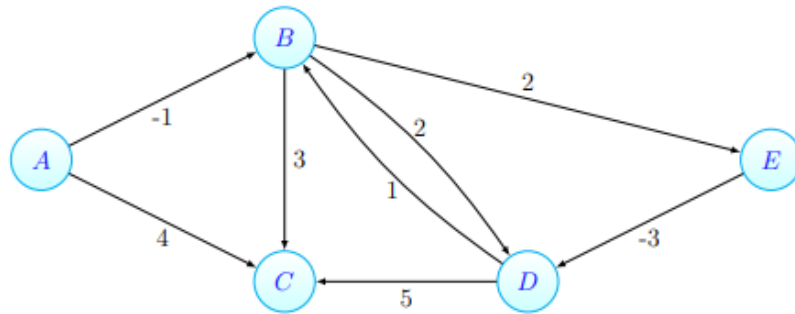


IE 400: Principles of Engineering Management

Study Set 5 Key

FALL 2022-2023

Question 1.

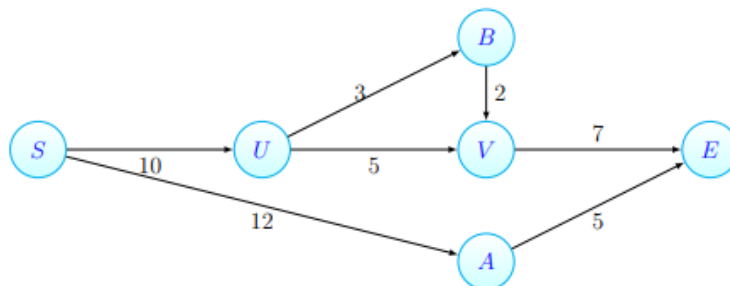


$$V_{[k]}^{(t)} = \min\{V_{[i]}^{t-1} + c_{ik}\}$$

t	$V_{(A)}^t$	$V_{(B)}^t$	$V_{(C)}^t$	$V_{(D)}^t$	$V_{(E)}^t$	d(A)	d(B)	d(C)	d(D)	d(E)
0	0	$+\infty$	$+\infty$	$+\infty$	$+\infty$					
1	0	-1	4	$+\infty$	$+\infty$		A	A		
2	0	-1	2	1	1			B	B	B
3	0	-1	2	-2	1				E	

Question 2.

- (a) The principal of optimality does not apply. See the following **counter example**:



Since we are trying to minimize the highest height encountered: Shortest path from S to E is S-U-V-E=10 (OR S-U-B-V-E =10) **but** shortest path from U to V is U-B-V=3, not U-V=5.

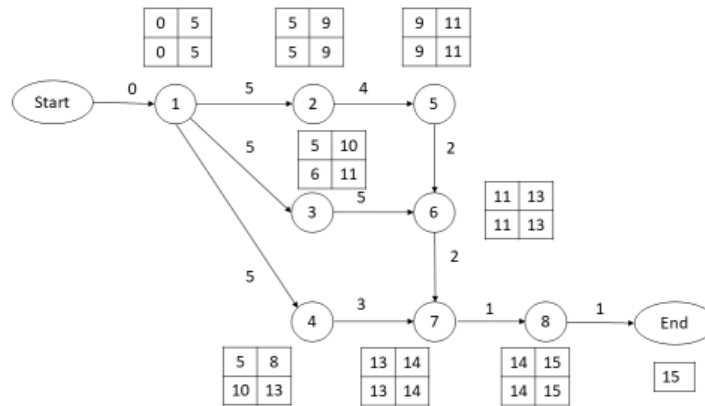
- (b)

$$V[S] = 0$$

$$V[i] = \min_{(i,k) \in A} \{\max\{V[k], c_{ik}\}\}$$

Question 3.

Activity	Predecessors	Duration
1 (Design)	-	5
2 (make A)	1	4
3 (make B)	1	5
4 (make C)	1	3
5 (test A)	2	2
6 (assembly A&B)	3, 5	2
7 (attach C)	6, 4	1
8 (testing)	7	1



Critical path: 1 - 2 - 5 - 6 - 7 - 8

Duration: 15

Activity	1	2	3	4	5	6	7	8
Slack	0	0	1	5	0	0	0	0

For the LP define decision variable x_j : start time of activity $j, j = 1, \dots, 10$, start, end

$$\begin{aligned}
 \min \quad & x_{end} \\
 \text{s.t.} \quad & x_1 \geq x_{start} \\
 & x_2 \geq x_1 + 5 \\
 & x_3 \geq x_1 + 5 \\
 & x_4 \geq x_1 + 5 \\
 & x_5 \geq x_2 + 4 \\
 & x_6 \geq x_3 + 5 \\
 & x_6 \geq x_5 + 2 \\
 & x_7 \geq x_6 + 2 \\
 & x_7 \geq x_4 + 3 \\
 & x_8 \geq x_7 + 1 \\
 & x_{end} \geq x_8 + 1 \\
 & x_{start} = 0
 \end{aligned}$$

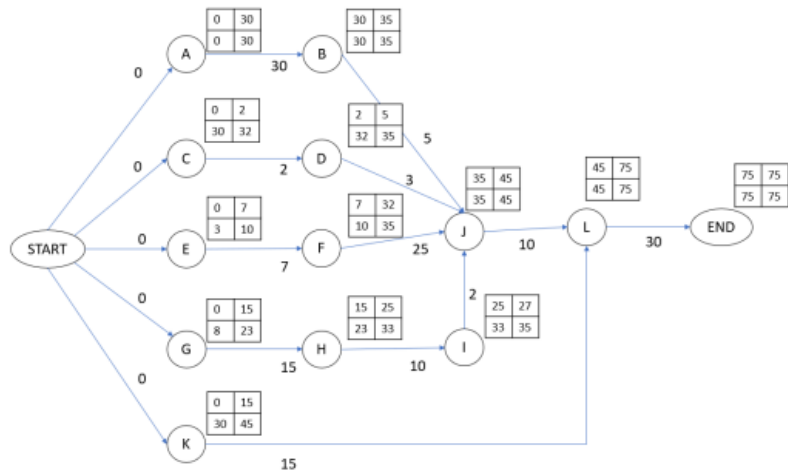
The tight constraints of the optimal solution for this LP, which hold at equality, will give you the critical path.

Question 4.

ES: Earliest Start, EF: Earliest Finish, LS: Latest Start, LF: Latest Finish

ES	EF
LS	LF

(a)-(b)



Activity	Slack
A	0
B	0
C	30
D	30
E	3
F	3
G	8
H	8
I	8
J	0
K	30
L	0

- (c) Since the slack for cutting the onions and mushrooms is 3, the dinner will be delayed 6-3=3 minutes.

If the cutting time is 2 minutes, slack is 10 (latest finish time) – 2 (task duration) = 8 minutes. Since the slack time is more than the delay time, it does not delay the dinner.

- (d) Let N be the set of activities and A be the set of arc in the project network.

Parameters:

a_i : Duration of activity i , for all $i \in N$

d : Project duration (75)

Decision Variables:

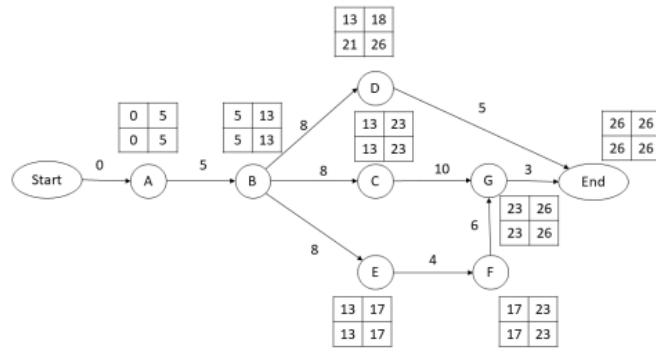
x_i : Start time of activity i , for all $i \in N$

Model:

$$\begin{aligned}
 & \max \sum_{i \in N} x_i \\
 & \text{s.t. } x_j \geq x_i + a_i & (i, j) \in A \\
 & \quad x_{end} \leq d \\
 & \quad x_i \geq 0 & i \in N \\
 & \quad x_{start} = 0
 \end{aligned}$$

Question 5.

(a)



Critical path: A - B - C - G and A - B - E - F - G
Duration: 26

(b) D can be delayed for 8 days.

(c) Decision Variables:

x_j : start time of activity j

y_j : number of days duration of activity j is reduced

Parameters:

c_j : cost of reduction

r_j : max reduction

d_j : duration of activity

$$\min \quad \sum_j (c_j y_j)$$

$$\text{subject to } y_j \leq r_j$$

$$x_j \geq x_i + d_i - y_i \quad \forall (i, j) \in A$$

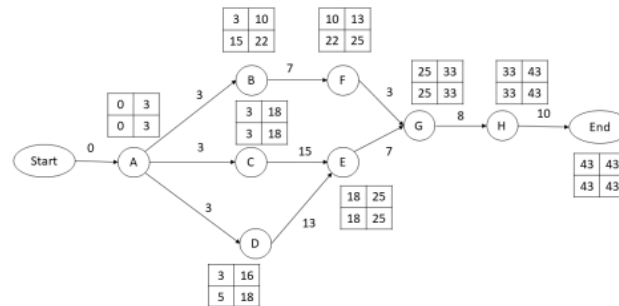
$$x_{start} = 0$$

$$x_{end} \leq 20$$

$$y_j \geq 0 \quad \forall j$$

Question 6.

(a)



Activity	A	B	C	D	E	F	G	H
Slack	0	12	0	2	0	12	0	0

Critical path: A - C - E - G - H

(b) N : set of activities

Decision Variables:

x_i : start time of activity i , $i \in N$

Model:

$$\begin{aligned} \min \quad & \sum_{i \in N} x_i \\ \text{s.t.} \quad & x_{end} \leq 43 \\ & x_{start} = 0 \\ & x_A \geq x_{start} \\ & x_B \geq x_A + 3 \\ & x_C \geq x_A + 3 \\ & x_D \geq x_A + 3 \\ & x_E \geq x_C + 15 \\ & x_E \geq x_D + 13 \\ & x_F \geq x_B + 7 \\ & x_G \geq x_F + 3 \\ & x_G \geq x_F + 7 \\ & x_H \geq x_G + 8 \\ & x_{end} \geq x_H + 10 \\ & x_i \geq 0 \quad \forall i \in N \end{aligned}$$

(c) No, the critical path does not change. The increase in the path A - B - F - G - H is $4 + 7 = 11$, which is smaller than the slack of activities B and F.

(d) **Additional Decision Variable:**

y_i : number of days that the duration of activity i is reduced, $i \in N$

Model:

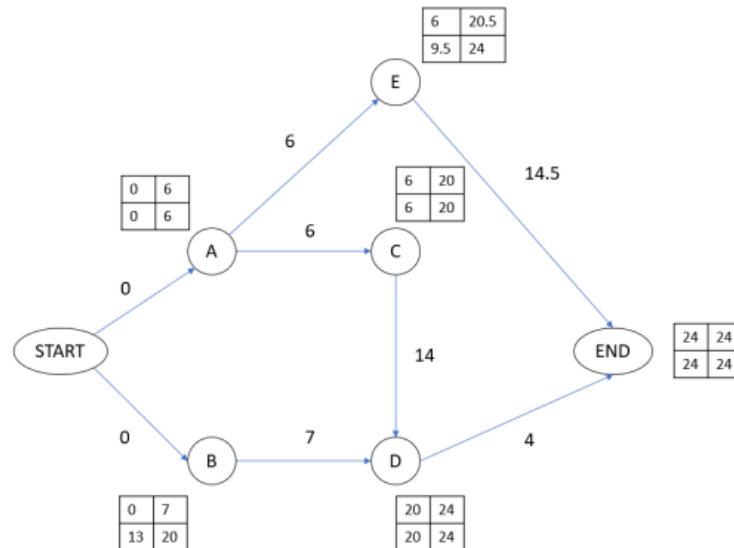
$$\min \quad 10y_A + 21y_B + 15y_C + 23y_D + 12y_E + 14y_F + 13y_G + 20y_H$$

$$\begin{aligned} \text{s.t.} \quad & x_A \geq x_{start} \\ & x_B \geq x_A + 3 - y_A \\ & x_C \geq x_A + 3 - y_A \\ & x_D \geq x_A + 3 - y_A \\ & x_E \geq x_C + 15 - y_C \\ & x_E \geq x_D + 13 - y_D \\ & x_F \geq x_B + 7 - y_B \\ & x_G \geq x_F + 3 - y_F \\ & x_G \geq x_F + 7 - y_E \\ & x_H \geq x_G + 8 - y_G \\ & x_{end} \geq x_H + 10 - y_H \\ & x_{end} \leq 40 \\ & y_A \leq 1 \\ & y_B \leq 2 \\ & y_C \leq 2 \\ & y_D \leq 1 \\ & y_E \leq 2 \\ & y_F \leq 1 \\ & y_G \leq 1 \\ & y_H \leq 3 \\ & y_i \geq 0, \text{ integer } \forall i \in N \\ & x_i \geq 0 \forall i \in N \end{aligned}$$

Question 7.

(a)

Activity	Mean	Variance
A	6	4/36
B	7	196/36
C	14	256/36
D	4	4/36
E	14.5	169/36



The critical path is A-C-D.

(b) Note that

$$\mathbb{P}(18 \leq t \leq 26) = \mathbb{P}(t \leq 26) - \mathbb{P}(t \leq 18)$$

We have, $\sigma_p = 2.708$

Note that $Z_{18} = \frac{18-24}{2.708} = -2.2156$

Then, $\mathbb{P}(t \leq 18) = \mathbb{P}(Z \leq -2.2156) = 0.013$

$Z_{26} = \frac{26-24}{2.708} = 0.739$

$\mathbb{P}(t \leq 26) = \mathbb{P}(Z \leq 0.739) = 0.770$

Hence,

$$\mathbb{P}(18 \leq t \leq 26) = \mathbb{P}(t \leq 26) - \mathbb{P}(t \leq 18) = 0.757$$

Question 8.

$$\begin{aligned}
 & \max \sum_{j=1}^n c_j x_j \\
 & \text{s.t.} \quad \sum_{j=1}^n a_j x_j \leq b_1 \\
 & \quad \quad \sum_{j=1}^n d_j x_j \leq b_2 \\
 & \quad \quad x_j \geq 0 \quad \quad \quad j = 1, \dots, n
 \end{aligned}$$

x_j : Decision at stage j , $j = 1, \dots, n$ (amount of type j items taken)

States: R_1, R_2 (Amount of resources available)

$f_j(R_1, R_2)$: Maximum value of the objective function considering stages (items) $j, j + 1, \dots, n$ with available resources R_1 and R_2 .

So the **recursion function** is as follows:

$$f_j(R_1, R_2) = \max_{\substack{a_j x_j \leq R_1 \\ d_j x_j \leq R_2 \\ x_j \geq 0}} \{c_j x_j + f_{j+1}(R_1 - a_j x_j, R_2 - d_j x_j)\} \quad j = 1, \dots, n$$

Base Case: $f_{n+1}(k_1, k_2) = 0$ for all k_1, k_2

Optimal Solution Value: $f_1(b_1, b_2)$

Then, for the LP given in the question;

Base Case: $f_3(430 - 2x_1 - x_2, 230 - x_2) = 0$

Optimal Solution Value: $f_1(430, 230)$

$$\begin{aligned}
 f_2(430 - 2x_1, 230) &= \max_{\substack{x_2 \leq 430 - 2x_1 \\ x_2 \leq 230 \\ x_2 \geq 0}} \{5x_2 + \underbrace{f_3(430 - 2x_1 - x_2, 230 - x_2)}_0\} \\
 &= \max_{\substack{x_2 \leq 430 - 2x_1 \\ x_2 \leq 230 \\ x_2 \geq 0}} \{5x_2\} \\
 &= \min\{5(430 - 2x_1), 5 \cdot 230\} \\
 &= \begin{cases} -10x_1 + 2150 & \text{if } 100 \leq x_1 \leq 215 \\ 1150 & \text{if } 0 \leq x_1 \leq 100 \end{cases}
 \end{aligned}$$

The minimization in the third equality follows as x_2 cannot be greater than the minimum of $430 - 2x_1$ and 230.

$$\begin{aligned}
 f_1(430, 230) &= \max_{\substack{2x_1 \leq 430 \\ x_1 \geq 0}} \{2x_1 + f_2(430 - 2x_1, 230)\} \\
 &= \max_{x_1} \begin{cases} 2x_1 + 1150 & \text{if } 0 \leq x_1 \leq 100 \\ -8x_1 + 2150 & \text{if } 100 \leq x_1 \leq 215 \end{cases} \\
 &= 1350
 \end{aligned}$$

(You can also graphically verify that $x_1^* = 100$ and $x_2^* = 230$. Therefore, $f_1(430, 230) = 2x_1^* + 5x_2^*$.)

Question 9.

State: s (Capacity available in the beginning of a year)

Stage: t (Years $1, \dots, T$)

Decision: x (Change in the capacity)

$f_t(s)$: the minimum cost of meeting power requirements from year t to T having capacity s in the beginning of year t .

Base Case: $f_{T+1}(s) = 0$ for all $s \geq 0$.

Optimal Solution Value: $f_1(100)$

Recursion Function: $f_t(s) = \min_{s+x \geq r_t} \{c_t(x) + m_t(s+x) + f_{t+1}(0.9(s+x))\}$

Question 10.

b_i : the minimum number of workforce needed for year i

x_i : the number of workers in year i

Salaries: $\$30000x_i$

Hiring cost: $\$10000(x_i - \lfloor 0.9x_{i-1} \rfloor)^+$

Firing cost: $\$20000(\lfloor 0.9x_{i-1} \rfloor - x_i)^+$

Stages: Years $i = 1, \dots, 5$

States: Number of workers in the previous year

$f_i(x_{i-1})$: minimal cost of maintaining a workforce from year i to 5 with workforce (x_{i-1}) in the previous year

Base Case: $f_6(x_5) = 0$

Optimal Solution Value: $f_1(20)$ (Since we started with 20 workers)

Recursion Function:

$$f_i(x_{i-1}) = \min_{b_i \leq x_i \leq \max\{b_1, \dots, b_n\}} \{f_{i+1}(x_i) + 30000x_i + \$10000(x_i - \lfloor 0.9x_{i-1} \rfloor)^+ + \$20000(\lfloor 0.9x_{i-1} \rfloor - x_i)^+\}$$

for $i = 1, \dots, 5$

Question 11.

Stages: Years $i=1, 2, 3, 4, 5$

f_t : minimum cost incurred during periods $t, t+1, \dots, T$, given that at the beginning of period t , the inventory level is zero.

$$f_t = \min(c_{tj} + f_{t+j+1})$$

$$c_{tj} = K + c(d_t + d_{t+1} + \dots + d_{t+j}) + h[j * d_{t+j} + (j-1) * d_{t+j-1} + \dots + d_{t+1}]$$

$$f_6 = 0$$

$$f_5 = 200 + 180 * 2 + f_6$$

$$f_4 = \min \left\{ \begin{array}{l} \text{Produce for period 4} \\ 270 + 200 * 2 + f_5 = 1030* \\ \text{Produce for periods 4, 5} \\ 270 + (200 + 180) * 2 + 180 * 1 + f_6 = 1170 \end{array} \right.$$

$$f_3 = \min \left\{ \begin{array}{l} \text{Produce for period 3} \\ 180 + 240 * 2 + f_4 = 1690* \\ \text{Produce for periods 3, 4} \\ 180 + (240 + 200) * 2 + 200 * 1 + f_5 = 1820 \\ \text{Produce for periods 3, 4, 5} \\ 180 + (240 + 200 + 180) * 2 + 200 * 1 + 180 * 2 + f_6 = 1980 \end{array} \right.$$

$$f_2 = \min \left\{ \begin{array}{l} \text{Produce for period 2} \\ 250 + 220 * 2 + f_3 = 2380* \\ \text{Produce for periods 2, 3} \\ 250 + (220 + 240) * 2 + 240 * 1 + f_4 = 2440 \\ \text{Produce for periods 2, 3, 4} \\ 250 + (220 + 240 + 200) * 2 + 240 * 1 + 200 * 2 + f_5 = 2770 \\ \text{Produce for periods 2, 3, 4, 5} \\ 250 + (220 + 240 + 200 + 180) * 2 + 240 * 1 + 200 * 2 + 180 * 3 + f_6 = 3110 \end{array} \right.$$

$$f_1 = \min \left\{ \begin{array}{l} \text{Produce for period 1} \\ 200 + 210 * 2 + f_2 = 3000 \\ \text{Produce for periods 1, 2} \\ 200 + (210 + 220) * 2 + 220 * 1 + f_3 = 2440* \\ \text{Produce for periods 1, 2, 3} \\ 200 + (210 + 220 + 240) * 2 + 220 * 1 + 240 * 2 + f_4 = 3270 \\ \text{Produce for periods 1, 2, 3, 4} \\ 200 + (210 + 220 + 240 + 200) * 2 + 220 * 1 + 240 * 2 + 200 * 3 + f_5 = 3800 \\ \text{Produce for periods 1, 2, 3, 4, 5} \\ 200 + (210 + 220 + 240 + 200 + 180) * 2 + 220 * 1 + 240 * 2 + 200 * 3 + 180 * 4 + f_6 = 4320 \end{array} \right.$$

If we begin with 0 inventory, it's optimal to produce $d_1 + d_2 = 430$ units in period 1. $d_3 = 240$ units in period 3, $d_4 = 200$ units in period 4, and $d_5 = 180$ units in period 5.