IE 400: Principles of Engineering Management

Simplex Method Continued

Agenda

Simplex for min problems

Alternative optimal solutions

Unboundedness

Degeneracy

Big M method

Two phase method

Simplex for min Problems

Simplex for min Problems

Alternative 1: Use the algorithm for max problems

Remember,

min $f(x_1, x_2, \ldots, x_n) \equiv -\max -f(x_1, x_2, \ldots, x_n)$

minimize $z = 2x_1 - 3x_2$ $\equiv$ -maximize $z = -2x_1 + 3x_2$

subject to

$x_1 + x_2 \leq 4$

$x_1 - x_2 \leq 6$

$x_1, x_2 \geq 0$

subject to

$x_1 + x_2 \leq 4$

$x_1 - x_2 \leq 6$

$x_1, x_2 \geq 0$

Don't forget to negate the optimal value when you solve it as max problem!

Simplex for min Problems

Alternative 2: Direct way

In Row 0 format, choose the variable with the most positive coefficent as the entering variable.

Alternate Optimal Solutions

An Example*

The Dakota Furniture Company manufactures desks, tables, and chairs. The manufacture of each type of furniture requires lumber and two types of skilled labor: finishing and carpentry. The amount of each resource needed to make each type of furniture is given in Table 4.

Currently, 48 board feet of lumber, 20 finishing hours, and 8 carpentry hours are available. A desk sells for $60, a table for $30, and a chair for $20. Dakota believes that demand for desks and chairs is unlimited, but at most five tables can be sold. Because the available resources have already been purchased, Dakota wants to maximize total revenue.

Resource Requirements for Dakota Furniture

Resource	Desk	Table	Chair
Lumber (board ft)	8	6	1
Finishing hours	4	2	1.5
Carpentry hours	2	1.5	0.5

*from our textbook: "Operations Research: Applications and Algorithms" by Wayne Winston

Model

Resource Requirements for Dakota Furniture

Resource	Desk	Table	Chair
Lumber (board ft)	8	6	1
Finishing hours	4	2	1.5
Carpentry hours	2	1.5	0.5

Defining the decision variables as

$$x_1 = \text{number of desks produced}$$
$$x_2 = \text{number of tables produced}$$
$$x_3 = \text{number of chairs produced}$$

it is easy to see that Dakota should solve the following LP:

$$\begin{array}{llll}
\max z = 60x_1 + 30x_2 + 20x_3 & & & \\
\text{s.t.} & 8x_1 + 6x_2 + x_3 \le 48 & & \text{(Lumber constraint)} \\
& 4x_1 + 2x_2 + 1.5x_3 \le 20 & & \text{(Finishing constraint)} \\
& 2x_1 + 1.5x_2 + 0.5x_3 \le 8 & & \text{(Carpentry constraint)} \\
& x_2 \le 5 & & \text{(Limitation on table demand)} \\
& x_1, x_2, x_3 \ge 0 & &
\end{array}$$

Simplex Iterations

Canonical Form 0

Row		Basic Variable
0	$z - 60x_1 - 30x_2 - 20x_3 = 0$	$z = 0$
1	$8x_1 + 6x_2 + x_3 + s_1 = 48$	$s_1 = 48$
2	$4x_1 + 2x_2 + 1.5x_3 + s_2 = 20$	$s_2 = 20$
3	$2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8$	$s_3 = 8$
4	$x_2 + s_4 = 5$	$s_4 = 5$

Canonical Form 1

Row		Basic Variable
Row 0′	$z + 15x_2 - 5x_3 + 30s_3 = 240$	$z = 240$
Row 1′	$- x_3 + s_1 - 4s_3 = 16$	$s_1 = 16$
Row 2′	$- x_2 + 0.5x_3 + s_2 - 2s_3 = 4$	$s_2 = 4$
Row 3′	$x_1 + 0.75x_2 + 0.25x_3 + 0.5s_3 = 4$	$x_1 = 4$
Row 4′	$x_2 + s_4 = 5$	$s_4 = 5$

Canonical Form 2

Row		Basic Variable
0″	$z + 5x_2 + 10s_2 + 10s_3 = 280$	$z = 280$
1″	$- 2x_2 + s_1 + 2s_2 - 8s_3 = 24$	$s_1 = 24$
2″	$- 2x_2 + x_3 + 2s_2 - 4s_3 = 8$	$x_3 = 8$
3″	$x_1 + 1.25x_2 - 0.5s_2 + 1.5s_3 = 2$	$x_1 = 2$
4″	$x_2 + s_4 = 5$	$s_4 = 5$

Alternate Optimal Solutions

Now, reconsider the example with the modification that tables sell for $35 instead of $30.

TABLE 13

Initial Tableau for Dakota Furniture ($35/Table)

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	rhs	Basic Variable	Ratio
1	−60	$\boxed{-35}$	−20	0	0	0	0	0	$z = 0$	
0	8	6	1	1	0	0	0	48	$s_1 = 48$	$\frac{48}{8} = 6$
0	4	2	1.5	0	1	0	0	20	$s_2 = 20$	$\frac{20}{4} = 5$
0	2	1.5	0.5	0	0	1	0	8	$s_3 = 8$	$\frac{8}{2} = 4$*
0	0	1	0	0	0	0	1	5	$s_4 = 5$	None

TABLE 14

First Tableau for Dakota Furniture ($35/Table)

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	rhs	Basic Variable	Ratio
1	0	10	−5	0	0	30	0	240	$z = 240$	
0	0	0	−1	1	0	−4	0	16	$s_1 = 16$	None
0	0	−1	0.5	0	1	−2	0	4	$s_2 = 4$	$\frac{4}{0.5} = 8$*
0	1	0.75	0.25	0	0	0.5	0	4	$x_1 = 4$	$\frac{4}{0.25} = 16$
0	0	1	0	0	0	0	1	5	$s_4 = 5$	None

Alternate Optimal Solutions

TABLE 15

Second (and Optimal) Tableau for Dakota Furniture ($35/Table)

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	rhs	Basic Variable
1	0	0	0	0	10	10	0	280	$z = 280$
0	0	−2	0	1	2	−8	0	24	$s_1 = 24$
0	0	−2	1	0	2	−4	0	8	$x_3 = 8$
0	1	1.25	0	0	−0.5	1.5	0	2	$x_1 = 2$*
0	0	1	0	0	0	0	1	5	$s_4 = 5$

Recall that all basic variables must have a zero coefficient in row 0 (or else they wouldn't be basic variables). However, in our optimal tableau, there is a nonbasic variable, x_2, which also has a zero coefficient in row 0. Let us see what happens if we enter x_2 into the basis. The

TABLE 16

Another Optimal Tableau for Dakota Furniture ($35/Table)

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	rhs	Basic Variable
1	0	0	0	0	10	10	0	280	$z = 280$
0	1.6	0	0	1	1.2	−5.6	0	27.2	$s_1 = 27.2$
0	1.6	0	1	0	1.2	−1.6	0	11.2	$x_3 = 11.2$
0	0.8	1	0	0	−0.4	1.2	0	1.6	$x_2 = 1.6$
0	−0.8	0	0	0	0.4	−1.2	1	3.4	$s_4 = 3.4$

Remember,

change in objective value=|coefficient of entering variable| * ratio test result

Alternate Optimal Solutions

Note that their convex combinations are also optimal.

	x1	**x2**	**x3**	**ObjFnVal**
ObjCoeff	60	35	20	-
opt1	2.00	0.00	8.00	280
opt2	0.00	1.60	11.20	280

lambda	Convex Combinations			ObjFnVal
0.0	0.00	1.60	11.20	280
0.1	0.20	1.44	10.88	280
0.2	0.40	1.28	10.56	280
0.3	0.60	1.12	10.24	280
0.4	0.80	0.96	9.92	280
0.5	1.00	0.80	9.60	280
0.6	1.20	0.64	9.28	280
0.7	1.40	0.48	8.96	280
0.8	1.60	0.32	8.64	280
0.9	1.80	0.16	8.32	280
1.0	2.00	0.00	8.00	280

Alternate Optimal Solutions - Remark

- In Simplex algorithm, alternative solutions are detected when there are *0* valued coefficients for nonbasic variables in row-0 of the optimal tableau.

- If there is no nonbasic variable with a zero coefficient in row 0 of the optimal tableau, the LP has a unique optimal solution.

- Even if there is a nonbasic variable with a zero coefficient in row 0 of the optimal tableau, it is possible that the LP may not have alternative optimal solutions.

Alternate Optimal Solutions

<u>Practice example:</u>

maximize $z = 2x_1 + 4x_2$

subject to

$$x_1 + 2x_2 \leq 5$$

$$x_1 + x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

Alternate Optimal Solutions

<u>Practice example:</u>

maximize $z = 2x_1 + 4x_2$

subject to

$$x_1 + 2x_2 \leq 5$$

$$x_1 + x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

Set of alternate optimal solutions=

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ \frac{5}{2} \\ 0 \\ \frac{3}{2} \end{pmatrix} + (1-\lambda) \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} \; where\ \lambda \in [0,1] \right\}$$

Unboundedness

Unbounded LPs

For some LPs, there exist points in the feasible region for which *z* assumes arbitrarily large (in max problems) or arbitrarily small (in min problems) values. When this occurs, we say the LP is unbounded.

Consider the following LP:

$$
\begin{aligned}
&\text{maximize } z = x_1 + 2x_2 \\
&\text{subject to} \\
&\qquad x_1 - x_2 \le 10 \\
&\qquad x_1 \le 40 \\
&\qquad x_1, x_2 \ge 0
\end{aligned}
$$

Unbounded LPs

Practice Example:

In standard form:

maximize $z = x_1 + 2x_2$

subject to

$$x_1 - x_2 + s_1 \qquad = 10$$

$$x_1 \qquad\qquad + s_2 = 40$$

$$x_1, x_2, s_1, s_2 \geq 0$$

Apply Simplex Method.

Consider $x_1 = 0$; $s_2 = 40$; $x_2 = a$; $s_1 = 10+a$.

The objective function value is then $2a$ for any $a \in \mathbb{R}^+$.

Unbounded LPs

- An unbounded LP occurs in a max (min) problem if there is a nonbasic variable with a negative (positive) coefficient in row 0 and there is no constraint that limits how large we can make this nonbasic variable.
- Specifically, an unbounded LP for a max (min) problem occurs when a variable with a negative (positive) coefficient in row 0 has a non positive coefficient in each constraint.

There is an entering variable but no leaving variable, since ratio test does not give a finite bound!

Degeneracy

Degeneracy

- An LP is a degenerate LP if in a basic feasible solution, one of the basic variables takes on a zero value. This bfs is called degenerate bfs.
- Degeneracy could cost simplex method extra iterations.
- When degeneracy occurs, obj fn value will not increase.
- A cycle in the simplex method is a sequence of K+1 iterations with corresponding bases B_0, …, B_K, B_0 and $K \geq 1$.
- If cycling occurs, then the algorithm will loop, or cycle, forever among a set of basic feasible solutions and never get to an optimal solution.

Example of Cycling

Example 1 *(Degenerate Pivoting)*
Pivot rules:

- *Choose entering variable with largest reduced cost.*
- *Choose leaving variable with smallest subscript.*

$$
\begin{array}{rrcrcrcrcl}
\textit{maximize} & 10x_1 & - & 57x_2 & - & 9x_3 & - & 24x_4 & & \\
\textit{subject to} & 0.5x_1 & - & 5.5x_2 & - & 2.5x_3 & + & 9x_4 & \leq & 0 \\
 & 0.5x_1 & - & 1.5x_2 & - & 0.5x_3 & + & x_4 & \leq & 0 \\
 & x_1 & & & & & & & \leq & 1 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0.
\end{array}
$$

Introduce slacks in initial tableau. Initial basis: $\{5,6,7\}$.

$$
\begin{array}{crrrrrrrcl}
z & -10x_1 & +57x_2 & +9x_3 & +24x_4 & & & & = & 0 \\
 & \boxed{0.5x_1} & -5.5x_2 & -2.5x_3 & +9x_4 & +x_5 & & & = & 0 \\
 & 0.5x_1 & -1.5x_2 & -0.5x_3 & +x_4 & & +x_6 & & = & 0 \\
 & x_1 & & & & & & +x_7 & = & 1.
\end{array}
$$

$$
\begin{array}{crrrrrrrcl}
z & & -53x_2 & -41x_3 & +204x_4 & +20x_5 & & & = & 0 \\
 & x_1 & -11x_2 & -5x_3 & +18x_4 & +2x_5 & & & = & 0 \\
 & & \boxed{4x_2} & +2x_3 & -8x_4 & -x_5 & +x_6 & & = & 0 \\
 & & 11x_2 & +5x_3 & -18x_4 & -2x_5 & & +x_7 & = & 1.
\end{array}
$$

$$
\begin{array}{crrrrrrrcl}
z & & & -14.5x_3 & +98x_4 & +6.75x_5 & +13.25x_6 & & = & 0 \\
 & x_1 & & \boxed{+0.5x_3} & -4x_4 & -0.75x_5 & +2.75x_6 & & = & 0 \\
 & & x_2 & +0.5x_3 & -2x_4 & -0.25x_5 & +0.25x_6 & & = & 0 \\
 & & & -0.5x_3 & +4x_4 & +0.75x_5 & -2.75x_6 & +x_7 & = & 1.
\end{array}
$$

$$
\begin{array}{crrrrrrrcl}
z & +29x_1 & & & -18x_4 & -15x_5 & +93x_6 & & = & 0 \\
 & 2x_1 & & +x_3 & -8x_4 & -1.5x_5 & +5.5x_6 & & = & 0 \\
 & -x_1 & +x_2 & & \boxed{+2x_4} & +0.5x_5 & -2.5x_6 & & = & 0 \\
 & x_1 & & & & & & +x_7 & = & 1.
\end{array}
$$

$$
\begin{array}{crrrrrrrcl}
z & +20x_1 & +9x_2 & & & -10.5x_5 & +70.5x_6 & & = & 0 \\
 & -2x_1 & +4x_2 & +x_3 & & \boxed{+0.5x_5} & -4.5x_6 & & = & 0 \\
 & -0.5x_1 & +0.5x_2 & & +x_4 & +0.25x_5 & -1.25x_6 & & = & 0 \\
 & x_1 & & & & & & +x_7 & = & 1.
\end{array}
$$

$$
\begin{array}{crrrrrrrcl}
z & -22x_1 & +93x_2 & +21x_3 & & & -24x_6 & & = & 0 \\
 & -4x_1 & +8x_2 & +2x_3 & & +x_5 & -9x_6 & & = & 0 \\
 & +0.5x_1 & -1.5x_2 & -0.5x_3 & +x_4 & & \boxed{+x_6} & & = & 0 \\
 & x_1 & & & & & & +x_7 & = & 1.
\end{array}
$$

New basis, $\{5,6,7\}$*, is identical with the first basis, so now we* ***CYCLE!***

*Lecture 13 by Prof. David P. Williamson of Cornell Uni. ORIE Dept.

Degeneracy

- Consider the following example*:

$$\max z = 5x_1 + 2x_2$$
$$\text{s.t.} \quad x_1 + x_2 \le 6$$
$$x_1 - x_2 \le 0$$
$$x_1, x_2 \ge 0$$

A Degenerate LP

z	x_1	x_2	s_1	s_2	rhs	Basic Variable	Ratio
1	-5	-2	0	0	0	$z = 0$	
0	1	1	1	0	6	$s_1 = 6$	6
0	1	-1	0	1	0	$s_2 = 0$	0*

First Tableau for (16)

z	x_1	x_2	s_1	s_2	rhs	Basic Variable	Ratio
1	0	-7	0	5	0	$z = 0$	
0	0	2	1	-1	6	$s_1 = 6$	$\frac{6}{2} = 3$*
0	1	-1	0	1	0	$x_1 = 0$	None

*from our textbook: "Operations Research: Applications and Algorithms" by Wayne Winston

Degeneracy

A Degenerate LP

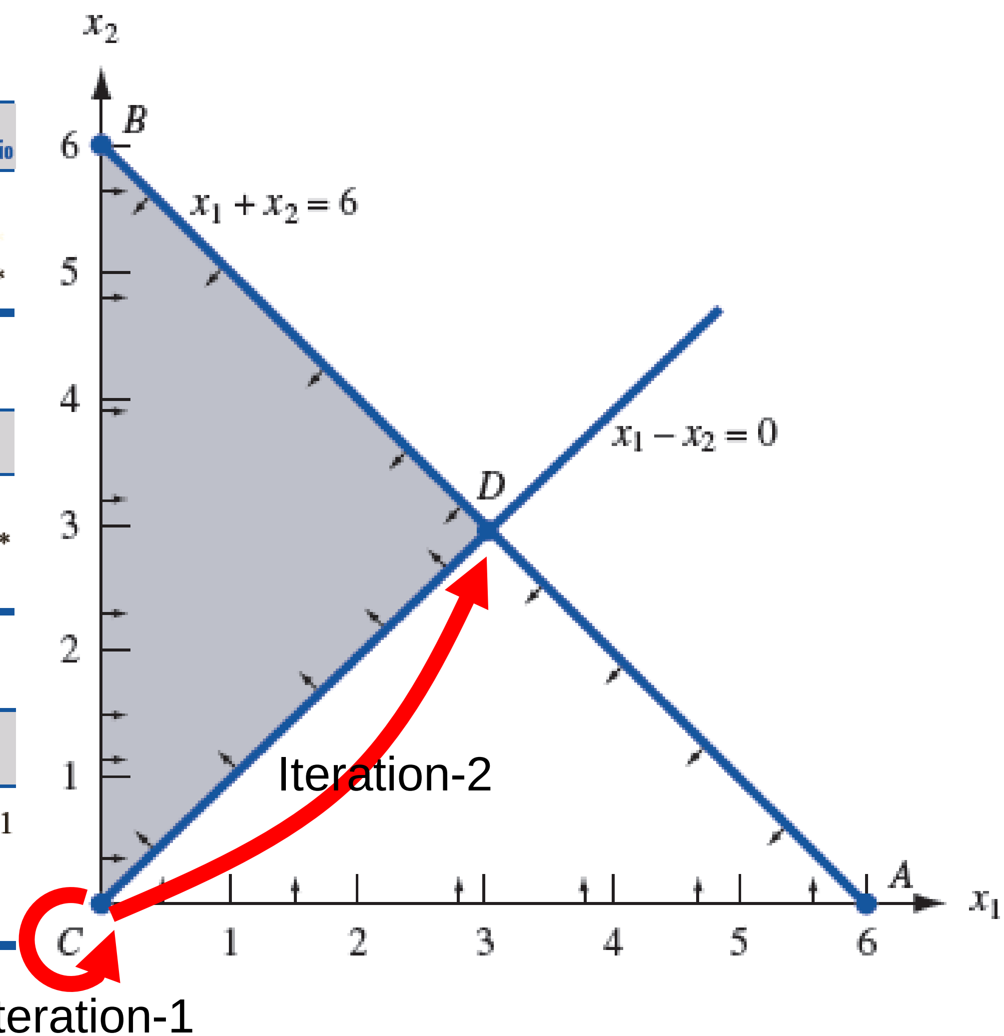
z	x_1	x_2	s_1	s_2	rhs	Basic Variable	Ratio
1	−5	−2	0	0	0	$z = 0$	
0	1	1	1	0	6	$s_1 = 6$	6
0	1	−1	0	1	0	$s_2 = 0$	0*

First Tableau for (16)

z	x_1	x_2	s_1	s_2	rhs	Basic Variable	Ratio
1	0	−7	0	5	0	$z = 0$	
0	0	2	1	−1	6	$s_1 = 6$	$\frac{6}{2} = 3$*
0	1	−1	0	1	0	$x_1 = 0$	None

Optimal Tableau for (16)

z	x_1	x_2	s_1	s_2	rhs	Basic Variable
1	0	0	3.5	1.5	21	$z = 21$
0	0	1	0.5	−0.5	3	$x_2 = 3$
0	1	0	0.5	0.5	3	$x_1 = 3$



Degeneracy

- In the simplex algorithm, degeneracy is detected when there is a tie for the minimum ratio test. In the following iteration, the solution is degenerate.

- <u>Example (for practice):</u>

$$\text{maximize } z = 3x_1 + 9x_2$$

subject to

$$x_1 + 4x_2 \leq 8$$
$$x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Degeneracy – Bland’s Rule

- When degeneracy occurs, obj fn value will not increase and algorithm cycles same basic feasible solutions. To prevent this:
- Bland showed that cycling can be avoided by applying the following rules (assume that the slack and excess variables are numbered x_{n+1}, x_{n+2} etc.)
- Choose an entering variable (in a max problem) the variable with a negative coefficient in row 0 that has the smallest index
- If there is a tie in the ratio test, then break the tie by choosing the winner of the ratio test so that the variable leaving the basis has the smallest index
- Using Bland’s rule, the Simplex Algorithm terminates in finite time with optimal solution (i.e. no cycling)

Start Applying Bland’s rule when a degenerate bfs is encountered

Big-M Method

Alternative 1 for finding and initial bfs.

Big M Method

- The simplex method algorithm requires a starting bfs.
- Previous problems have found starting bfs by using the slack variables as our basic variables.
  - If an LP has ≥ or = constraints, however, a starting bfs may not be readily apparent.
- In such a case, the Big M method may be used to solve the problem.

Big M Method

- Consider the following LP:

$$
\begin{aligned}
\text{minimize} \quad & z = 2x_1 + 3x_2 \\
\text{subject to} \quad & 0.5x_1 + 0.25x_2 \leq 4 \\
& x_1 + 3x_2 \geq 20 \\
& x_1 + x_2 = 10 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Big M Method

- Consider the following LP:

$$\begin{array}{ll} \text{minimize} & z = 2x_1 + 3x_2 \\ \text{subject to} & 0.5x_1 + 0.25x_2 \leq 4 \\ & x_1 + 3x_2 \geq 20 \\ & x_1 + x_2 = 10 \\ & x_1, x_2 \geq 0 \end{array} \equiv \begin{array}{ll} -\text{maximize} & z = -2x_1 - 3x_2 \\ \text{subject to} & 0.5x_1 + 0.25x_2 \leq 4 \\ & x_1 + 3x_2 \geq 20 \\ & x_1 + x_2 = 10 \\ & x_1, x_2 \geq 0 \end{array}$$

Big M Method

- The LP in standard form has z and s_1 which could be used for BVs but row 2 would violate sign restrictions and row 3 no readily apparent basic variable.

$$
\begin{aligned}
&\text{Row 0:} \quad z + 2x_1 + 3x_2 = 0 \\
&\text{Row 1:} \quad 0.5x_1 + 0.25x_2 + s_1 = 4 \\
&\text{Row 2:} \quad x_1 + 3x_2 - e_2 = 20 \\
&\text{Row 3:} \quad x_1 + x_2 = 10
\end{aligned}
$$

- In order to use the simplex method, a bfs is needed.
  - To remedy the predicament, **artificial variables** are created.
  - The variables will be labeled according to the row in which they are used.

Big M Method

$$
\begin{array}{llr}
\text{Row 0:} & z + 2x_1 + 3x_2 & = 0 \\
\text{Row 1:} & 0.5x_1 + 0.25x_2 + s_1 & = 4 \\
\text{Row 2:} & x_1 + 3x_2 - e_2 + a_2 & = 20 \\
\text{Row 3:} & x_1 + x_2 + a_3 & = 10
\end{array}
$$

- In the optimal solution, all artificial variables must be set equal to zero.
  - To accomplish this, in a min LP, a term Ma_i is added to the objective function for each artificial variable a_i.
  - For a max LP, the term $-Ma_i$ is added to the objective function for each a_i.
  - M represents some very large number.

Big M Method

- The modified LP in standard form then becomes:

$$\text{Row 0: } z + 2x_1 + 3x_2 + Ma_2 + Ma_3 = 0$$

$$\text{Row 1: } 0.5x_1 + 0.25x_2 + s_1 = 4$$

$$\text{Row 2: } x_1 + 3x_2 - e_2 + a_2 = 20$$

$$\text{Row 3: } x_1 + x_2 + a_3 = 10$$

- Modifying the objective function this way makes it extremely costly for an artificial variable to be positive. The optimal solution should force $a_2 = a_3 = 0$ (whenever possible!)

Big M Method

Row	**Basic Variable**	**z**	x_1	x_2	s_1	e_2	a_2	a_3	**RHS**
0	z	1	2	3	0	0	M	M	0
1	s_1	0	1/2	1/4	1	0	0	0	4
2	a_2	0	1	3	0	-1	1	0	20
3	a_3	0	1	1	0	0	0	1	10

Because basic variables a_2 and a_3 have nonzero Row 0 coefficients, do elementary row operations to zero them out: Add -M(Row2) and –M(Row 3) to Row 0 to achieve a proper Row 0 for simplex to start

Big M Method

Row	**Basic Variable**	**z**	x_1	x_2	s_1	e_2	a_2	a_3	**RHS**
~~0~~	~~z~~	~~1~~	~~2~~	~~3~~	~~0~~	~~0~~	~~M~~	~~M~~	~~0~~
0	z	1	2-2M	3-4M	0	M	0	0	-30M
1	s_1	0	1/2	1/4	1	0	0	0	4
2	a_2	0	1	3	0	-1	1	0	20
3	a_3	0	1	1	0	0	0	1	10

Big M Method

Row	Basic Variable	z	x_1	x_2	s_1	e_2	a_2	a_3	RHS	Min Ratio Test
0	z	1	2-2M	3-4M	0	M	0	0	-30M	
1	s_1	0	1/2	1/4	1	0	0	0	4	16
2	a_2	0	1	3	0	-1	1	0	20	20/3
3	a_3	0	1	1	0	0	0	1	10	10

Big M Method

Row	Basic Variable	z	x_1	x_2	s_1	e_2	a_2	a_3	RHS	Min Ratio Test
0	z	1	1-2M/3	0	0	1-M/3		0	-20-10M/3	
1	s_1	0	5/12	0	1	1/12		0	7/3	28/5
2	x_2	0	1/3	1	0	-1/3		0	20/3	20
3	a_3	0	2/3	0	0	1/3		1	10/3	5

Since a_2 has left the basis, we can forget about that column for good!

Big M Method

Row	Basic Variable	z	x_1	x_2	s_1	e_2	a_2	a_3	RHS
0	z	1	0	0	0	1/2			-20-10M/3
1	s_1	0	0	0	1	-1/8			1/4
2	x_2	0	0	1	0	-1/2			5
3	x_1	0	1	0	0	1/2			5

Since a_3 has left the basis, we can also forget about that column for good!

Big M Method

Row	Basic Variable	z	x_1	x_2	s_1	e_2	~~a_2~~	~~a_3~~	RHS
0	z	1	0	0	0	1/2			-25
1	s_1	0	0	0	1	-1/8			1/4
2	x_2	0	0	1	0	-1/2			5
3	x_1	0	1	0	0	1/2			5

Final Tableau!
The optimal solution is $z=-25$, $x_1=x_2 = 5$, $s_1=1/4$, $e_2=0$.

Big M Method

- The optimal solution (for the original min problem) is $z=25$, $x_1=x_2 = 5$, $s_1=1/4$, $e_2=0$.
- *Remark*: once an artificial variable is NB, it can be dropped from the future tableaus since it will never become basic again.
- *Remark*: when choosing the entering variable, remember that M is a very large number. For example,
    - $4M-2 > 3M + 5000$,
    - $-6M-5 < -3M - 10000$.

Big M Method

- Another example LP:

maximize $z = x_1 + x_2$

subject to $x_1 - x_2 \geq 1$

$-x_1 + x_2 \geq 1$

$x_1, x_2 \geq 0$

Big M Method

Row	Basic Variable	z	x_1	x_2	e_1	e_2	a_1	a_2	RHS
0	z	1	-1	-1	0	0	M	M	0
1	a_1	0	1	-1	-1	0	1	0	1
2	a_2	0	-1	1	0	-1	0	1	1

Because basic variables a_1 and a_2 have nonzero Row 0 coefficients, do elementary row operations to zero them out: Add -M(Row1) and –M(Row 2) to Row 0 to achieve a proper Row 0 for simplex to start

Big M Method

Row	Basic Variable	z	x_1	x_2	e_1	e_2	a_1	a_2	RHS
~~0~~	~~z~~	~~1~~	~~-1~~	~~-1~~	~~0~~	~~0~~	~~M~~	~~M~~	~~0~~
0	z	1	-1	-1	M	M	0	0	-2M
1	a_1	0	1	-1	-1	0	1	0	1
2	a_2	0	-1	1	0	-1	0	1	1

Because basic variables a_1 and a_2 have nonzero Row 0 coefficients, do elementary row operations to zero them out: Add -M(Row1) and –M(Row 2) to Row 0 to achieve a proper Row 0 for simplex to start

Big M Method

Row	Basic Variable	z	x_1	x_2	e_1	e_2	a_1	a_2	RHS
0	z	1	-1	-1	M	M	0	0	-2M
1	a_1	0	1	-1	-1	0	1	0	1
2	a_2	0	-1	1	0	-1	0	1	1

Big M Method

Row	Basic Variable	**z**	x_1	x_2	e_1	e_2	a_1	a_2	**RHS**
0	z	1	0	-2	M-1	M		0	-2M+1
1	x_1	0	1	-1	-1	0		0	1
2	a_2	0	0	0	-1	-1		1	2

The final tableau indicates that the solution is unbounded (no exiting variable) and one of the artificial variables is nonzero.

Thus, the original LP is infeasible.

Big M Method

1. Modify the constraints so that the rhs of each constraint is nonnegative. Identify each constraint that is now an = or ≥ constraint.
2. Convert each inequality constraint to standard form (add a slack variable for ≤ constraints, add an excess variable for ≥ constraints).
3. For each ≥ or = constraint, add artificial variables. Add sign restriction $a_i \geq 0$.
4. Let M denote a very large positive number. Add (for each artificial variable) Ma_i to min problem objective functions or $-Ma_i$ to max problem objective functions.
5. Since each artificial variable will be in the starting basis, all artificial variables must be eliminated from row 0 before beginning the simplex. Remembering M represents a very large number, solve the transformed problem by the simplex.

Big M Method

- If all artificial variables in the optimal solution equal zero, the solution is ?

Big M Method

- If all artificial variables in the optimal solution equal zero, the solution is optimal.
- If any artificial variables are positive in the optimal solution, the problem is ?

Big M Method

- If all artificial variables in the optimal solution equal zero, the solution is optimal.
- If any artificial variables are positive in the optimal solution, the problem is infeasible.
- When the LP (with the artificial variables) is solved, the final tableau may indicate that the LP is unbounded. If the final tableau indicates the LP is unbounded and all artificial variables in this tableau equal zero, then the original LP is ?

Big M Method

- If all artificial variables in the optimal solution equal zero, the solution is optimal.
- If any artificial variables are positive in the optimal solution, the problem is infeasible.
- When the LP (with the artificial variables) is solved, the final tableau may indicate that the LP is unbounded. If the final tableau indicates the LP is unbounded and all artificial variables in this tableau equal zero, then the original LP is unbounded. If the final tableau indicates that the LP is unbounded and at least one artificial variable is positive, then the original LP is ?

Big M Method

- If all artificial variables in the optimal solution equal zero, the solution is optimal.
- If any artificial variables are positive in the optimal solution, the problem is infeasible.
- When the LP (with the artificial variables) is solved, the final tableau may indicate that the LP is unbounded. If the final tableau indicates the LP is unbounded and all artificial variables in this tableau equal zero, then the original LP is unbounded. If the final tableau indicates that the LP is unbounded and at least one artificial variable is positive, then the original LP is infeasible.

Big M Method - Remark

For computer programs, it is difficult to determine how large M should be. Generally, M is chosen to be at least 100 times larger than the largest coefficient in the original objective function. The introduction of such large numbers into the problem can cause roundoff errors and other computational difficulties. For this reason, most computer codes solve LPs by using the two-phase simplex method.

Two-Phase Simplex

Alternative 2 for finding and initial bfs.

Two-Phase Simplex Method - Example

- Solve the same LP with the two-phase method

$$\begin{array}{ll} \text{minimize} & z = 2x_1 + 3x_2 \\ \text{subject to} & 0.5x_1 + 0.25x_2 \le 4 \\ & x_1 + 3x_2 \ge 20 \\ & x_1 + x_2 = 10 \\ & x_1, x_2 \ge 0 \end{array} \quad \equiv \quad \begin{array}{ll} -\ \text{maximize} & z = -2x_1 - 3x_2 \\ \text{subject to} & 0.5x_1 + 0.25x_2 \le 4 \\ & x_1 + 3x_2 \ge 20 \\ & x_1 + x_2 = 10 \\ & x_1, x_2 \ge 0 \end{array}$$

Two-Phase Simplex Method - Example

- Solve the same LP with the two-phase method

$$
\begin{aligned}
\text{maximize} \quad & z = -2x_1 - 3x_2 \\
\text{subject to} \quad & 0.5x_1 + 0.25x_2 \le 4 \\
& x_1 + 3x_2 \ge 20 \\
& x_1 + x_2 = 10 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

$$
\begin{aligned}
\text{Row 0:} \quad & z + 2x_1 + 3x_2 = 0 \\
\text{Row 1:} \quad & 0.5x_1 + 0.25x_2 + s_1 = 4 \\
\text{Row 2:} \quad & x_1 + 3x_2 - e_2 + a_2 = 20 \\
\text{Row 3:} \quad & x_1 + x_2 + a_3 = 10
\end{aligned}
$$

Two-Phase Simplex Method - Example

Phase I: Change objective function and solve the following LP

$$
\begin{aligned}
\text{Min} \quad & w = a_2 + a_3 \\
\text{s.t.} \quad & 0.5x_1 + 0.25x_2 + s_1 = 4 \\
& x_1 + 3x_2 - e_2 + a_2 = 20 \\
& x_1 + x_2 + a_3 = 10
\end{aligned}
$$

Two-Phase Simplex Method – Phase I

Row	Basic Variable	w	x_1	x_2	s_1	e_2	a_2	a_3	RHS
0	w	1	0	0	0	0	-1	-1	0
1	s_1	0	1/2	1/4	1	0	0	0	4
2	a_2	0	1	3	0	-1	1	0	20
3	a_3	0	1	1	0	0	0	1	10

Because basic variables a_2 and a_3 have nonzero Row 0 coefficients, do elementary row operations to zero them out: Add (Row2) and (Row 3) to Row 0 to achieve a proper Row 0 for simplex to start

Two-Phase Simplex Method – Phase I

Row	Basic Variable	w	x_1	x_2	s_1	e_2	a_2	a_3	RHS
~~0~~	~~w~~	~~1~~	~~0~~	~~0~~	~~0~~	~~0~~	~~-1~~	~~-1~~	~~0~~
0	w	1	2	4	0	-1	0	0	30
1	s_1	0	1/2	1/4	1	0	0	0	4
2	a_2	0	1	3	0	-1	1	0	20
3	a_3	0	1	1	0	0	0	1	10

Because basic variables a_2 and a_3 have nonzero Row 0 coefficients, do elementary row operations to zero them out: Add (Row2) and (Row 3) to Row 0 to achieve a proper Row 0 for simplex to start

Two-Phase Simplex Method – Phase I

Row	Basic Variable	w	x_1	x_2	s_1	e_2	a_2	a_3	RHS	Min Ratio Test
0	w	1	2	4	0	-1	0	0	30	
1	s_1	0	1/2	1/4	1	0	0	0	4	16
2	a_2	0	1	3	0	-1	1	0	20	20/3
3	a_3	0	1	1	0	0	0	1	10	10

Two-Phase Simplex Method – Phase I

Row	Basic Variable	w	x_1	x_2	s_1	e_2	a_2	a_3	RHS	Min Ratio Test
0	w	1	2/3	0	0	1/3		0	10/3	
1	s_1	0	5/12	0	1	1/12		0	7/3	28/5
2	x_2	0	1/3	1	0	-1/3		0	20/3	20
3	a_3	0	2/3	0	0	1/3		1	10/3	5

Since a_2 has left the basis, we can forget about that column for good!

Two-Phase Simplex Method – Phase I

Row	**Basic Variable**	**w**	x_1	x_2	s_1	e_2	a_2	a_3	**RHS**
0	w	1	0	0	0	0			0
1	s_1	0	0	0	1	-1/8			1/4
2	x_2	0	0	1	0	-1/2			5
3	x_1	0	1	0	0	1/2			5

Since a_3 has left the basis, we can also forget about that column for good! This is the end of Phase I. Since w=0, move to Phase II with this bfs.

Two-Phase Simplex Method – Phase II

$$\begin{aligned}
\text{maximize} \quad & z = -2x_1 - 3x_2 \\
\text{subject to} \quad & 0.5x_1 + 0.25x_2 \le 4 \\
& x_1 + 3x_2 \ge 20 \\
& x_1 + x_2 = 10 \\
& x_1, x_2 \ge 0
\end{aligned}$$

Row 0: $z + 2x_1 + 3x_2 = 0$

Row 1: $0.5x_1 + 0.25x_2 + s_1 = 4$

Row 2: $x_1 + 3x_2 - e_2 + a_2 = 20$

Row 3: $x_1 + x_2 + a_3 = 10$

Two-Phase Simplex Method – Phase II

Row	**Basic Variable**	**z**	x_1	x_2	s_1	e_2	**RHS**
0	z	1	2	3	0	0	0
1	s_1	0	0	0	1	-1/8	1/4
2	x_2	0	0	1	0	-1/2	5
3	x_1	0	1	0	0	1/2	5

Bring in the original objective.
Zero out the nonzero coefficients of basic variables in Row 0.
Add -2(Row3) – 3(Row2) to Row 0

Two-Phase Simplex Method – Phase II

Row	**Basic Variable**	**z**	x_1	x_2	s_1	e_2	**RHS**
~~0~~	~~z~~	~~1~~	~~2~~	~~3~~	~~0~~	~~0~~	~~0~~
0	z	1	0	0	1	1/2	-25
1	s_1	0	0	0	1	-1/8	1/4
2	x_2	0	0	1	0	-1/2	5
3	x_1	0	1	0	0	1/2	5

Bring in the original objective.
Zero out the nonzero coefficients of basic variables in Row 0.
Add -2(Row3) – 3(Row2) to Row 0

Two-Phase Simplex Method – Phase II

Row	Basic Variable	**z**	x_1	x_2	s_1	e_2	**RHS**
0	z	1	0	0	1	1/2	-25
1	s_1	0	0	0	1	-1/8	1/4
2	x_2	0	0	1	0	-1/2	5
3	x_1	0	1	0	0	1/2	5

This is a max problem so the current tableau is optimal!
End of Phase II
The optimal solution is $z=-25$, $x_1=x_2 = 5$, $s_1=1/4$, $e_2=0$.

Two-Phase Simplex Method - Summary

- When a basic feasible solution is not readily available, the two-phase simplex method may be used as an alternative to the Big M method.
- In this method, artificial variables are added to the same constraints, then a bfs to the original LP is found by solving Phase I LP.
- In Phase I LP, the objective function is to minimize the sum of all artificial variables.
- At completion, reintroduce the original LPs objective function and determine the optimal solution to the original LP.

Two-Phase Simplex Method – Phase I

- Replace the objective function with:

  min w = (sum of all artificial variables).

- The act of solving the Phase I LP will force the artificial variables to be zero.
- Since the artificial variables are in the starting basis, we should create zeros for each artificial variables in row 0 and then solve the minimization problem.
- Solving the Phase I LP will result in one of the following three cases:

Two-Phase Simplex Method – Phase I cont'

- CASE 1: The optimal value of w is greater than zero. In this case, the original LP has no feasible solution (which means at least one of the $a_i > 0$).
- CASE 2: The optimal value of w is equal to zero, and no artificial a_i's are in the optimal Phase I basis. Then a basic feasible solution to the original problem is found. Continue to Phase II by bringing in the original objective function.
- CASE 3: The optimal value of w is zero and at least one artificial variable is in the optimal Phase I basis. Recall that we wanted a bfs of the original problem. But this means that we don't want the basis to contain any artificial variables. Then either we can perform an additional pivot and get rid of the artificial variable, or there was a redundant constraint and we can delete the constraint with the artificial variable.

So that in the end, we will get w is zero and no artificial variables are in the optimal Phase I basis.

Two-Phase Simplex Method – Phase II

- Drop all columns in the optimal Phase I tableau that correspond to the artificial variables. And combine the original objective function with the constraints from the optimal Phase I tableau.
- Make sure that all basic variables have zero in row 0 by performing elementary row operations.
- Solve the problem starting with this tableau. The optimal solution to the Phase II LP is the optimal solution to the original LP.

Why does it work?

- Suppose the original LP is feasible. Then this feasible solution (with all a_i's being zero) is feasible in the Phase I LP with w=0. w=0 is the lowest value that w can get. Hence, it is optimal to Phase I. Therefore, if the original LP has a feasible solution then the optimal Phase I solution will have w = 0.
- If the original LP is infeasible then the only way to obtain a feasible solution to the Phase I LP is to let at least one artificial variable to be positive. In this situation, w > 0, hence optimal w will be greater than zero.

Two-Phase Simplex Method - Remarks

- As with the Big M method, the column for any artificial variable may be dropped from future tableaus as soon as the artificial variable leaves the basis.
- The Big M method and Phase I of the two-phase method make the same sequence of pivots. The two-phase method does not cause roundoff errors and other computational difficulties.