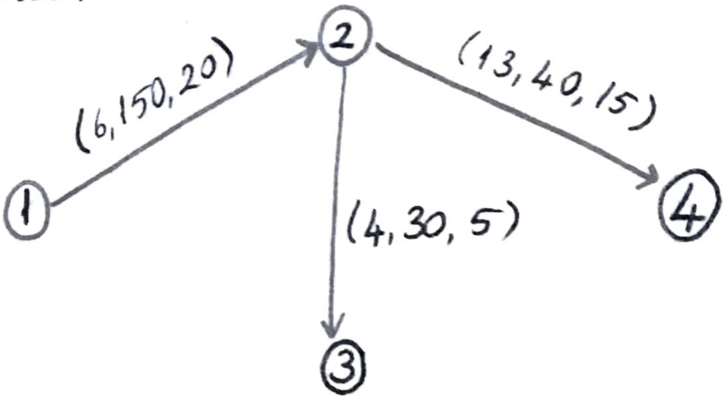
QUESTION 2

I use this notation : (c, u, x) as cost, capacity and flow respectively.

→ a basic feasible solution (bfs) is as follows:



Tree arcs: (1,2), (2,4), (2,3)

Co-tree arcs: (1,3), (3,4)

For tree arcs:

$\pi_1 - \pi_2 = c_{12} = 6$

$\pi_2 - \pi_3 = c_{23} = 4$

$\pi_2 - \pi_4 = c_{24} = 13$

Let $\pi_4 = 0$. Then,

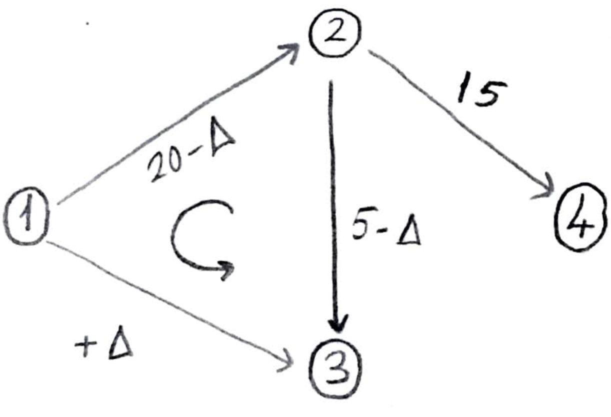
$\pi_2 = 13,\ \pi_3 = 9,\ \pi_1 = 19$.

For co-tree arcs, $x_{13} = 0$ and $x_{34} = 0$. So, check:

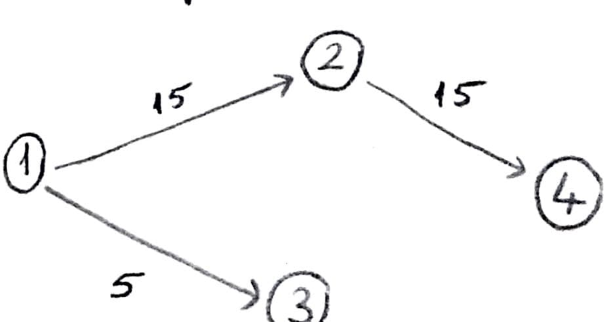
$\pi_1 - \pi_3 \overset{?}{\leq} c_{13} \rightarrow 19 - 9 = 10 \rightarrow 10 \nleq 2$ ✗

$\pi_3 - \pi_4 \overset{?}{\leq} c_{34} \rightarrow 9 - 0 = 9 \rightarrow 9 \nleq 5$ ✗

Arc (1,3) will be the entering arc. From now on, the numbers on the arcs show the flows only.



$\Delta = \min\{5, 20, 50\} = 5$



Tree arcs: (1,2), (2,4), (1,3)

Co-tree arcs: (3,4), (2,3)

For tree arcs: $\pi_1 - \pi_2 = c_{12} = 6,\ \pi_2 - \pi_4 = c_{24} = 13,\ \pi_1 - \pi_3 = c_{13} = 2$

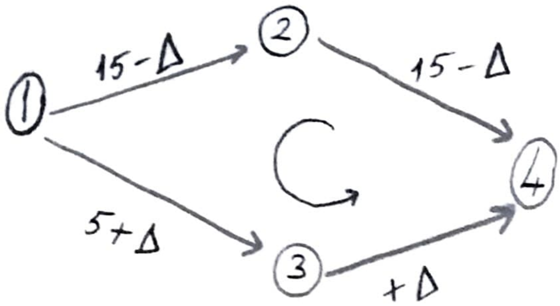
Let $\pi_4 = 0$. Then: $\pi_2 = 13$, $\pi_1 = 19$, $\pi_3 = 17$

For co-tree arcs: $x_{23} = 0$ and $x_{34} = 0$. So, check:

$$\pi_2 - \pi_3 \overset{?}{\leq} c_{23} \rightarrow 13 - 17 \leq 4 \quad ✓$$

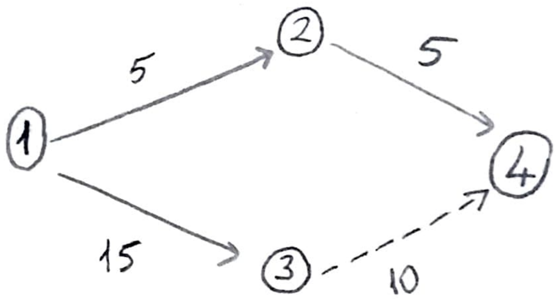
$$\pi_3 - \pi_4 \overset{?}{\leq} c_{34} \rightarrow 17 - 0 \nleq 5 \quad \times$$

Arc (3,4) will be the entering arc.



$\Delta = \min\{45, 10, 15, 15\} = 10$

$\downarrow$



Tree arcs: (1,2), (1,3), (2,4)

Co-tree arcs: (2,3), (3,4)

For tree arcs: $\pi_1 - \pi_2 = c_{12} = 6$

$\pi_1 - \pi_3 = c_{13} = 2$

$\pi_2 - \pi_4 = c_{24} = 13$

Let $\pi_4 = 0$

Then, $\pi_2 = 13$, $\pi_1 = 19$, $\pi_3 = 17$

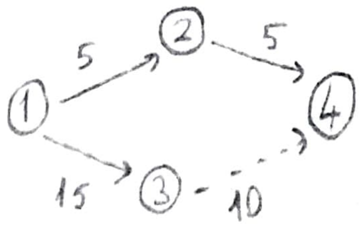
For co-tree arcs: $x_{23} = 0 \rightarrow$ check: $\pi_2 - \pi_3 \overset{?}{\leq} c_{23} \rightarrow 13 - 17 \leq 4$ ✓

$x_{34} = 10 = U_{34} \rightarrow$ check: $\pi_3 - \pi_4 \overset{?}{\geq} 5 \rightarrow 17 - 0 \geq 5$ ✓

Optimal is found.

$\boxed{\text{optimal Cost}: (5)(6) + (5)(13) + (15)(2) + (10)(5) = 175}$

Initial cost was $(20)(6) + (4)(5) + (15)(13) = 335$



Question 3:

6 possible location with

$$f = \begin{array}{cccccc} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \\ \end{array}$$
$f = [5, 6, 4, 4, 2, 3]$, and 5 customers with $P_j = 3$.

A location covers a customer if it within 55 km dist.

Let say if Location (L_J) covers $c_i = 1$ $i = (1, \dots, 5)$
otherwise $= 0$ $J = (1, \dots, 6)$

Loc. \ Cust	1	2	3	4	5
1	0	1	1	0	1
2	0	0	1	0	1
3	0	1	0	0	0
4	1	1	0	1	0
5	0	0	0	1	1
6	1	0	0	1	0

Solve

$$\max \sum_J w_j$$

s.t. $\sum_{j}^{5} a_{ij} w_j \leq f_i \quad \forall i$ $i \in \{0, \dots, 6\}$

$0 \leq w_j \leq P_j$

$$\max \quad w_1 + \dots + w_5$$

s.t

$w_2 + w_3 + w_5 \leq 5$

$w_3 + w_5 \leq 6$

$w_2 \leq 4$

$w_1 + w_2 + w_4 \leq 4$

$w_4 + w_5 \leq 2$

$w_1 + w_4 \leq 3$

$0 \leq w_j \leq 3$

L\C	1	2	3	4	5	f_i
1	0	1	1	0	1	$\cancel{5}\ (5-1) = \cancel{4}\ (4-3) = \cancel{1}\ 0$
2	0	0	1	0	1	$\cancel{6}\ (6-3) = \cancel{3}\ (3-1) = 2$
3	0	1	0	0	0	$\cancel{4}\ (4-1) = 3$
4	1	1	0	1	0	$\cancel{4}\ (4-3)\ \cancel{1}\ 0$
5	0	0	0	1	1	$\cancel{2}\ (2-1)\ 1$
6	1	0	0	1	0	$\cancel{3}\ (3-3)\ 0$
W	3	1	3	0	1	
P_j	3	3	3	3	3	

$$w^* = (3 \quad 1 \quad 3 \quad 0 \quad 1)$$

(1) if $w_J \neq 0$

$$\sum_i a_{iJ} y_i + z_j - 1 = 0$$

- For $J = 1, 2, 3, 5$

$$\sum_i a_{iJ} y_i + z_j - 1 = 0$$

(2) $(w_J - p_J) z_J = 0$

if $w_J < p_J \Rightarrow z_J = 0$

So, $z_2 = 0$

$z_4 = 0$

$z_5 = 0$

3) if $f_i - \sum_J a_{iJ} w_J > 0$

$\Rightarrow y_i = 0$

So, $y_2 = 0$

$y_3 = 0$

$y_5 = 0$

$J = 1$, $y_4 + y_6 + z_1 = 1$

$J = 2$, $y_1 + y_3 + y_4 + z_2 = 1$

$J = 3$, $y_1 + y_2 + z_3 = 1$

$J = 5$, $y_1 + y_2 + y_5 + z_5 = 1$

$$\left\{\begin{array}{l} y_4 + y_6 + z_1 = 1 \\ y_1 + y_4 = 1 \\ y_1 + z_3 = 1 \\ y_1 = 1 \end{array}\right.$$

So, $y_1 = 1$

$z_3 = 0$

$y_4 = 0$

$\Rightarrow y_6 + z_1 = 1$

$y = [1 \ 0 \ 0 \ 0 \ 0 \ -]$, $z = [- \ 0 \ 0 \ 0 \ 0]$

- $y_6 + z_1 = 1$

1) If $y_6 = 1$, $z_1 = 0$

2) If $y_6 = 0$, $z_1 = 1$

cost (f_1)

1) $5 + \overset{\to \text{location } 6}{3} = 8$

2) $5 + \overset{\to P_1}{3} = 8$

$\Big\}$ check feasibility.

2) Location y_1 only not covered $= 4$. So it's infeasible

1) Location y_1 & y_6 covered all customers.

Since there are no uncovered customers, it's feasible.

Therefore

$w^* = [3 \quad 1 \quad 3 \quad 0 \quad 1]$

$y = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]$

$z = [0 \quad 0 \quad 0 \quad 0 \quad 0]$

Location 1 & 6 cover all customers with cost 8.

Question 4: a)

Let V_1 : set of alternative locations (1, 2, 3, 4)

V_2 : set of customer (1, 2, 3, 4)

d_{ij} : distance btw $i \in V_1$ and $j \in V_2$

Question 4: a)

Optimum 1-center location problem.

Loc. \ Cust	1	2	3	4	max d_{ij}
1	0	3	3	4	4
2	3	0	5	7	7
3	3	5	0	7	7
4	4	7	7	0	7

pick min d_{ij}

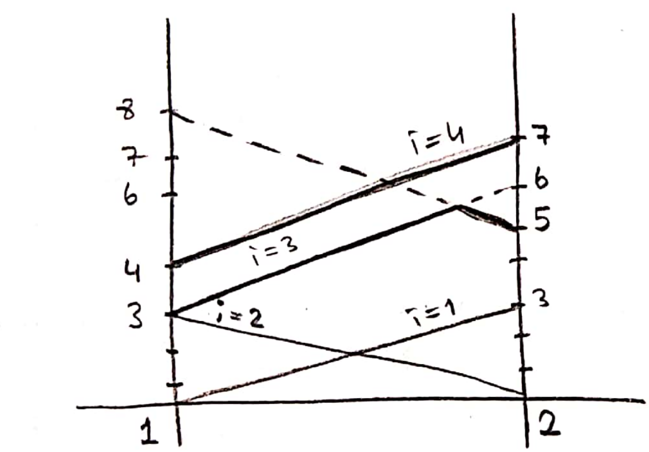
$= 4$

Therefore DC should open node 1 in order to minimize the longest distance to any customer.

b) $V = \{1, 2, 3, 4\}$

$E = \{(1,3), (1,2), (1,4), (2,3)\}$

Edge (1,2)



$i = 3$ from $1 : 3 \to 6^*$

$2 : 5 \to 8^*$

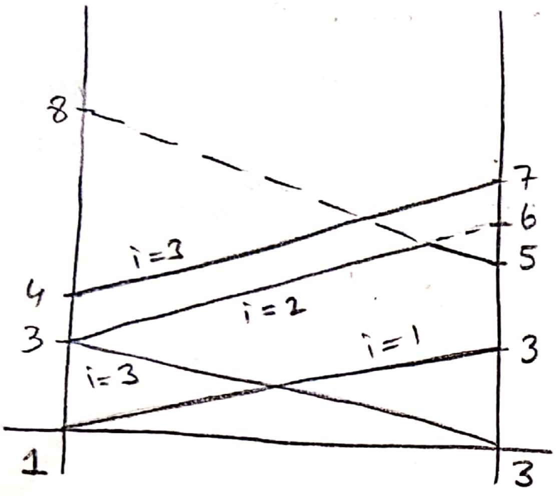
$i = 4$ from $1 : 4 \to 7^*$

$2 : 7 \to 10$ (meaningless)

Worst node is 4.

To minimize the distance to 4 locate at 1, cost is 4.

Edge (1,3)

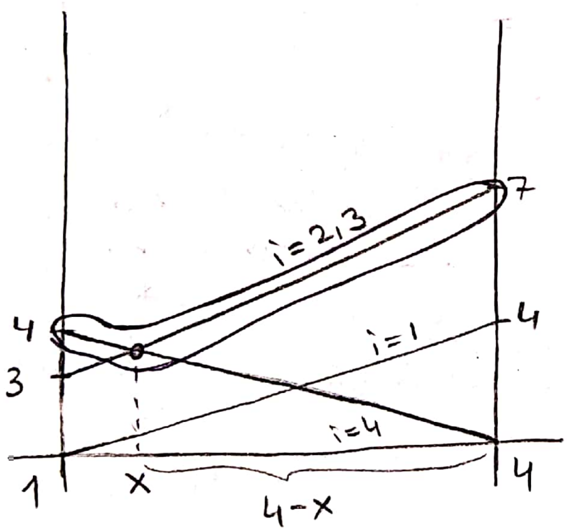


i=2 from 1: 3 → 6 *
3: 5 → 8

i=4 from 1: 4 → 7 *
3: 7 → 10

Worst node is 4.
To minimize the distance to 4
locate at 1, cost is $\underline{4}$.

Edge (1,4)



i=2 from 1: 3 → 7 *
4: 7 → 11

i=3 from 1: 3 → 7 *
4: 7 → 11

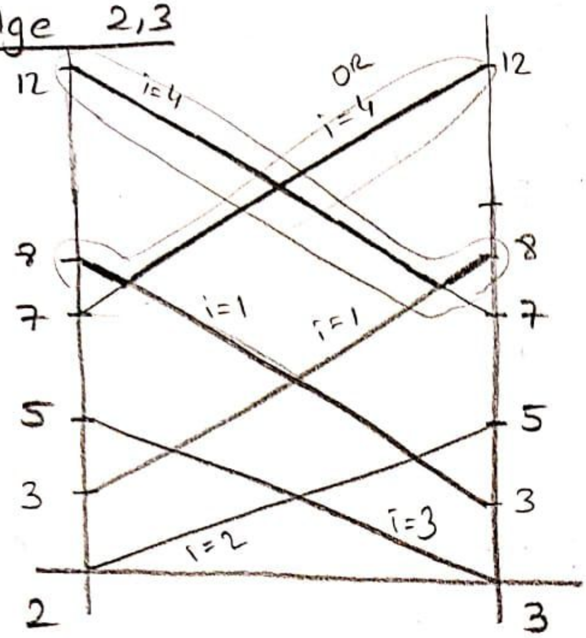
Upper envelope is determined by 3 nodes. Node 2 & 3 & 4.
For the upper envelope best is at a point where the distance to 2 & 3 = distance to 4

$3 + x = 4 - x$

$x = 0.5$

cost is $\boxed{3.5}$

Edge 2,3



i=1 from 2 : 3 → 8

3 : 3 → 8

i=4 from 2 : 7 → 12

3 : 7 → 12

Upper envelope is determined by 2 nodes. Node 1 & 4

(locate 2 or 3)

For the upper envelope best is

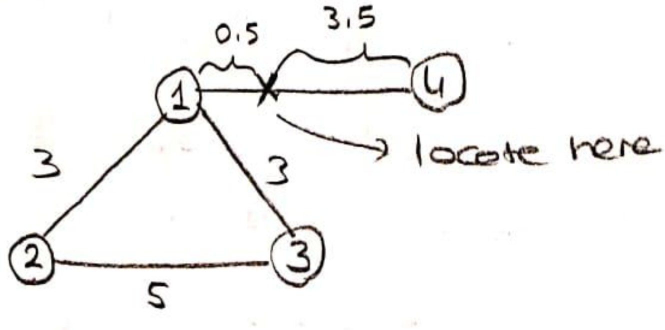
where the distance to 1 = distance to 4

→ (cost = 7)

$3 + x = 7 + x$ ⇒ There is no solution.

(So, choosing a point on the arc (2,3) will not give us a more improved solution than the choosing a point on node 2 & 3.)

→ Optimum 1-center location is 0.5 from node 1.
Because it gives min cost. (= 3.5)



Question 6:

a)

Customers $\Rightarrow N = \{1, 2, 5, 8, 9, 10\}$

Sites $\Rightarrow M = \{3, 4, 6, 7\}$

Coverage matrix $A = [a_{ij}]$ for $T = 5$

$$
\begin{array}{c|cccccc}
\text{Sites} \backslash \text{Customers} & 1 & 2 & 5 & 8 & 9 & 10 \\
\hline
3 & 1 & 1 & 1 & 0 & 0 & 0 \\
4 & 1 & 1 & 1 & 0 & 1 & 0 \\
6 & 1 & 0 & 0 & 0 & 0 & 1 \\
7 & 0 & 0 & 1 & 1 & 1 & 1
\end{array}
$$

b) Decision Variables

$$y_i = \begin{cases} 1 & \text{if } \exists \text{ a facility at } i \in M \\ 0 & \text{otw} \end{cases}$$

$$z_j = \begin{cases} 1 & \text{if customer } j \text{ is not covered}, \ j \in N \\ 0 & \text{otw} \end{cases}$$

Given $p_j = 3 \quad \forall j, j \in N$ and $f = (5, 9, 6, 8)$

Consider

$$\min \sum_{i \in M} f_i y_i + \sum_{j \in N} p_j z_j$$

st $$\sum_{i \in M} a_{ij} y_i + z_j \geq 1$$

$$y_i, z_j \in \{0, 1\} \quad \forall j, j \in m \quad \forall i, i \in N$$

take dual $\Longrightarrow$

$$\max \sum_{j \in N} w_j$$

st $$\sum_{j \in N} a_{ij} w_j \leq f_i \quad \forall i, i \in M$$

$$0 \leq w_j \leq p_j \quad \forall j, j \in N$$

(cont'd dual)

$$\max \quad w_1 + w_2 + w_5 + w_8 + w_9 + w_{10}$$

s.t

$$w_1 + w_2 + w_5 \le f_1 = 5$$

$$w_1 + w_2 + w_5 + w_9 \le f_2 = 9$$

$$w_1 + w_{10} \le f_3 = 6$$

$$w_8 + w_9 + w_{10} \le f_4 = 8$$

$$0 \le w_j \le p_j \quad \forall j, \; j = 1, 2, 5, 8, 9, 10$$

sites \ customer

	1	2	5	8	9	10	f_i
3	1	1	1	0	0	0	~~5~~ ~~2~~ 0
4	1	1	1	0	1	0	~~9~~ 6 4 1
6	1	0	0	0	0	1	~~6~~ 3 1
7	0	0	0	1	1	1	~~8~~ 5 2 0
w_j	3	2	0	3	3	2	
p_j	3	3	3	3	3	3	

Star $w_1 \Rightarrow \overset{f_1}{(5-3)}, \overset{f_2}{(9-3)}, \overset{f_3}{(6-3)} \Rightarrow w_1 = 3$

then $w_2 \Rightarrow (2-2), (6-2) \Rightarrow w_2 = 2$

" $w_5 \Rightarrow f_3 = 0 \Rightarrow w_5 = 0$

" $w_8 \Rightarrow (8-3) \Rightarrow w_8 = 3$

" $w_9 \Rightarrow (4-3), (5-3) \Rightarrow w_9 = 3$

" $w_{10} \Rightarrow (2-2), (3-2) \Rightarrow w_{10} = 2$

$$w^* = [3 \; 2 \; 0 \; 3 \; 3 \; 2]$$

Apply greedy approach

(1) if $w_j \neq 0 \Rightarrow \sum_{i \in m} a_{ij} y_i + z_j - 1 = 0 \quad \forall j,\ j = 1, 2, 8, 9, 10$

for $j = 1 \Rightarrow y_3 + y_4 + y_6 + z_1 = 1$

" $j = 2 \Rightarrow y_3 + y_4 + z_2 = 1$

" $j = 8 \Rightarrow y_7 + z_8 = 1$

" $j = 9 \Rightarrow y_4 + y_7 + z_9 = 1$

" $j = 10 \Rightarrow y_6 + y_7 + z_{10} = 1$

Considering values from (2) & (3)

- $y_3 + z_1 = 1 \Rightarrow \boxed{z_1 = 0}$
- $\boxed{y_3 = 1}$
- $y_7 + z_8 = 1 \Rightarrow \boxed{z_8 = 1}$
- $y_7 + z_9 = 1 \Rightarrow \boxed{z_9 = 1}$
- $\boxed{y_7 = 1}$

(2) $(w_j - p_j) z_j = 0 \quad \forall j,\ j = 1, 2, 8, 9, 10$

if $w_j < p_j \Rightarrow z_j = 0$

$\Rightarrow \boxed{z_2 = z_5 = z_{10} = 0}$

(3) $\left(\sum_{i \in m} a_{ij} w_j - f_i \right) y_i = 0$, if $f_i - \sum_{i \in m} a_{ij} w_j > 0 \Rightarrow y_i = 0$

$\Rightarrow \boxed{y_4 = y_6 = 0}$

At the end

$w^* = [3 \ 2 \ 0 \ 3 \ 3 \ 2]$

$$\begin{array}{c} \quad y_3 \ \ y_4 \ \ y_6 \ \ y_7 \\ y = [1 \ \ 0 \ \ 0 \ \ 1] \end{array}$$

$$\begin{array}{c} \quad z_1 \ \ z_2 \ \ z_5 \ \ z_8 \ \ z_9 \ \ z_{10} \\ z = [0 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 0] \end{array}$$

So, sites 3 and 7 should be opened. Because they cover all of customers.

Total cost $= f_3 + f_7 = 5 + 8 = \underline{13}$

Question 7:

Sets V_1 = "set of alternative locations" (1, 2, 3, 4)

V_2 = "set of customers" (1, 2, 3, 4)

d_{ij} = distance btw $i \in V_1$ and $j \in V_2$

a) Vertex - Restricted p-center problem (p=1)

$$\begin{array}{c|cccc} L \backslash C & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 7 & 5 & 9 \\ 2 & 7 & 0 & 3 & 6 \\ 3 & 5 & 3 & 0 & 4 \\ 4 & 9 & 6 & 4 & 0 \end{array} \qquad \begin{array}{c} \max d_{ij} \\ \hline 9 \\ 7 \\ \boxed{5} \\ 9 \end{array} \Bigg\} = \underline{\underline{5}} \quad \text{pick min } d_{ij}$$

Therefore DC should open node 3 in order to minimize the longest distance to any customer

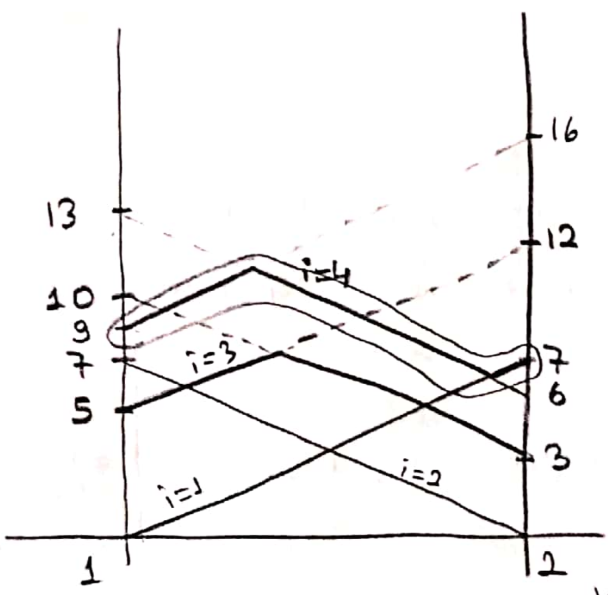
b) Absolute 1-center problem

Using Hakimi's Algorithm.

$V = \{1, 2, 3, 4\}$ $(= V_1 = V_2)$

$E = \{(1,2), (1,3), (2,3), (2,4), (3,4)\}$

Edge (1,2)



i=3 from 1 : 5 → 12

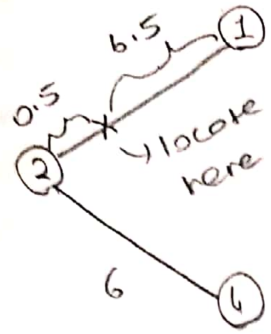
2 : 3 → 10

i=4 from 1 : 9 → 16

2 : 6 → 13

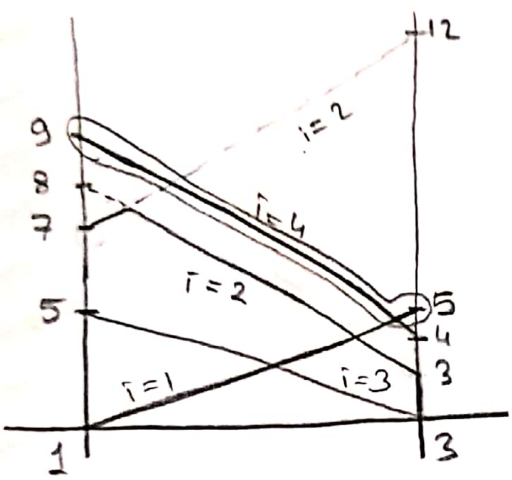
Upper envelope is determined by 2 nodes. Nodes 1 & 4. For the upper envelope best is at a point where the distance to 1 = distance 4

$7 - x = 6 + x \Rightarrow x = 0.5$



t is 6.5

Edge (1,3)

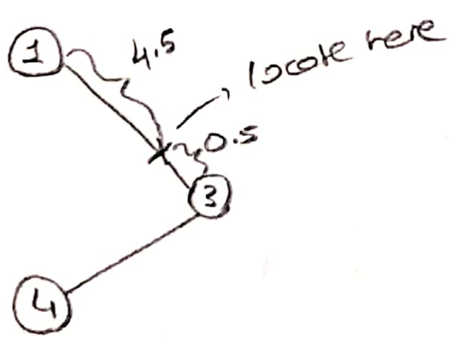


i=2 from 1 : 7 → 12
3 : 3 → 8

i=4 from 1 : 9 → 14
3 : 4 → 9 *

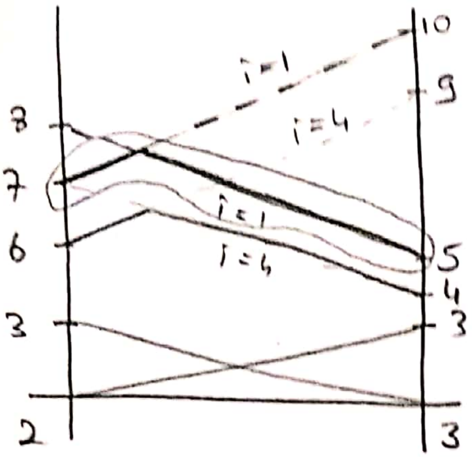
Upper envelope is determined by 2 nodes. Nodes (1 & 4) for the upper envelope best is at a point where the distance 1 = distance 4

$\Rightarrow 5 - x = 4 + x \quad \Rightarrow x = 0.5$



t is 4.5

Edge (2,3)



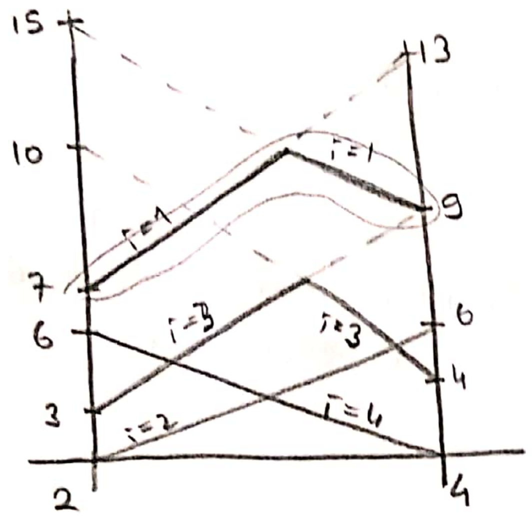
i=1 from 2: 7→10
3: 5→8

i=4 from 2: 6→9
3: 4→7

Worst node is 1, locate at 3

t is 5

Edge (2,4)



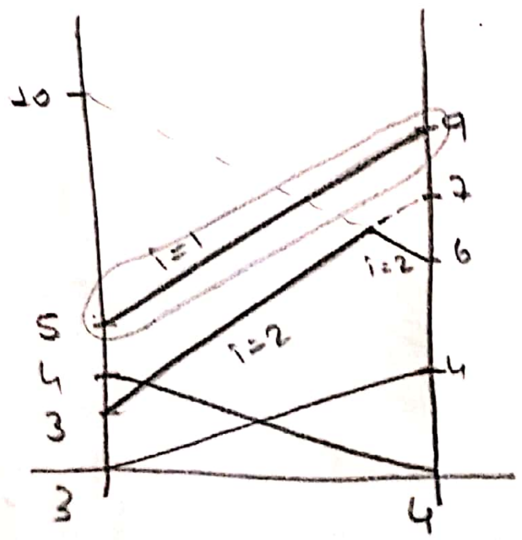
i=1 from 2: 7→13
4: 9→15

i=3 from 2: 3→9
4: 4→10

Worst node is 1, locate at 2

t is $\underline{\underline{7}}$

Edge 3,4



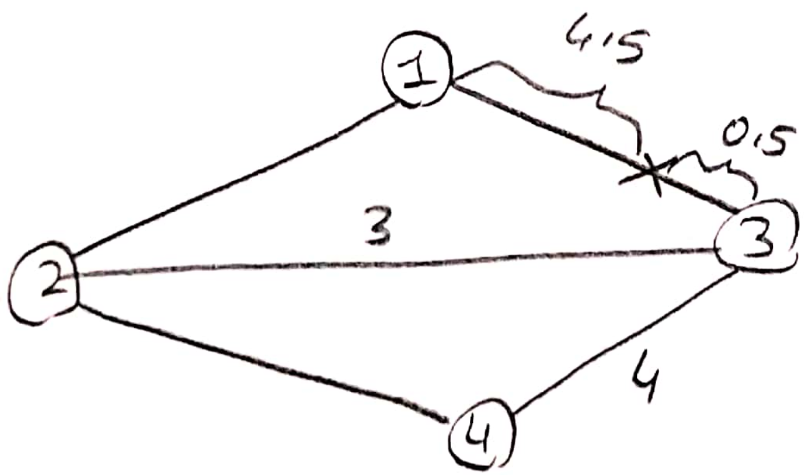
i=2 from 3: 3→7
4: 6→10

i=1 from 3: 5→9 *
4: 9→13

Worst node is 1, locate at 3

⇒ t is $\underline{\underline{5}}$

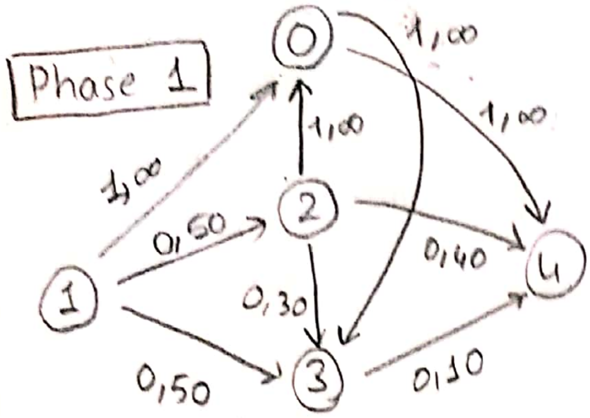
Finally when we consider Edge (1,3), it's the minimum distance and locate as

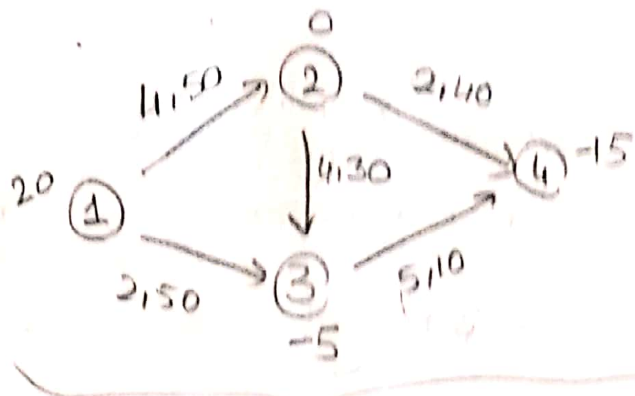


$\Rightarrow t = \underline{\underline{4{,}5}}$

Question 8:

Phase 1





Add an artificial node (node 0) and artificial arcs.

$c_{ij} = 0$ for all $(i,j) \in E$

$c_{ij} = 1$ for all artificial arcs

u_{ij} = as given for orginal arcs

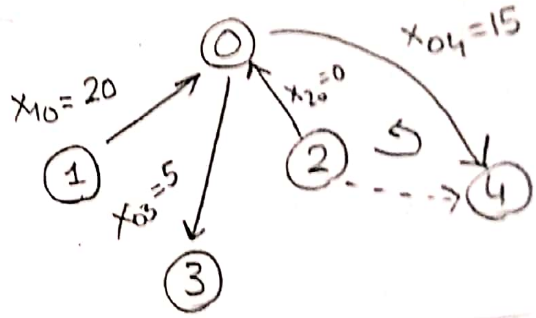
$u_{ij} = \infty$ for artificial arcs

If $b_i > 0$ then add $(i,0)$ with flow b_i

If $b_i < 0$ then add $(0,i)$ with flow $-b_i$

If $b_i = 0$ then add $(i,0)$ or $(0,i)$ w/flow 0

Immediate bfs:



Apply network simplex.

Consider non-basic arc $(2,4)$

$x_{24} = 0$ so it should be a forward arc.

change in cost $= c_{24} - (c_{20} + c_{04})$

$= 0 - (1+1) = -2 < 0$ violates.

So improving and enter $(2,4)$

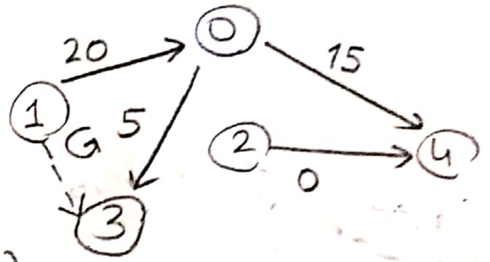
Ratio test

$$\Delta = \min \{ (u_{24} - x_{24}), x_{04}, x_{20} \}$$

$$= \min(40, 15, 0) = 0$$

so arc $(2,0)$ leaves.

Consider non-basic arc (1,3)



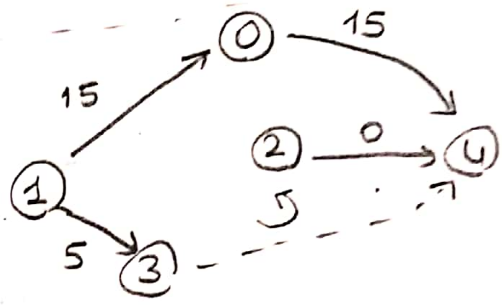
(1,3) → it should be forward

Change in cost = 0 - 1 - 1 = -2 (violates)

$\Delta = \min\{50, 5, 20\}$

$\Delta = 5$

Arc (0,3) leaves.



Consider arc (3,4)

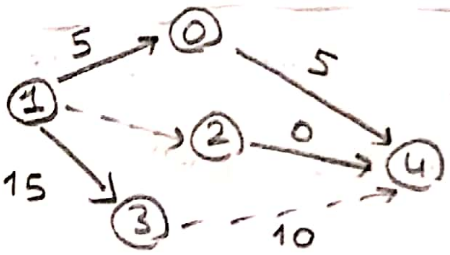
→ improving and forward

Change in cost = 0 - 1 - 1 = -2

$\Delta = \min\{(u_{34} - x_{34}), (u_{13} - x_{13}), x_{04}, x_{10}\}$

$= \min\{(10-0), (50-5), 15, 15\} = 10$

Arc (3,4) remains nonbasic (enters and leaves immediately)



Consider arc (1,2)

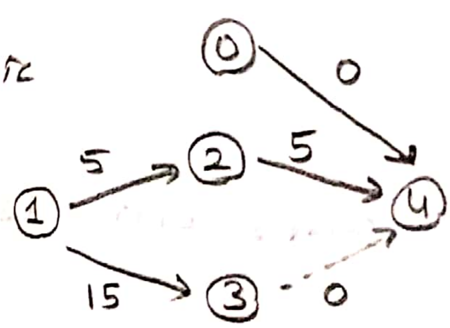
Forward

Change in cost $= (c_{12} + c_{24} - (c_{04} + c_{01}))$
$= 0 + 0 - 1 - 1 = -2$

$\Delta = \min\{50, 40, 5, 5\} = 5$

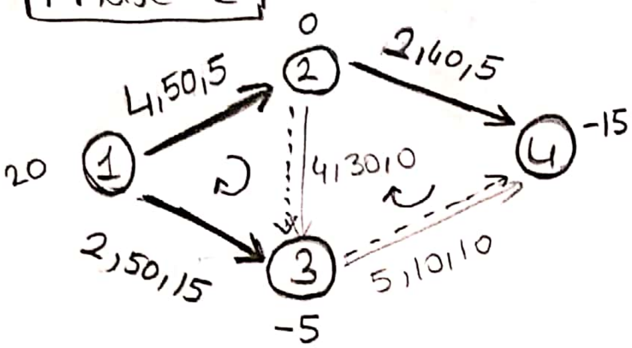
(0,4) or (1,0) becomes non basic

say (1,0) leaves



One artificial arc left with zero flow. Eliminate it and artificial node 0.

Phase 2



Consider non basic arc (2,3),

It should be forward arc

change in cost = $4-2+4=6>0$

So, no improvement

Consider non basic arc (3,4)

It should be backword arc.

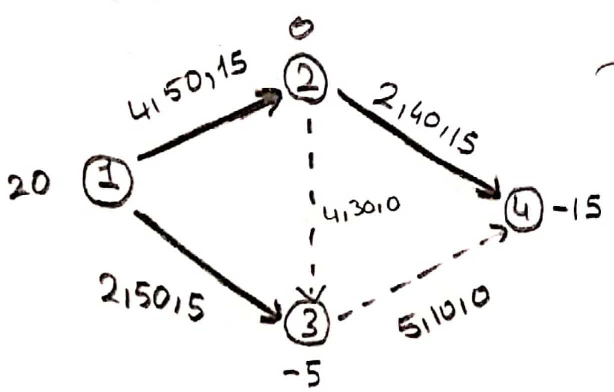
Change in cost = $4+2-5-2=-1<0$

improvement ✓

$\theta = \min\{(50-5), (40-5), 10, 15\}$

$= 10$

→ Arc (3,4) remains non-basic.



There is no more improvement

→ Optimum shipment structure

Basic arcs: (1,2), (1,3), (2,4)

$x_{12} = 15$, $x_{13} = 5$, $x_{24} = 15$

Non-basic arcs: (2,3), (3,4)

$x_{23} = 0$, $x_{34} = 0$

Total cost = $(4\times15)+(2\times15)+(2\times5) = \boxed{100}$