following network.



Question 9.

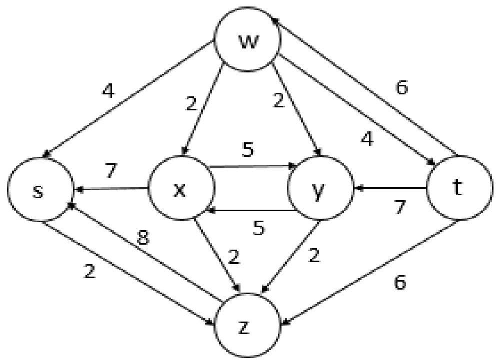
(a) The value of this flow is 17. (The total amount of flow passing from s to t)

(b) No, it is not a maximum flow. We can find a maximum flow by looking at the residual network. In this case, from the given flow, we can get a max flow with only a single augmenting path, as shown below.
First residual network for the given flow network:



Augment flow along path s-z-y-x-w-t. Capacity of this path is 2, so send 2 units of flow on this path.

New residual network:



So, the maximum flow has value 19.

(c) We can find a minimum cut using the residual network for the final maximum flow. Starting at s, find all vertices that are reachable from s in the residual network. this forms one set in the minimum cut. The other vertices form the other set part of the cut. So, a minimum cut in this case are the two sets $S^* = \{s, z\}$ and $T^* = \{x, w, y, t\}$.