MODELING

(Integer Programming Examples)
**Integer Programs**

*Integer programs*: a linear program plus the additional constraints that some or all of the variables must be integer valued.

We also permit “$x_j \in \{0, 1\}$,” or equivalently, “$x_j$ is binary”

This is a shortcut for writing the constraints:

$$0 \leq x_j \leq 1 \text{ and } x_j \text{ integer.}$$
Why integer programs?

- Advantages of restricting variables to take on integer values
  - More realistic
  - More flexibility

- Disadvantages
  - More difficult to model
  - Can be much more difficult to solve
A 2-Variable Integer program

maximize \quad 3x + 4y

subject to \quad 5x + 8y \leq 24

\quad x, y \geq 0 \text{ and integer}

What is the optimal solution?
The Feasible Region

Question: What is the optimal integer solution?

What is the optimal linear solution?

Can one use linear programming to solve the integer program?
A rounding technique that sometimes is useful, and sometimes not.

Solve LP (ignore integrality) get \( x=24/5, y=0 \) and \( z = 14 \ 2/5 \).
Round, get \( x=5, y=0 \), infeasible!
Truncate, get \( x=4, y=0 \), and \( z = 12 \).
Same solution value at \( x=0, \ y=3 \).
Optimal is \( x=3, \ y=1 \), and \( z = 13 \).
Integer Programming:

So far, we have considered problems under the following assumptions:

i. Proportionality & Additivity
ii. Divisibility
iii. Certainty

While many problems satisfy these assumptions, there are other problems in which we will need to either relax these assumptions.

For example, consider the following bus scheduling problem:
Each bus starts to operate at the beginning of a period and operates for 8 consecutive hours and then receives 16 hours off. For example, a bus operating from 4 AM to 12 PM must be off between 12 PM and 4 AM.

Find the minimum # of busses to provide the required service.
Integer Programming:

Bus Scheduling:

<table>
<thead>
<tr>
<th>Periods</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hours</td>
<td>0-4 AM</td>
<td>4-8 AM</td>
<td>8 AM – 12 PM</td>
<td>12 - 4 PM</td>
<td>4 – 8 PM</td>
<td>8 PM-0AM</td>
</tr>
<tr>
<td>Min. # of buses required</td>
<td>4</td>
<td>8</td>
<td>10</td>
<td>7</td>
<td>12</td>
<td>4</td>
</tr>
</tbody>
</table>

How do you define your decision variables here?

a) $x_i$: # of busses operating in period $i$, $i=1,2,...,6$

or

b) $x_i$: # of busses starting to operate in period $i$, $i=1,2,...,6$
Integer Programming:

The alternative “b” makes modeling of this problem much easier. It is not possible to keep track of the total number of busses under alternative “a”. (WHY?)

The model is as follows:
The alternative “b” makes modeling of this problem much easier. It is not possible to keep track of the total number of busses under alternative “a”. (WHY?)

The model is as follows:

\[
\begin{align*}
& \min \ x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \\
\text{s.t.} & \\
& x_1 + x_6 \geq 4 \\
& x_1 + x_2 \geq 8 \\
& x_2 + x_3 \geq 10 \\
& x_3 + x_4 \geq 7 \\
& x_4 + x_5 \geq 12 \\
& x_5 + x_6 \geq 4 \\
& x_i \geq 0, \ x_i \text{ integer, } i = 1, \ldots, 6
\end{align*}
\]
Integer Programming:

Note that each decision variable represents the number of busses. Clearly, it does not make sense to talk about 3.5 busses because busses are not divisible. Hence, we should impose the additional condition that each of $x_1, \ldots, x_6$ can only take integer values.

If some or all of the variables are restricted to be integer valued, we call it an integer program (IP). This last example is hence an IP problem.
Work Scheduling:
Burger Queen would like to determine the min # of kitchen employees. Each employee works six days a week and takes the seventh day off.

<table>
<thead>
<tr>
<th>Days</th>
<th>Mon</th>
<th>Tue</th>
<th>Wed</th>
<th>Thu</th>
<th>Fri</th>
<th>Sat</th>
<th>Sun</th>
</tr>
</thead>
<tbody>
<tr>
<td>Min. # of workers</td>
<td>8</td>
<td>6</td>
<td>7</td>
<td>6</td>
<td>9</td>
<td>11</td>
<td>9</td>
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</tbody>
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How do you formulate this problem?

The objective function?

The constraints?
Work Scheduling:

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</table>

\( x_i : \# \text{ of workers who are not working on day } i, \)
\( i = 1, \ldots, 7 \) (1: Mon, 7: Sun)
Work Scheduling:

Decision Variables:

IP Model:
The Assignment Problem:

- There are $n$ people and $n$ jobs to carry out.
- Each person is assigned to carry out exactly one job.
- If person $i$ is assigned to job $j$, then the cost is $c_{ij}$.

Find an assignment of $n$ people to $n$ jobs with minimum total cost.
The Assignment Problem:

\[ x_{ij} = \begin{cases} 
1, & \text{person } i \text{ is assigned to job } j \\
0, & \text{otherwise}
\end{cases} \]

\[ i = 1, 2, \ldots, n; \ j = 1, 2, \ldots, n \]

Such variables are called BINARY VARIABLES and offer great flexibility in modeling. They are ideal for YES/NO decisions.

So how is the problem formulated?
The Assignment Problem:

The problem is formulated as follows:

\[
\min \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}x_{ij}
\]

s.t.

\[
\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1,...,n \quad \text{(person i is assigned to exactly one job)}
\]

\[
\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1,...,n \quad \text{(job j is assigned to exactly one person)}
\]

\[
x_{ij} \in \{0,1\}, \ i = 1,...,n; \ j = 1,...,n
\]

This is still an example of an IP problem since \(x_{ij}\) can only take one of the two integer values 0 and 1.
The 0-1 Knapsack Problem

You are going on a trip. Your suitcase has a capacity of \( b \) kg’s.

- You have \( n \) different items that you can pack into your suitcase 1, \ldots, \( n \).
- Item \( i \) has a weight of \( a_i \) and gives you a utility of \( c_i \).

How should you pack your suitcase to maximize total utility?
The 0-1 Knapsack Problem

Decision Variables:

IP Model:
Workforce Scheduling

Monthly demand for a shoe company

<table>
<thead>
<tr>
<th>Month</th>
<th>Demand</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>300</td>
</tr>
<tr>
<td>2</td>
<td>500</td>
</tr>
<tr>
<td>3</td>
<td>100</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
</tr>
</tbody>
</table>

- A pair of shoes requires 4 hours of labor and $5 worth of raw materials
- 50 pairs of shoes in stock
- Monthly inventory cost is charged for shoes in stock at the end of each month at $30/pair
- Currently 3 workers are available
- Salary $1500/month
- Each worker works 160 hours/month
- Hiring a worker costs $1600
- Firing a worker costs $2000
Workforce Scheduling

Monthly demand for a shoe company

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- Hiring a worker costs $1600
- Firing a worker costs $2000

How to meet demand with minimum total cost?
IP Model For Workforce Scheduling

Decision Variables:

LP Model:
The Cutting Stock Problem:
A paper company manufactures and sells rolls of paper of fixed width in 5 standard lengths: 5, 8, 12, 15 and 17 m. Demand for each size is given below.
It produces 25 m length rolls and cuts orders from its stock of this size. What is the min # of rolls to be used to meet the total demand?

<table>
<thead>
<tr>
<th>Length (meters)</th>
<th>5</th>
<th>8</th>
<th>12</th>
<th>15</th>
<th>17</th>
</tr>
</thead>
<tbody>
<tr>
<td>Demand</td>
<td>40</td>
<td>35</td>
<td>30</td>
<td>25</td>
<td>20</td>
</tr>
</tbody>
</table>
The Cutting Stock Problem:

First, identify the cutting patterns:

1. 17 8
2. 17 5
3. 15 8
4. 15 5 5
5. 12 12
6. 12 8 5
7. 12 5 5
8. 8 8 8
9. 8 8 5
10. 8 5 5 5
11. 5 5 5 5 5
The Cutting Stock Problem:

Then how do you define the decision variable?

\[ x_j : \text{# of rolls cut according to pattern } j. \]
The Cutting Stock Problem:

Then how do you define the decision variable?

\[ x_j : \# \text{ of rolls cut according to pattern } j. \]

What about the constraints?

i.e. How do you write the constraint for the demand of 12 m rolls of paper?

We look at the patterns that contain 12 m pieces, and then write the constraint based on \( x_j \).

<table>
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The Cutting Stock Problem:

The cutting patterns:

1. 17 8
2. 17 5
3. 15 8
4. 15 5 5
5. 12 12
6. 12 8 5
7. 12 5 5
8. 8 8 8
9. 8 8 5
10. 8 5 5 5
11. 5 5 5 5 5
The Cutting Stock Problem:

The cutting patterns:

1. 17 8
2. 17 5
3. 15 8
4. 15 5 5
5. 12 12
6. 12 8 5
7. 12 5 5
8. 8 8 8
9. 8 8 5
10. 8 5 5 5
11. 5 5 5 5 5

\[2x_5 + x_6 + x_7 \geq 30\]

Why do we have “≥” sign?
The Cutting Stock Problem:

Then how do you define the decision variable?

\[ x_j : \# \text{ of rolls cut according to pattern } j. \]

What about the constraints?

i.e. What about demand for 5 m?

<table>
<thead>
<tr>
<th>Length (meters)</th>
<th>5</th>
<th>8</th>
<th>12</th>
<th>15</th>
<th>17</th>
</tr>
</thead>
<tbody>
<tr>
<td>Demand</td>
<td>40</td>
<td>35</td>
<td>30</td>
<td>25</td>
<td>20</td>
</tr>
</tbody>
</table>
The Cutting Stock Problem:

The cutting patterns:

1. 17 8
2. 17 5
3. 15 8
4. 15 5 5
5. 12 12
6. 12 8 5
7. 12 5 5
8. 8 8 8
9. 8 8 5
10. 8 5 5 5
11. 5 5 5 5 5
The Cutting Stock Problem:

The cutting patterns:

1. 17 8
2. 17 5
3. 15 8
4. 15 5 5
5. 12 12
6. 12 8 5
7. 12 5 5
8. 8 8 8
9. 8 8 5
10. 8 5 5 5
11. 5 5 5 5 5

\[ x_2 + 2x_4 + x_6 + 2x_7 + x_9 + 3x_{10} + 5x_{11} \geq 40 \]
The Cutting Stock Problem:

The problem is formulated as follows:

$$\min \sum_{j=1}^{11} x_j$$

s.t.

$$x_2 + 2x_4 + x_6 + 2x_7 + x_9 + 3x_{10} + 5x_{11} \geq 40$$
$$x_1 + x_3 + x_6 + 3x_8 + 2x_9 + x_{10} \geq 35$$
$$2x_5 + x_6 + x_7 \geq 30$$
$$x_3 + x_4 \geq 25$$
$$x_1 + x_2 \geq 20$$

$$x_j \geq 0, \ j = 1, \ldots, 11, \ x_j \text{ integer.}$$
Project Selection

A manager should choose from among 10 different projects:

\[ p_j : \text{profit if we invest in project } j, j = 1, \ldots, 10 \]
\[ c_j : \text{cost of project } j, j = 1, \ldots, 10 \]
\[ q : \text{total budget available} \]

There are also some additional requirements:

i. Projects 3 and 4 cannot be chosen together

ii. Exactly 3 projects are to be selected

iii. If project 2 is selected, then so is project 1

iv. If project 1 is selected, then project 3 should not be selected

v. Either project 1 or project 2 is chosen, but not both

vi. Either both projects 1 and 5 are chosen or neither are chosen

vii. At least two and at most four projects should be chosen from the set \{1, 2, 7, 8, 9, 10\}

viii. Neither project 5 nor 6 can be chosen unless either 3 or 4 is chosen

ix. At least two of the investments 1, 2, 3 are chosen, or at most three of the investments 4, 5, 6, 7, 8 are chosen.

x. If both 1 and 2 are chosen, then at least one of 9 and 10 should also be chosen.
Project Selection

Under these requirements, the objective is to maximize the Profit.

Decision Variables

\[ x_j = \begin{cases} 
1, & \text{if project } j \text{ is chosen} \\
0, & \text{otherwise} 
\end{cases} \]

\( j = 1, \ldots, 10 \)
Project Selection

Objective function is:

\[
\text{max} \sum_{j=1}^{10} p_j x_j
\]
Objective function is:

$$\max \sum_{j=1}^{10} p_j x_j$$

First the budget constraint:
Project Selection

Objective function is:

\[ \text{max} \sum_{j=1}^{10} p_j x_j \]

First the budget constraint:

\[ \sum_{j=1}^{10} c_j x_j \leq q \]
Project Selection

Objective function is:

$$\max \sum_{j=1}^{10} p_j x_j$$

First the budget constraint:

$$\sum_{j=1}^{10} c_j x_j \leq q$$

i. Projects 3 and 4 cannot be chosen together:
Project Selection

Objective function is:

$$\max \sum_{j=1}^{10} p_jx_j$$

First the budget constraint:

$$\sum_{j=1}^{10} c_jx_j \leq q$$

i. Projects 3 and 4 cannot be chosen together:

$$x_3 + x_4 \leq 1$$
Project Selection

ii Exactly 3 projects are to be chosen:

\[ x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} = 3 \]

iii. If project 2 is chosen then so is project 1:

\[ x_2 \leq x_1 \]
Project Selection

iv If project 1 is selected, then project 3 should not be selected

\[ x_1 \leq 1 - x_3 \]

v. Either project 1 or project 2 is chosen, but not both

\[ x_1 + x_2 = 1 \]
vi. Either both projects 1 and 5 are chosen or neither are chosen

\[ x_1 = x_5 \quad \text{or} \quad x_1 - x_5 = 0 \]

vii. At least two and at most four projects should be chosen from the set \{1, 2, 7, 8, 9, 10\}:

\[ x_1 + x_2 + x_7 + x_8 + x_9 + x_{10} \geq 2 \]

\[ x_1 + x_2 + x_7 + x_8 + x_9 + x_{10} \leq 4 \]
viii. Neither project 5 nor 6 can be chosen unless either 3 or 4 is chosen:

Equivalently, \[ x_3 + x_4 = 0 \] then \[ x_5 = x_6 = 0 \]

\[ x_5 \leq x_3 + x_4 \]
\[ x_6 \leq x_3 + x_4 \]

Another possibility,

\[ x_5 + x_6 \leq 2(x_3 + x_4) \]
ix. At least two of the investments 1, 2, 3 are chosen, or at most three of the investments 4, 5, 6, 7, 8 are chosen:

It is more difficult to express this requirement in mathematical terms due to “or”!

\[ x_1 + x_2 + x_3 \geq 2 \quad \text{or} \quad x_4 + x_5 + x_6 + x_7 + x_8 \leq 3 \]

So we want to ensure that at least one of (I) and (II) is satisfied.
Either-Or Constraints

The following situation commonly occurs in mathematical programming problems. We are given two constraints of the form,

\[ f(x_1, x_2, \ldots, x_n) \leq 0 \]  \hspace{1cm} (I)

\[ g(x_1, x_2, \ldots, x_n) \leq 0 \]  \hspace{1cm} (II)

Either-Or Constraints

The following situation commonly occurs in mathematical programming problems. We are given two constraints of the form,

$$f(x_1, x_2, \ldots, x_n) \leq 0 \quad (I)$$

$$g(x_1, x_2, \ldots, x_n) \leq 0 \quad (II)$$

We want to make sure that at least one of (I) and (II) is satisfied. These are often called either-or constraints.
Either-Or Constraints

The following situation commonly occurs in mathematical programming problems. We are given two constraints of the form,

\[ f(x_1, x_2, \ldots, x_n) \leq 0 \quad \text{(I)} \]
\[ g(x_1, x_2, \ldots, x_n) \leq 0 \quad \text{(II)} \]

We want to make sure that at least one of (I) and (II) is satisfied. These are often called either-or constraints. Adding two new constraints as below to the formulation will ensure that at least one of (I) or (II) is satisfied,

\[ f(x_1, x_2, \ldots, x_n) \leq M y \quad \text{(I')} \]
\[ g(x_1, x_2, \ldots, x_n) \leq M (1 - y) \quad \text{(II')} \]

where \( y \in \{0, 1\} \).
Either-Or Constraints

\( M \) is a number chosen large enough to ensure that 
\( f(x_1, x_2, \ldots, x_n) \leq M \) and \( g(x_1, x_2, \ldots, x_n) \leq M \) are satisfied 
for all values of \( x_1, x_2, \ldots, x_n \) that satisfy the other 
constraints in the problem.
Either-Or Constraints

\[ f(x_1, x_2, \ldots, x_n) \leq My \quad (I') \]
\[ g(x_1, x_2, \ldots, x_n) \leq M(1 - y) \quad (II') \]

where \( y \in \{0, 1\} \).
Either-Or Constraints

\[ f(x_1, x_2, \ldots, x_n) \leq My \quad (\text{I'}) \]
\[ g(x_1, x_2, \ldots, x_n) \leq M(1 - y) \quad (\text{II'}) \]

where \( y \in \{0, 1\} \).

a) If \( y=0 \), then (I’) and (II’) becomes:
\[ f(x_1, x_2, \ldots, x_n) \leq 0, \]
and
\[ g(x_1, x_2, \ldots, x_n) \leq M. \]

Hence, this means that (I) must be satisfied, and possibly (II) is satisfied.
Either-Or Constraints

\[
\begin{align*}
f(x_1, x_2, \ldots, x_n) & \leq My & (I') \\
g(x_1, x_2, \ldots, x_n) & \leq M(1 - y) & (II')
\end{align*}
\]

where \( y \in \{0, 1\} \).
Either-Or Constraints

\[ f(x_1, x_2, ...., x_n) \leq My \]  
\[ g(x_1, x_2, ...., x_n) \leq M(1 - y) \]

where \( y \in \{0, 1\} \).

b) If \( y=1 \), then (I’) and (II’) becomes:

\[ f(x_1, x_2, \ldots, x_n) \leq M, \]
\[ g(x_1, x_2, \ldots, x_n) \leq 0. \]

Hence, this means that (II) must be satisfied, and possibly (I) is satisfied.
Either-Or Constraints

\[ f(x_1, x_2, \ldots, x_n) \leq My \quad (I') \]
\[ g(x_1, x_2, \ldots, x_n) \leq M(1 - y) \quad (II') \]

where \( y \in \{0, 1\} \).

Therefore whether \( y=0 \) or \( y=1 \), (I') and (II') ensure that at least one of (I) and (II) is satisfied.

\[ f(x_1, x_2, \ldots, x_n) \leq 0 \quad (I) \]
\[ g(x_1, x_2, \ldots, x_n) \leq 0 \quad (II) \]
Now, let us return to our problem:

ix. At least two of the investments 1, 2, 3 are chosen, or at most three of the investments 4, 5, 6, 7, 8 are chosen:

\[
\begin{align*}
x_1 + x_2 + x_3 & \geq 2 \quad \text{or} \quad x_4 + x_5 + x_6 + x_7 + x_8 \leq 3 \\
\text{(I)} & \quad \text{(II)}
\end{align*}
\]

This is an example of either-or constraint. Hence by defining the \( f(.) \) and \( g(.) \) functions, (I) and (II) becomes:

\[
\begin{align*}
2 - (x_1 + x_2 + x_3) & \leq 0 \\
-3 + x_4 + x_5 + x_6 + x_7 + x_8 & \leq 0
\end{align*}
\]
Now, let us return to our problem:

ix. At least two of the investments 1, 2, 3 are chosen, or at most three of the investments 4, 5, 6, 7, 8 are chosen:

\[
\begin{align*}
 x_1 + x_2 + x_3 & \geq 2 \quad \text{or} \quad x_4 + x_5 + x_6 + x_7 + x_8 \leq 3 \\
 (I) & \quad (II)
\end{align*}
\]

Hence by defining a binary variable \( y \), the formulation becomes:

\[
\begin{align*}
 2 - (x_1 + x_2 + x_3) & \leq M \ y \\
-3 + x_4 + x_5 + x_6 + x_7 + x_8 & \leq M (1 - y) \\
y & \in \{0,1\}
\end{align*}
\]

We have to find \( M \) large enough to ensure that \( f(.) \leq M \) and \( g(.) \leq M \) for all values of \( x_1, \ldots, x_n \).

\( M=2 \) works!
Now, let us return to our problem:

ix. At least two of the investments 1, 2, 3 are chosen, or at most three of the investments 4, 5, 6, 7, 8 are chosen:

\[ x_1 + x_2 + x_3 \geq 2 \quad \text{or} \quad x_4 + x_5 + x_6 + x_7 + x_8 \leq 3 \]

(I) \hspace{1cm} (II)

Putting \( M=2 \) and rearranging the terms, we have:

\[ x_1 + x_2 + x_3 \geq 2(1 - y) \]
\[ x_4 + x_5 + x_6 + x_7 + x_8 \leq 5 - 2y \]
\[ y \in \{0, 1\} \]
Project Selection

x. If both 1 and 2 are chosen, then at least one of 9 and 10 should also be chosen.

Both projects 1 and 2 are chosen if \( x_1 + x_2 = 2 \).

At least one of projects 9 and 10 is chosen if \( x_9 + x_{10} \geq 1 \).

Hence, we have \( x_1 + x_2 = 2 \Rightarrow x_9 + x_{10} \geq 1 \).

So, how to express this as a linear constraint?
If-Then Constraints

The following situation commonly occurs in mathematical programming problems. We want to ensure that:

If a constraint $f(x_1,\ldots,x_n) > 0$ is satisfied,
   then $g(x_1,\ldots,x_n) \leq 0$ must be satisfied;

while if a constraint $f(x_1,\ldots,x_n) > 0$ is not satisfied,
   then $g(x_1,\ldots,x_n) \leq 0$ may or may not be satisfied.
If-Then Constraints

In other words, we want to ensure that $f(x_1,\ldots,x_n) > 0$ implies $g(x_1,\ldots,x_n) \leq 0$.

Logically, this is equivalent to:

$$f(x_1, x_2, \ldots, x_n) \leq 0 \quad (I)$$

or

$$g(x_1, x_2, \ldots, x_n) \leq 0 \quad (II)$$
If-Then Constraints

\[ f(x_1, x_2, \ldots, x_n) \leq 0 \quad (I) \]

or

\[ g(x_1, x_2, \ldots, x_n) \leq 0 \quad (II) \]

Now, these are either-or type constraints and we can handle them as before by defining a binary variable, say \( z \):

\[
\begin{align*}
  f(x_1, x_2, \ldots, x_n) & \leq Mz \\
g(x_1, x_2, \ldots, x_n) & \leq M(1 - z)
\end{align*}
\]

where \( z \in \{0,1\} \)

Where as before \( M \) is a large enough number such that \( f(.) \leq M \) and \( g(.) \leq M \) holds for all values \( x_1, \ldots, x_n \).
Now, let us return to our problem:

x. If both 1 and 2 are chosen, then at least one of 9 and 10 should also be chosen.

\[ x_1 + x_2 = 2 \implies x_9 + x_{10} \geq 1 \]

We need to define \( f(.) > 0 \) and \( g(.) \leq 0 \) constraints:

\[ x_1 + x_2 > 1 \implies x_9 + x_{10} \geq 1 \]

equivalently

\[ x_1 + x_2 - 1 > 0 \implies 1 - (x_9 + x_{10}) \leq 0 \]
Project Selection

x. If both 1 and 2 are chosen, then at least one of 9 and 10 should also be chosen.

\[ x_1 + x_2 = 2 \implies x_9 + x_{10} \geq 1 \]

\[ x_1 + x_2 - 1 > 0 \implies 1 - (x_9 + x_{10}) \leq 0 \]

Now converting to either-or, we have:

\[ x_1 + x_2 - 1 \leq 0 \quad (I) \]

or

\[ 1 - (x_9 + x_{10}) \leq 0 \quad (II) \]
Project Selection

\[ x_1 + x_2 - 1 \leq 0 \]  \hspace{1cm} (I)

\text{or}

\[ 1 - (x_9 + x_{10}) \leq 0 \]  \hspace{1cm} (II)

To linearize the above constraints, define a binary variable, say \( z \) and a large enough value \( M \):

\[ x_1 + x_2 - 1 \leq Mz \]

\[ 1 - (x_9 + x_{10}) \leq M(1 - z) \]

\( z \in \{0,1\} \)

M=1 works and after rearranging we have:

\[ x_1 + x_2 \leq 1 + z \]

\[ x_9 + x_{10} \geq z \]

\( z \in \{0,1\} \)
Project Selection

x. If both 1 and 2 are chosen, then at least one of 9 and 10 should also be chosen.

\[ x_1 + x_2 \leq 1 + z \]
\[ x_9 + x_{10} \geq z \]
\[ z \in \{0,1\} \]
Project Selection

Finally our integer programming model with linear constraints becomes:

\[
\begin{align*}
\max & \quad \sum_{j=1}^{10} p_j x_j \\
\sum_{j=1}^{10} c_j x_j & \le q \\
x_3 + x_4 & \le 1 \quad \text{(i)} \\
x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} & = 3 \quad \text{(ii)} \\
x_2 & \le x_1 \quad \text{(iii)} \\
x_1 & \le 1 - x_3 \quad \text{(iv)} \\
x_1 + x_2 & = 1 \quad \text{(v)} \\
x_1 - x_2 & = 0 \quad \text{(vi)}
\end{align*}
\]
Project Selection
Model cont.

\[
\begin{align*}
&x_1 + x_2 + x_7 + x_8 + x_9 + x_{10} \geq 2 \\
&x_1 + x_2 + x_7 + x_8 + x_9 + x_{10} \leq 4
\end{align*}
\]  
\[(vii)\]

\[
\begin{align*}
&x_5 \leq x_3 + x_4 \\
&x_6 \leq x_3 + x_4
\end{align*}
\]  
\[(viii)\]

\[
\begin{align*}
&x_1 + x_2 + x_3 \geq 2(1 - y) \\
&x_4 + x_5 + x_6 + x_7 + x_8 \leq 5 - 2y
\end{align*}
\]  
\[(ix)\]

\[
\begin{align*}
&x_1 + x_2 \leq 1 + z \\
&x_9 + x_{10} \geq z
\end{align*}
\]  
\[(x)\]

\[
x_1, x_2, \ldots, x_{10} \in \{0,1\}
\]

\[
y, z \in \{0,1\}
\]
Modeling Fixed Costs

- A set of potential warehouses: 1, . . . , m
- A set of clients : 1, . . . , n

- If warehouse $i$ is used, then there is a fixed cost of $f_i$, $i = 1,...,m$
- $c_{ij}$: unit cost of transportation from warehouse $i$ to client $j$, $i = 1,...,m$; $j = 1,...,n$
- $a_i$: capacity of warehouse $i$ in terms of # of units, $i = 1,...,m$
- $d_j$: # of units demanded by client $j$, $j = 1,...,n$

Note: More than one warehouse can supply the client.
Modeling Fixed Costs

How to formulate the mathematical model in order to minimize total fixed costs and transportation costs while meeting the demand and capacity constraints?

Decision Variables:

IP Model:
The Set Covering Problem
You wish to decide where to locate hospitals in a city. Suppose that there are 5 different towns in the city.

<table>
<thead>
<tr>
<th>Town</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>15</td>
<td>20</td>
<td>10</td>
<td>16</td>
</tr>
<tr>
<td>2</td>
<td>15</td>
<td>10</td>
<td>8</td>
<td>14</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>6</td>
<td>12</td>
<td>15</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
<td>17</td>
<td>7</td>
<td>9</td>
<td>10</td>
</tr>
</tbody>
</table>

Travel time between each town and the nearest hospital should not be more than 10 minutes. What is the minimum number of hospitals?
The Set Covering Problem

Decision Variables:

IP Model:
Location Problems (Cover, Center, Median)

Ankara municipality is providing service to 6 regions. The travel times (in minutes) between these regions, say $t_{ij}$ values are:

<table>
<thead>
<tr>
<th></th>
<th>R1</th>
<th>R2</th>
<th>R3</th>
<th>R4</th>
<th>R5</th>
<th>R6</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>0</td>
<td>10</td>
<td>20</td>
<td>30</td>
<td>30</td>
<td>20</td>
</tr>
<tr>
<td>R2</td>
<td>10</td>
<td>0</td>
<td>25</td>
<td>35</td>
<td>20</td>
<td>10</td>
</tr>
<tr>
<td>R3</td>
<td>20</td>
<td>25</td>
<td>0</td>
<td>15</td>
<td>30</td>
<td>20</td>
</tr>
<tr>
<td>R4</td>
<td>30</td>
<td>35</td>
<td>15</td>
<td>0</td>
<td>15</td>
<td>25</td>
</tr>
<tr>
<td>R5</td>
<td>30</td>
<td>20</td>
<td>30</td>
<td>15</td>
<td>0</td>
<td>14</td>
</tr>
<tr>
<td>R6</td>
<td>20</td>
<td>10</td>
<td>20</td>
<td>25</td>
<td>14</td>
<td>0</td>
</tr>
</tbody>
</table>
Location Problems (Cover, Center, Median)

a) Cover:
The municipality wants to build fire stations in some of these regions. Find the minimum number of fire stations required if the municipality wants to ensure that there is a fire station within 15 minute drive time from each region. *(Similar to ??)*

**Decision Variables:**

**IP Model:**
b) Median:
The municipality now wants to build 2 post offices. There is going to be a back and forth trip each day from each post office to each one of the regions assigned to this post office. Formulate a model which will minimize the total trip time.

Decision Variables:

IP Model:
Location Problems (Cover, Center, Median)

c)

What if additional to part (b), you have the restriction that a post office cannot provide service to more than 4 regions including itself?

Add the following constraint to above model:
Location Problems (Cover, Center, Median)

d) Center:
The municipality wants to build 2 fire stations so that the time to reach the farthest region is kept at minimum.

Decision Variables:

IP Model:
Location Problems (Cover, Center, Median)
e) Consider the median problem in part (b). Assume that in addition to the back and forth trips between the post offices and the assigned regions, there will be a trip between the two post offices. The municipality wants to locate the offices in such a way that the total trip time is minimized.

Decision Variables:

IP Model:
Traveling Salesman Problem (TSP):

A sales person lives in city 1. He has to visit each of the cities $2, \ldots, n$ exactly once and return home.

Let $c_{ij}$ be the travel time from city $i$ to city $j$, $i = 1, \ldots, n; j = 1, \ldots, n$.

What is the order in which she should make her tour to finish as quickly as possible?

What should be the decision variables here?
Traveling Salesman Problem (TSP):

A sales person lives in city 1. He has to visit each of the cities 2,...,n exactly once and return home.

Let $c_{ij}$ be the travel time from city $i$ to city $j$, $i = 1, \ldots, n$; $j = 1, \ldots, n$.

What is the order in which she should make her tour to finish as quickly as possible?

What should be the decision variables here?

$$x_{ij} = \begin{cases} 
1, & \text{if he goes directly from city } i \text{ to city } j \\
0, & \text{otherwise}
\end{cases}$$

$i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n$
Traveling Salesman Problem (TSP):

Since each city has to be visited only once, the formulation is:

\[
\min \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}
\]

s.t.

\[
\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, \ldots, n
\]

\[
\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, \ldots, n
\]

\[
x_{ii} = 0, \quad i = 1, \ldots, n
\]

\[
x_{ij} \in \{0, 1\}, \quad \forall \ i, j = 1, \ldots, n
\]
Traveling Salesman Problem (TSP):

With this formulation, a feasible solution is:

\[ x_{12} = x_{23} = x_{31} = 1, \]
\[ x_{45} = x_{54} = 1. \]

But this is a “subtour”! How can we eliminate “subtours”?

We need to impose the constraints:

\[ \sum_{i \in S} \sum_{j \not\in S} x_{ij} \geq 1 \quad \forall S \subseteq N, \ S \neq \emptyset \]
Traveling Salesman Problem (TSP):

For example, for n=5 and S={1,2,3} this constraint implies:

\[ x_{14} + x_{15} + x_{24} + x_{25} + x_{34} + x_{35} \geq 1 \]

The difficulty is that there are a total of \(2^n - 1\) such constraints. And that is a lot!

- For \(n = 100\), \(2^n - 1 \approx 1.27 \times 10^{30}\).

How about the total number of valid tours? That is \((n - 1)!\)

- For \(n = 100\), \((n - 1)! \approx 9.33 \times 10^{155}\).

TSP is one of the hardest problems. But several efficient algorithms exist that can find “good” solutions.
A company produces animal feed mixture. The mix contains 2 active ingredients and a filler material. 1 kg of feed mix must contain a minimum quantity of each of the following 4 nutrients:

<table>
<thead>
<tr>
<th>Nutrients</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Required amount (in grams) in 1 kg of mix</td>
<td>90</td>
<td>50</td>
<td>20</td>
<td>2</td>
</tr>
</tbody>
</table>

The ingredients have the following nutrient values and costs:

<table>
<thead>
<tr>
<th>Ingredients</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>Cost/kg</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ingredient1 (gram/kg)</td>
<td>100</td>
<td>80</td>
<td>40</td>
<td>10</td>
<td>$40</td>
</tr>
<tr>
<td>Ingredient2 (gram/kg)</td>
<td>200</td>
<td>150</td>
<td>20</td>
<td>N/A</td>
<td>$60</td>
</tr>
</tbody>
</table>
Class Exercises (Blending problem revisited with binary variables):

a) What should be the amount of active ingredients and filler material in 1 kg of feed mix in order to satisfy the nutrient requirements at minimum total cost?

Decision Variables:

Model (LP or IP?)
Class Exercises (Blending problem revisited with binary variables):

b) Suppose now that we have the following additional limitations

   i. If we use any of ingredient 2, we incur a fixed cost of $15 (in addition to the $60 unit cost).

   ii. It is enough to only satisfy 3 of the nutrition requirements rather than 4.

Decision Variables (Additional to those for Part a):

Model: (LP or IP?)
Some Comments on Integer Programming:

- There are often multiple ways of modeling the same integer program.
- Solvers for integer programs are extremely sensitive to the formulation (not true for LPs).
- Not as easy to model.
- Not as easy to solve.